Symmetries in Physics
Lecture 13

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Lecture contents

Chapter 4. Special unitary groups

- IV.1. Internal symmetries: Isospin
- IV.2. Strange quark and the eight-fold way
- IV.3. Gauge theories of elementary particles

IV.1. Internal symmetries: Isospin
IV.1.2. Proton-neutron degeneracy

- The study of the states allowed under the irreps of $SO(3)$ showed a remarkable property of elementary particles: internal angular momentum (spin).
- In addition, elementary particles posses other internal degrees of freedom.
- The discovery of the neutron n by Chadwick (1932) prompted physicists (Heisenberg, Fermi, Yukawa, ...) to speculate a tight relation with the proton p, since:

$$\frac{M_n - M_p}{M_n} \simeq \frac{939.6 - 938.3}{939.6} \simeq 0.00138. \tag{1}$$

- Wigner (1935) proposed that the nuclear force does not distinguish between n and p, who form a doublet $N = \begin{pmatrix} p \\ n \end{pmatrix}$ of an $SU(2)$ symmetry group: the "isotopic spin," or *isospin*.
- Yukawa (1935) proposed that the nuclear interaction between n and p must be mediated by mesons called *pions* (found in 1947).

IV.1.2. Nuclear theory of $SU(2)_I$ isospin

- Consider $\psi_N = \begin{pmatrix} p \\ n \end{pmatrix}$ the nucleon wavefunction, with

$$\psi_N^\alpha \to \psi_N'^\alpha = U^\alpha{}_\beta \psi_N^\beta, \quad I_3 \begin{pmatrix} p \\ n \end{pmatrix} = \begin{pmatrix} \frac{1}{2} p \\ -\frac{1}{2} n \end{pmatrix}, \tag{2}$$

  such that $I_p = \frac{1}{2}$ and $I_n = -\frac{1}{2}$.
- As postulated by Yukawa, the strong force between nucleons is mediated by $\pi^\pm$:

$$p \to n + \pi^+, \qquad n \to p + \pi^-. \tag{3}$$

- Conservation of isospin entails $I_{\pi^\pm} = \pm 1 \Rightarrow \pi^\pm$ must correspond to the $I = 1$ irrep of $SO(2)_I$.
- The predicted π^0 ($I_{\pi^0} = 0$) was discovered in 1950.
- Yukawa's proposal was for a coupling of the form $ig\overline{\psi}_N \gamma^5 \phi \psi_N$, with f the coupling constant and

$$\phi = \boldsymbol{\pi} \cdot \boldsymbol{\tau} = \frac{1}{2} \begin{pmatrix} \pi_3 & \pi_1 - i\pi_2 \\ \pi_1 + i\pi_2 & -\pi_3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix}, \tag{4}$$

  where $\boldsymbol{\tau} = \frac{1}{2}\boldsymbol{\sigma}$ are the $SU(2)$ generators for $I = 1/2$.

IV.1.3. Subnuclear theory of isospin: quarks

- $\pi^+ + p \to \pi^+ + p$ showed a peak at $\sqrt{s} \simeq 1232$ MeV $\Rightarrow$ new resonant channel: $\pi^+ + p \to \Delta^{++} \to \pi^+ + p$.
- since $I_{3;\mathrm{in}} = I_{3;\mathrm{out}} = \frac{3}{2}$, Δ^{++} is part of the $I = 3/2$ family.
- Δ^+, Δ^0 and Δ^{*-} were shortly discovered.
- Similar particle sets falling within the same isospin family were observed: Δ baryons; kaons; etc.
- At low energies, the spectrum of observed particles can be explained via the u and d quarks, forming a doublet w.r.t. the strong interaction: $\psi_q = \begin{pmatrix} u \\ d \end{pmatrix}$.
- Baryons (p, n,...) are formed of 3 quarks; mesons are formed by $\bar{q}q$.
- Assumption: The isospin symmetry of (p, n) and $(\pi^\pm, \pi^0)$ is inherited from that of u, d. Assume:

$$I_3 \begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} \frac{1}{2} u \\ -\frac{1}{2} d \end{pmatrix}, \qquad Q \begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} \frac{2}{3} u \\ -\frac{1}{3} d \end{pmatrix}. \tag{5}$$

- Then $p = uud$ ($Q = 1$, $I_3 = 1/2$) and $n = udd$ ($Q = 0$, $I_3 = -1/2$).
- Similarly, $\pi^+ = u\bar{d}$, $\pi^- = \bar{\pi}^+ = d\bar{u}$ and $\pi^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$.

IV.1.4. Hadrons as direct-product representations

- The direct product of three $I = 1/2$ quarks leads to

$$\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = (1 \oplus 0) \otimes \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{2} \oplus \frac{1}{2}. \tag{6}$$

- The $I = \frac{3}{2}$ irrep corresponds to the Δ baryons, with

$$\begin{aligned} |\Delta^{++}\rangle &= |uuu\rangle\,, \\ |\Delta^{+}\rangle &= \frac{1}{\sqrt{3}}(|uud\rangle + |udu\rangle + |duu\rangle), \\ |\Delta^{0}\rangle &= \frac{1}{\sqrt{3}}(|ddu\rangle + |dud\rangle + |udd\rangle), \\ |\Delta^{-}\rangle &= |ddd\rangle\,. \end{aligned} \tag{7}$$

- The p and n inhabit one of the $\frac{1}{2}$ representations:

$$\begin{aligned} |p\rangle &= \frac{1}{\sqrt{6}}(2\,|duu\rangle - |udu\rangle - |uud\rangle), \\ |n\rangle &= \frac{1}{\sqrt{6}}(2\,|ddu\rangle - |dud\rangle - |udd\rangle). \end{aligned} \tag{8}$$

- The observed particle spectra includes many resonances that lie at higher mass-energy.

IV.1.5. QCD color singlets

- Pauli exclusion principle requires that fermionic ($J = \frac{1}{2}, \frac{3}{2}, \dots$) wavefunctions be antisymmetric w.r.t. the interchange of identical constituents.
- $\Delta^{++} = uuu$ is viable only if there is a hidden structure giving it its anti-symmetry: the color structure.
- QCD is an $SU(3)$ interaction between quarks, that posses one of three colours.
- Aside from spin quarks possess flavour and colour quantum numbers: $q \to |f, s, c\rangle$, where $f \in \{u, d, \dots\}$, $s = \pm 1/2$ (spin) and $c \in \{r, g, b\}$.
- Hadrons are in color-singlet states. Baryons must have

$$|B\rangle \to |qqq\rangle \sim \frac{1}{\sqrt{6}}(|rgb\rangle + |gbr\rangle + |brg\rangle - |rbg\rangle - |bgr\rangle - |grb\rangle). \quad (9)$$

IV.1.6. Structure of Δ baryons

- Coming back to $\Delta^{++} = |uuu\rangle$, the flavour $\otimes$ spin part must be totally symmetric, therefore

$$|\Delta^{++}_{3/2}\rangle = |u_\uparrow u_\uparrow u_\uparrow\rangle \,. \tag{10}$$

- Note that $J = 3/2$ for the Δ baryons. $J = 1/2$ would require a non-symmetric wavefunction.
- $\Delta^{+}_{3/2}$, $\Delta^{0}_{3/2}$ and $\Delta^{-}_{3/2}$ can be obtaind by applying the isospin I_-:

$$\begin{aligned}
|\Delta^{+}_{3/2}\rangle &= \frac{1}{\sqrt{3}}(|u_\uparrow u_\uparrow d_\uparrow\rangle + |u_\uparrow d_\uparrow u_\uparrow\rangle + |d_\uparrow u_\uparrow u_\uparrow\rangle),\\
|\Delta^{0}_{3/2}\rangle &= \frac{1}{\sqrt{3}}(|d_\uparrow d_\uparrow u_\uparrow\rangle + |d_\uparrow u_\uparrow d_\uparrow\rangle + |u_\uparrow d_\uparrow d_\uparrow\rangle),\\
|\Delta^{-}_{3/2}\rangle &= |d_\uparrow d_\uparrow d_\uparrow\rangle \,.
\end{aligned} \tag{11}$$

- $\Delta^{\cdots}_{1/2}$ can be obtaind by applying the rotational J_- on $\Delta^{\cdots}_{3/2}$.
- The $(I, J) = (\frac{1}{2}, \frac{3}{2})$ irrep can be obtained from the above simply as:

$$\begin{aligned}
|N^{+}_{3/2}\rangle &= \frac{1}{\sqrt{6}}(2\,|u_\uparrow u_\uparrow d_\uparrow\rangle - |u_\uparrow d_\uparrow u_\uparrow\rangle - |d_\uparrow u_\uparrow u_\uparrow\rangle),\\
|N'^{+}_{3/2}\rangle &= \frac{1}{\sqrt{2}}(|u_\uparrow d_\uparrow u_\uparrow\rangle - |d_\uparrow u_\uparrow u_\uparrow\rangle).
\end{aligned} \tag{12}$$

IV.1.7. Proton and neutron quark structure

- Applying $J_- \Delta^+_{3/2} = \Delta^+_{1/2}\sqrt{3}$ gives

$$|\Delta^+_{1/2}\rangle = \frac{1}{3}\begin{pmatrix}|uud\rangle & |udu\rangle & |duu\rangle\end{pmatrix}\begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix}\begin{pmatrix}|\uparrow\uparrow\downarrow\rangle\\ |\uparrow\downarrow\uparrow\rangle\\ |\downarrow\uparrow\uparrow\rangle\end{pmatrix}. \qquad (13)$$

- $p_{\uparrow/\downarrow}$ corresponding to $I = I_3 = \frac{1}{2}$ and $J = \pm s = \frac{1}{2}$ is

$$|p_\uparrow\rangle = \frac{1}{\sqrt{18}}\begin{pmatrix}|duu\rangle & |udu\rangle & |uud\rangle\end{pmatrix}\begin{pmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ -1 & -1 & 2\end{pmatrix}\begin{pmatrix}|\uparrow\uparrow\downarrow\rangle\\ |\uparrow\downarrow\uparrow\rangle\\ |\downarrow\uparrow\uparrow\rangle\end{pmatrix},$$

$$|p_\downarrow\rangle = \frac{1}{\sqrt{18}}\begin{pmatrix}|duu\rangle & |udu\rangle & |uud\rangle\end{pmatrix}\begin{pmatrix}-2 & 1 & 1\\ 1 & -2 & 1\\ 1 & 1 & -2\end{pmatrix}\begin{pmatrix}|\uparrow\downarrow\downarrow\rangle\\ |\downarrow\uparrow\downarrow\rangle\\ |\downarrow\downarrow\uparrow\rangle\end{pmatrix}. \qquad (14)$$

- Applying I_- gives the neutron:

$$|n_\uparrow\rangle = \frac{1}{\sqrt{18}}\begin{pmatrix}|udd\rangle & |dud\rangle & |ddu\rangle\end{pmatrix}\begin{pmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ -1 & -1 & 2\end{pmatrix}\begin{pmatrix}|\uparrow\uparrow\downarrow\rangle\\ |\uparrow\downarrow\uparrow\rangle\\ |\downarrow\uparrow\uparrow\rangle\end{pmatrix},$$

$$|n_\downarrow\rangle = \frac{1}{\sqrt{18}}\begin{pmatrix}|udd\rangle & |dud\rangle & |ddu\rangle\end{pmatrix}\begin{pmatrix}-2 & 1 & 1\\ 1 & -2 & 1\\ 1 & 1 & -2\end{pmatrix}\begin{pmatrix}|\uparrow\downarrow\downarrow\rangle\\ |\downarrow\uparrow\downarrow\rangle\\ |\downarrow\downarrow\uparrow\rangle\end{pmatrix}. \qquad (15)$$

IV.1.8. Anti-quarks

- Mesons are $m \sim \bar{q}q$ states of integer spin (bosons).
- The requirement of color singlet imposes $m \sim |\bar{r}r\rangle\,, |\bar{g}g\rangle\,, |\bar{b}b\rangle$.
- $\psi^i = \begin{pmatrix} u \\ d \end{pmatrix}$ and $\bar{\psi}_i \equiv \psi^{*i} = \begin{pmatrix} \bar{u} & \bar{d} \end{pmatrix}$.
- In general, ψ^i refers to quarks and $\bar{\psi}_i$ refers to anti-quarks.
- Under $SU(2)_I$, $\psi^i \to \psi'^i = U^i{}_j \psi^j$ and

$$\bar{\psi}_i \to \bar{\psi}'_i = U^{*;i}{}_j \psi^{*;j} = \bar{\psi}_j U^{\dagger;j}{}_i. \tag{16}$$

- For $SU(2)$, $U = e^{-\frac{i}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\xi}}$ and U^* are related by a symmetry transformation:

$$\sigma_2 U \sigma_2 = e^{-\frac{i}{2}\sigma_2(\boldsymbol{\sigma}\cdot\boldsymbol{\xi})\sigma_2} = e^{\frac{i}{2}\boldsymbol{\sigma}^*\cdot\boldsymbol{\xi}} = U^*. \tag{17}$$

- For $SU(2)_I$, q and $\bar{q}$ transform under equiv. reps., and the lower index $\bar{\psi}_i$ can be raised by multiplication with $i\sigma^2$:

$$\bar{\psi}_i \to \bar{\psi}^i = i(\sigma_2)^{ij}\bar{\psi}_j = \varepsilon^{ij}\bar{\psi}_j = \begin{pmatrix} \bar{d} \\ -\bar{u} \end{pmatrix}. \tag{18}$$

- Now $\bar{\psi}'^i = \varepsilon^{ij}\bar{\psi}_k (U^\dagger)^k{}_j = \varepsilon^{lj}\bar{\psi}_k U^{\dagger\,k}{}_j U^{\dagger\,m}{}_l U^i{}_m = U^i{}_m \bar{\psi}^m$, since $\varepsilon^{lj} U^{\dagger\,m}{}_l U^{\dagger\,k}{}_j = \varepsilon^{mk}\det(U^\dagger)$ and $\det(U^\dagger) = \det(U) = 1$.

IV.1.9. Meson structure

- Therefore, we can obtain the irreps of $q\bar{q}$ states as $\frac{1}{2}\otimes\left(\frac{1}{2}\right)^* = 1\oplus 0$.
- The $I=1$ representation corresponding to $\varphi^i_j = \psi^i\bar{\psi}_j - \delta^i_j(\psi^k\bar{\psi}_k)$ is known as the *adjoint representation*.
- The state with $J=I=1$ and $s=I_3=1$ is $\rho^+ = \psi^u\tilde{\psi}^u = u\bar{d}$,

$$|\rho^+_{s=1}\rangle = |u_\uparrow\bar{d}_\uparrow\rangle\,,\quad Q\,|\rho^+_{+1}\rangle = |\rho^+_{+1}\rangle\left[\tfrac{2}{3}-\left(-\tfrac{1}{3}\right)\right] = |\rho^+_{+1}\rangle\,,$$
$$|\rho^0_{+1}\rangle = \frac{1}{\sqrt{2}}\left(|d_\uparrow\bar{d}_\uparrow\rangle - |u_\uparrow\bar{u}_\uparrow\rangle\right)\,,\quad |\rho^-_{+1}\rangle = -\,|d_\uparrow\bar{u}_\uparrow\rangle \tag{19}$$

- The missing $J=s=1$ and $I=I_3=0$ vector meson is $|\omega\rangle = \frac{1}{\sqrt{2}}(|u_\uparrow\bar{u}_\uparrow\rangle + |d_\uparrow\bar{d}_\uparrow\rangle)$.
- Since $|\rho^+_{s=0}\rangle = \frac{1}{\sqrt{2}}(|u_\downarrow\bar{d}_\uparrow\rangle + |u_\uparrow\bar{d}_\downarrow\rangle)$, we readily derive the $J=0$ and $I=1$ mesons:

$$|\pi^+\rangle = \frac{1}{\sqrt{2}}(|u_\uparrow\bar{d}_\downarrow\rangle - |u_\downarrow\bar{d}_\uparrow\rangle),$$
$$|\pi^0\rangle = \frac{1}{2}(|d_\uparrow\bar{d}_\downarrow\rangle - |d_\downarrow\bar{d}_\uparrow\rangle - |u_\uparrow\bar{u}_\downarrow\rangle + |u_\downarrow\bar{u}_\uparrow\rangle),$$
$$|\pi^-\rangle = -\frac{1}{\sqrt{2}}(|d_\uparrow\bar{u}_\downarrow\rangle - |d_\downarrow\bar{u}_\uparrow\rangle). \tag{20}$$

- For $J=I=0$, $|\eta\rangle = \frac{1}{2}(|d_\uparrow\bar{d}_\downarrow\rangle - |d_\downarrow\bar{d}_\uparrow\rangle + |u_\uparrow\bar{u}_\downarrow\rangle - |u_\downarrow\bar{u}_\uparrow\rangle)$.

IV.2. Strange quark and the eight-fold way

IV.2.1. Strange quark

- The $SU(2)_I$ isospin model was highly successful to explain the hadron spectrum at low energies.
- Experiments at higher energies revealed a plethora of new particles, some of them exhibiting isospin symmetry (e.g., kaons; Sigma and Xi baryons).
- Gell-mann proposed that the approximate $SU(2)_I$ symmetry was part of a larger, less exact, symmetry group: $SU(3)_f$, which implied the existence of a third quark: the strange quark.
- In practice, $(m_u, m_d) \simeq (2.3, 4.8)\ \mathrm{MeV/c^2}$, while $m_s \simeq 95\ \mathrm{MeV}/c^2$, so the $SU(3)_f$ symmetry is broken at the level of $m_s/m_p \simeq 10\%$.
- We have $\psi^i = (u\ d\ s)^T$ and $\bar{\psi}_i = (\bar{u}\ \bar{d}\ \bar{s})$, transforming under the fundamental and complex conjugate irreps:

$$\psi'^i = U^i{}_j \psi^j, \quad \bar{\psi}'_i = \bar{\psi}_j U^{\dagger j}{}_i. \tag{21}$$

- The lower indices can be raised with ε^{ijk}, since

$$\bar{\psi}^{ij} = \varepsilon^{ijk} \bar{\psi}_k \to \bar{\psi}'^{ij} = U^i{}_k U^j{}_l \bar{\psi}^{kl}. \tag{22}$$

IV.2.2. Mesons and $SU(3)$ adjoint representation

- $SU(3)$ irreps are labeled by their size: 3 (fundamental), $\bar{3}$ (complex conjugate irrep), ...
- Mesons $\equiv q\bar{q}$ correspond to $3 \otimes \bar{3}$, represented as $\varphi^i_j = \psi^i \bar{\psi}_j$.
- The trace φ^k_k forms a singlet:

$$\varphi'^k_k = U^k_i \psi^i \bar{\psi}_j U^{\dagger j}{}_k = \varphi^k_k. \tag{23}$$

- The remainder, $\varphi^{(i)}_{(j)} = \varphi^i{}_j - \frac{1}{3}\varphi^k_k \delta^i_j$, transforms irreducibly under $SU(3)$, s.t.:

$$3 \otimes \bar{3} = 8 \oplus 1. \tag{24}$$

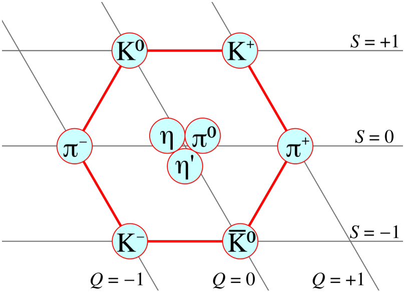
- $8 \equiv$ adjoint representation, corresponding also to gluons in QCD.
- **Theorem:** $\varphi^{(i_1 \dots i_m)}_{(j_1 \dots j_n)}$ denoting a tensor which is totally symmetric in the upper $(i_1, \dots i_m)$ and lower indices $(j_1, \dots j_n)$ and traceless w.r.t. upper-lower contractions: $\varphi^{(k i_2 \dots i_m)}_{(k j_2 \dots j_n)} = 0$ transforms irreducibly under $SU(3)$, under the irrep labelled (m, n), having dimension:

$$\dim(m, n) = \frac{1}{2}(m+1)(n+1)(m+n+2). \tag{25}$$

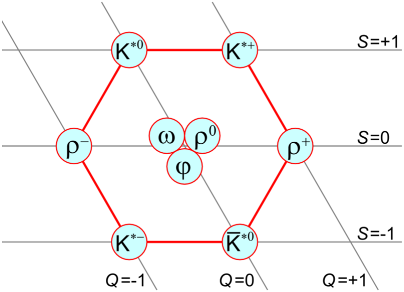
- *Test:* $\dim(1,1) = \frac{1}{2} \times 2 \times 2 \times 4 = 8$; $\dim(1,0) = \dim(0,1) = 3$.

Meson nonets

Scalar mesons ($J = 0$):



Vector mesons ($J = 1$):



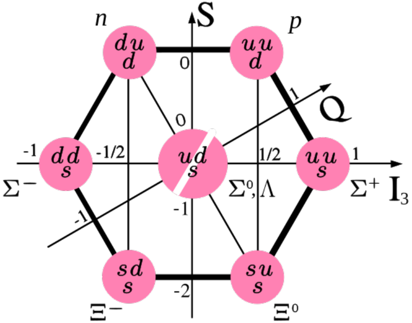
- The $3 \otimes \bar{3} = 8 \oplus 1$ irreps of the $SU(3)_f$ group correspond to mesonic families of $J = 0$ (left) and $J = 1$ (right).
- The families are arranged by strangeness (from top to bottom), taking the values $S_s = -1$ for the strange quark and $+1$ for the strange anti-quark.
- The tilted diagonal lines give the electric charge.
- The scalar mesons ($J = 0$) have a single degree of freedom each; the vector mesons ($J = 1$) form four-vectors, e.g. ω^μ.

IV.2.3. Baryons

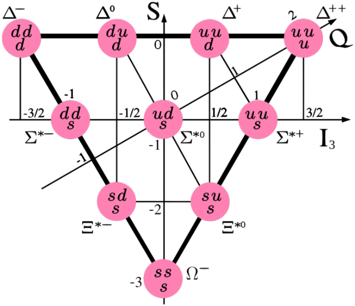
- Consider the tensor $\varphi^{ij} = \psi_1^i \psi_2^j$ representing a qq state. Wriitng
$$\varphi^{ij} = \frac{1}{2}(\varphi^{ij} + \varphi^{ji}) + \frac{1}{2}(\varphi^{ij} - \varphi^{ji}) = \varphi^{(ij)} + \varepsilon^{ijk}\varphi_{(k)}, \qquad (26)$$
we have $3 \otimes 3 = 6 \oplus \overline{3}$ [note that $\dim(2,0) = 6$].
- Baryons are composed of 3 quarks, characterized by $\varphi^{ijk} = \psi_1^i \psi_2^j \psi_3^k$.
- Since $3 \otimes 3 \otimes 3 = 3 \otimes 6 + 3 \otimes \overline{3}$, with $3 \otimes \overline{3} = 8 \oplus 1$, we only need to decompose
$$\varphi^{i(jk)} = \frac{1}{2}(\varphi^{ijk} + \varphi^{ikj}), \qquad \dim(\varphi^{i(jk)}) = 3 \times 6 = 18. \qquad (27)$$
- It can be seen that $\frac{1}{2}\varepsilon_{ijl}\varphi^{i(jk)} = \varphi_{(l)}^{(k)}$ is traceless and hence transforms as $(1,1) = 8$.
- Subtracting the degrees of freedom in $\varphi_{(l)}^{(k)}$, we arrive at
$$\varphi^{i(jk)} - \frac{2}{3}(\varepsilon^{ikl}\varphi_{(l)}^{(j)} + \varepsilon^{ijl}\varphi_{(l)}^{(k)}) = \varphi^{(ijk)}, \qquad (28)$$
i.e. we uncover the $(3,0) = 10$ irrep.
- The entire decomposition is then:
$$3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1. \qquad (29)$$

Baryon octet (8-fold way) and decuplet

$J = 1/2$:



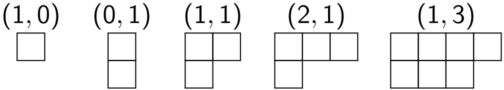
$J = 3/2$:



- The n and p ($S = 0$) form a $J = 1/2$, $SU(3)$ octet together with the $\Sigma^{\pm;0}$ ($S = -1$), Λ ($S = -1$) and $\Xi^{-;0}$ ($S = -2$) hyperons.
- The Δ baryons form a $J = 3/2$, $SU(3)$ decuplet, exhibiting at $S = -3$ the Ω^- baryon.

IV.2.4. Direct product decomposition by Young tableaux.

- The (m, n) irrep of $SU(3)$ is represented by the Young diagram with $m + n$ boxes on the first row and n boxes on the second one, e.g.:



- The decomposition of the direct products of two irreps T_1 and T_2 can be done straightforwardly using Young tableaux, in general for $SU(N)$.

a) Write T_1 and T_2, labelling the rows of T_2 successively with $a, b, \ldots$.

b) Add the boxes from T_2 one at a time, in order, according to the rules:

   (1) At each stage, T_1 must be a legal Young tableau.

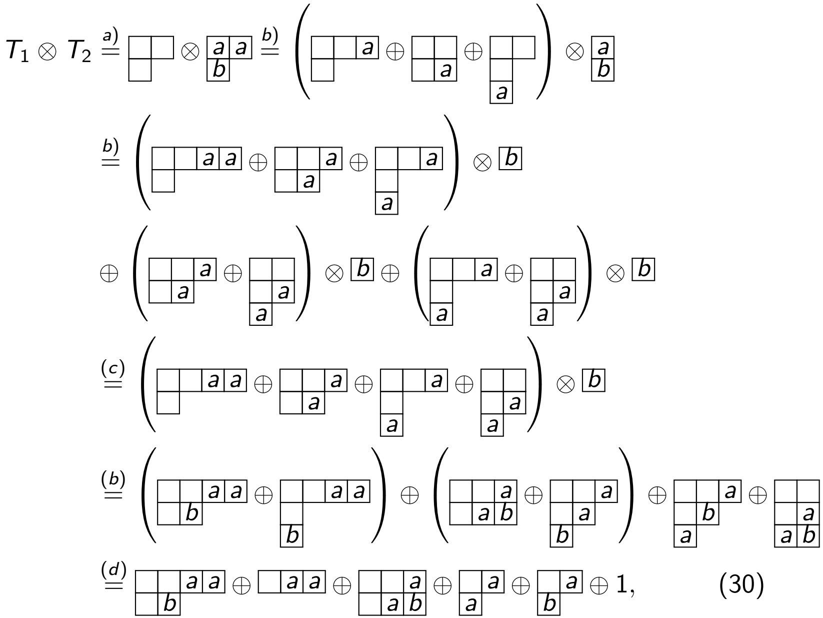
   (2) Boxes with the same label, e.g. a, must not appear on the same column.

   (3) For any box, define $n_a =$ no. of a boxes above and to the right of it. Then $n_a \geq n_b \geq n_c \geq \ldots$.

c) Two tableaux of the same shape are different if the labels are differently distributed.

d) Columns with N boxes, corresponding to the trivial irrep of $SU(N)$, are cancelled.

- Take $T_1 = (1,1)$ and $T_2 = (1,1)$ as an example:



(30)

i.e. $8 \otimes 8 = 27 \oplus 10 \oplus \overline{10} \oplus 8 \oplus 8 \oplus 1$.

IV.2.5. $SU(3)$: Lie algebra

- The generators of the $SU(N)$ Lie algebra satisfy $\mathrm{Tr}(T_a T_b) = \frac{1}{2}\delta_{ab}$.
- For $SU(3)$, we have $T_a = \frac{1}{2}\lambda_a$, where λ_a are the Gell-mann matrices:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{31}$$

- The commutators $[T^a, T^b] = if^{abc}T^c$ define the totally antisymmetric structure constants,

$$f^{123} = 1, \quad f^{458} = f^{678} = \frac{\sqrt{3}}{2},$$
$$f^{147} = f^{165} = f^{246} = f^{257} = f^{345} = f^{376} = \frac{1}{2}. \tag{32}$$

IV.2.6. Cartan subalgebra.

- **Def:** The set of generators of a Lie algebra that commute with all other generators form the **Cartan subalgebra**.
- For $SU(2)$, the Cartan subalgebra consists of $J_3 \equiv I_3$ (isospin).
- For $SU(3)$, the Cartan subalgebra consists of:

$$\text{Isospin: } I_3 = T_3, \qquad \text{Hypercharge: } Y = \frac{2}{\sqrt{3}} T_8. \tag{33}$$

- Considering $\psi = \begin{pmatrix} u & d & s \end{pmatrix}^T$, we have

$$I_u = -I_d = \frac{1}{2}, \quad I_s = 0, \quad Y_u = Y_d = \frac{1}{3}, \quad Y_s = -\frac{2}{3}. \tag{34}$$

- Defining the baryon number B as $B_u = B_d = B_s = \frac{1}{3}$, other quantum numbers can be obtained, as follows:

$$\begin{aligned} \text{Electric charge:} \qquad & Q_q = I_q + \frac{1}{2} Y_q, \\ \text{Strangeness:} \qquad & S_q = Y_s - B_s. \end{aligned} \tag{35}$$

- Antiquarks have exactly opposite charges:

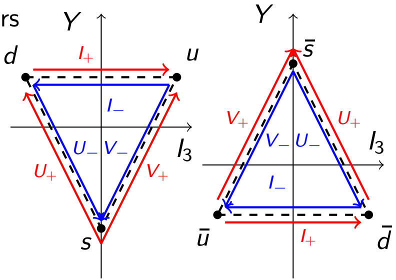
$$I_{\bar{q}} = -I_q, \ Y_{\bar{q}} = -Y_q, \ B_{\bar{q}} = -B_q, \ Q_{\bar{q}} = -Q_q, \ S_{\bar{q}} = -S_q. \tag{36}$$

IV.2.7. $SU(3)$ ladder operators.

- By analogy to $SU(2)$, we can construct 3×2 ladder operators:

$$
\begin{aligned}
I_\pm &= T_1 \pm i T_2, \quad [I_3, I_\pm] = \pm I_\pm, \quad [Y, I_\pm] = 0, \\
U_\pm &= T_6 \pm i T_7, \quad [I_3, U_\pm] = \mp \frac{1}{2} U_\pm, \quad [Y, U_\pm] = \pm U_\pm, \\
V_\pm &= T_4 \pm i T_5, \quad [I_3, V_\pm] = \pm \frac{1}{2} V_\pm, \quad [Y, V_\pm] = \pm V_\pm. \qquad (37)
\end{aligned}
$$

- The action of the ladder operators can be illustrated at the level of the fundamental representations, 3 and $\bar{3}$:



$$
\begin{aligned}
I_+ |d\rangle &= |u\rangle, \quad I_- |u\rangle = |d\rangle, \\
U_+ |s\rangle &= |d\rangle, \quad U_- |d\rangle = |s\rangle, \\
V_+ |s\rangle &= |u\rangle, \quad V_- |u\rangle = |s\rangle,
\end{aligned}
$$

with all other terms vanishing (e.g., $I_+ |u\rangle = I_+ |s\rangle = 0$).

- For antiquarks: $I_+ |\bar{u}\rangle = -|\bar{d}\rangle$, $U_+ |\bar{d}\rangle = -|\bar{s}\rangle$ and $V_+ |\bar{u}\rangle = -|\bar{s}\rangle$.

IV.2.8. Weight diagram.

- The ladder operators $I_\pm$, $U_\pm$, $V_\pm$ connect states in a given irrep.
- Expressing these operators as vectors in the $(T^3, T^8) = (I_3, \frac{\sqrt{3}}{2}Y)$ plane,

$$\vec{I}_\pm = (\pm 1, 0), \quad \vec{U}_\pm = \left(\mp\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right), \quad \vec{V}_\pm = \left(\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right), \quad (38)$$

  we obtain the canonical representation of the *root vectors* of the $su(3)$ Lie algebra.
- With respect to (T^3, T^8) all roots have unit length, making angles of 60° (or integer multiples) between them.
- Roots are *positive* if their first non-zero component is positive: $\vec{I}_+$, $\vec{U}_-$ and $\vec{V}_+$.
- A subset of roots is *simple* if any of the positive roots can be written as a linear combination of the simple roots with non-negative coefficients.
- Since $\vec{I}_+ = \vec{U}_- + \vec{V}_+ \Rightarrow \vec{U}_-$ and $\vec{V}_+$ are simple roots.

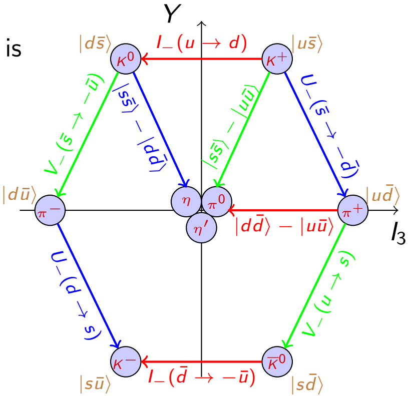
IV.2.9. Meson octet.

- Consider the $|q\bar{q}\rangle$ states.
- The state of maximum weight is

$$|K^{+}\rangle = |u\bar{s}\rangle \,.$$

- Using I_{-}, V_{-} and U_{-} gives:

$$|K^{0}\rangle = I_{-}\,|K^{+}\rangle = |d\bar{s}\rangle \,,$$
$$|\pi^{+}\rangle = U_{-}\,|K^{+}\rangle = |u\bar{d}\rangle \,,$$
$$|\pi^{-}\rangle = V_{-}\,|K^{0}\rangle = |d\bar{u}\rangle \,,$$
$$|\overline{K}^{0}\rangle = V_{-}\,|\pi^{+}\rangle = |s\bar{d}\rangle \,,$$
$$|K^{-}\rangle = I_{-}\,|\overline{K}^{0}\rangle = |s\bar{u}\rangle \,.$$



- At $I_3 = Y = 0$, the result depends on the direction employed.
- One of the 2 states from 8 is $\pi^{0} = \frac{1}{\sqrt{2}}(|d\bar{d}\rangle - |u\bar{u}\rangle)$.
- The second state is orthogonal to $|\pi^{0}\rangle$:

$$|\eta\rangle = -\frac{1}{\sqrt{6}}(V_{-}\,|K^{+}\rangle + U_{-}\,|K^{0}\rangle) = \frac{1}{\sqrt{6}}(|u\bar{u}\rangle + |d\bar{d}\rangle - 2\,|s\bar{s}\rangle). \quad (39)$$

- The singlet from 1 is orthogonal to both $|\pi^{0}\rangle$ and $|\eta\rangle$ and cannot be reached from the octet: $|\eta'\rangle = \frac{1}{\sqrt{3}}(|u\bar{u}\rangle + |d\bar{d}\rangle + |s\bar{s}\rangle)$