Symmetries in Physics
Lecture 12

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Lecture contents

Chapter 3. The Lorentz and Poincare groups

- III.1. The Lorentz and Poincare groups
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III.4.6. The complete Lorentz group

- In the $3+1$-D Minkowski space, spatial inversion and time reversal have the representation:

$$I_s = \mathrm{diag}(1, -1, -1, -1), \quad I_t = \mathrm{diag}(-1, 1, 1, 1), \tag{1}$$

with $I_s I_t = -E$ and, in general, $I_s \Lambda I_s^{-1} = I_t \Lambda I_t^{-1}$.

- At the level of Lorentz transformations,

$$I_s R I_s^{-1} = I_t R I_t = R, \quad I_s L I_s^{-1} = I_t L I_t^{-1} = L^{-1}. \tag{2}$$

- The transformations I_s and I_t split the complete Lorentz group in four classes:
  1. Proper, orthochronous: $\widetilde{L}_+^\uparrow$, with $\Lambda^0{}_0 \geq 1$ and $\det\Lambda = 1$;
  2. Improper, orthochronous: $\widetilde{L}_-^\uparrow = I_s \widetilde{L}_+^\uparrow$, with $\Lambda^0{}_0 \geq 1$ and $\det\Lambda = -1$;
  3. Improper, non-orthochronous: $\widetilde{L}_-^\downarrow = I_t \widetilde{L}_+^\uparrow : \Lambda^0{}_0 \leq -1, \det\Lambda = -1$.
  4. Proper, non-orthochronous: $\widetilde{L}_+^\downarrow = I_t I_s \widetilde{L}_+^\uparrow : \Lambda^0{}_0 \leq -1, \det\Lambda = -1$;
- The cosets $\{\widetilde{L}_+^\uparrow, I_s\widetilde{L}_+^\uparrow, I_t\widetilde{L}_+^\uparrow, I_s I_t\widetilde{L}_+^\uparrow\}$ form a group isomorphic to the dihedral group D_2, inducing 4 inequivalent, degenerate, $1D$ representations of $\widetilde{L}$.

III.4.7. Time reversal as an antilinear operator

- It is clear that under time reversal, $t \to -t$, which gives the matrix representation of I_t.
- In QM, the Schrödinger eq. demands $i\hbar\partial_t\psi(\mathbf{x}, t) = H\psi(\mathbf{x}, t)$.
- Considering $H = \mathbf{P}^2/2m + V(\mathbf{x})$, it is clear that $H \to H$ under I_t.
- In order for the Schrödinger eq. to be invariant under time reversal, it is not enough to impose $t \to -t$.
- The solution is to implement I_t as an antilinear operator,

$$\psi(\mathbf{x}, t) \xrightarrow{I_t} \psi'(\mathbf{x}', t') = \eta\psi^*(\mathbf{x}, -t), \tag{3}$$

which implies *antilinearity:* $\alpha_1\psi_1 + \alpha_2\psi_2 \xrightarrow{I_t} \alpha_1^*\psi_1' + \alpha_2^*\psi_2'$.

Antilinear and antiunitary operators

- Let A be an **antilinear operator**.
- A does not commute with c-numbers: $Ac = c^* A$.
- Inner products behave as: $\langle \phi | A\psi \rangle = \langle \psi | A^\dagger \phi \rangle = \langle A^\dagger \phi | \psi \rangle^*$.
- **Lemma:** The eigenvectors satisfying $A\left|\lambda\right\rangle = \left|\lambda\right\rangle \lambda$ of A are divided into classes characterized by $|\lambda|$. In each class, there is an infinity of eigenvectors, characterized by $0 \leq \theta = \arg(\lambda) < 2\pi$.
  Proof: Consider $Ae^{i\alpha/2}\left|\lambda\right\rangle = e^{-i\alpha/2}A\left|\lambda\right\rangle = e^{i\alpha/2}\left|\lambda\right\rangle \lambda e^{-i\alpha}$. Thus $\{e^{i\alpha/2}\left|\lambda\right\rangle, 0 \leq \alpha < \pi\}$ generates a one-parameter family of eigenvectors of A with magnitude $|\lambda|$.
- **Def:** An **antiunitary operator** additionally satisfies $AA^\dagger = E$, leading to $\langle A\phi | A\psi \rangle = \langle \psi | \phi \rangle = \langle \phi | \psi \rangle^*$.
- If A is antiunitary, then $|\lambda| = 1$.
- **Theorem:** The representation space of any group with time reversal must contain eigenstates of I_t in entire classes, characterized by the eigenvalue (phase) of I_t.

III.4.8. Complete Poincaré group

- For translations, we have

$$I_s T(b^0, \mathbf{b}) I_s^{-1} = T(b^0, -\mathbf{b}), \quad I_t T(b^0, \mathbf{b}) I_t^{-1} = T(-b^0, \mathbf{b}). \qquad (4)$$

- Imposing Eq. (2) implies for the generators:

$$\begin{aligned} I_s(\mathbf{J}, \mathbf{K}) I_s^{-1} &= (\mathbf{J}, -\mathbf{K}), & I_t(\mathbf{J}, \mathbf{K}) I_t^{-1} &= (-\mathbf{J}, \mathbf{K}), \\ I_s(P^0, \mathbf{P}) I_s^{-1} &= (P^0, -\mathbf{P}), & I_t(P^0, \mathbf{P}) I_t^{-1} &= (P^0, -\mathbf{P}), \end{aligned} \qquad (5)$$

  since $A e^{-iB} A^{-1} = e^{iABA^{-1}}$ for any antilinear A.
- At the level of the generators $\mathbf{M} = \frac{1}{2}(\mathbf{J} + i\mathbf{K})$ and $\mathbf{N} = \frac{1}{2}(\mathbf{J} - i\mathbf{K})$, we have

$$I_s(\mathbf{M}, \mathbf{N}) I_s^{-1} = (\mathbf{N}, \mathbf{M}), \qquad I_t(\mathbf{M}, \mathbf{N}) I_t^{-1} = (-\mathbf{N}, -\mathbf{M}), \qquad (6)$$

  implying that $I_{s,t}(\mathbf{M}^2, \mathbf{N}^2) I_{s,t}^{-1} = (\mathbf{N}^2, \mathbf{M}^2)$.

III.4.9. Finite-dimensional representations of $\widetilde{L}$

- Consider the basis $|kl\rangle \equiv |u,k;v,l\rangle \equiv |kl\rangle_{u,v}$, satisfying

$$(M_3, N_3, \mathbf{M}^2, \mathbf{N}^2)\,|kl\rangle_{u,v} = |kl\rangle_{u,v}\,(k, l, u(u+1), v(v+1)). \quad (7)$$

- Under space reflection, we have $I_s\,|kl\rangle_{u,v} = |lk\rangle_{v,u}\,\eta$, with $|\eta| = 1$.
- When $v \neq u$, the basis vectors can be redefined to give $\eta = 1$; for $v = u$, $\eta = \pm 1$ gives two inequiv. irreps:
- **Theorem:** The finite-D irreps of $\widetilde{L}$ belong to two classes:
  (i) the *self-conjugate* reps with $u = v$ are $(2u+1)^2$-dimensional and are characterized by (u, η), with $u = 0, 1/2, 1, \ldots$ and $\eta = \pm 1$.
  (ii) the *general* reps, with $u \neq v$, are $2(2u+1)(2v+1)$-dimensional and behave as $(u, v) \oplus (v, u)$.
- Examples: Dirac spinor: $(0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)$;
- Scalars and pseudoscalars: $(u, \eta) = (0, \pm 1)$;
- Vectors and axial vectors: $(u, \eta) = (\frac{1}{2}, \pm 1)$;
- Second-rank anti-symmetric tensors: $(1, 0) \oplus (0, 1)$.

III.4.10. Irreps of the complete Poincaré group
Time-like case $c_1 > 0$

- We consider states for which $c_1 = P^2 = M^2$ and $C_2 = -W^2 = M^2 s(s+1)$.
- Considering the intrinsic parity $\eta_p = \pm 1$, the rest-frame basis vector is defined by

$$(\mathbf{P}, J^z, I_s) |\mathbf{0}\lambda\rangle = |0\lambda\rangle (\mathbf{0}, \lambda, \eta_p), \tag{8}$$

- $J_3 I_t |\mathbf{0}\lambda\rangle = -I_t |\mathbf{0}\lambda\rangle \lambda$ shows that $I_t |\mathbf{0}\lambda\rangle \sim |\mathbf{0}, -\lambda\rangle$.
- The action of I_t can be "reverted" by employing the rotation $R_2(\pi)$:

$$R_2(\pi) I_t |\mathbf{0}\lambda\rangle = |\mathbf{0}\lambda\rangle \eta_T \;\Rightarrow\; I_t |\mathbf{0}\lambda\rangle = |\mathbf{0}, -\lambda\rangle \eta_T (-1)^{s+\lambda}, \tag{9}$$

  where the extra phase η_T can be absorbed into the basis vectors, by defining $|\mathbf{0}\lambda\rangle' = |\mathbf{0}\lambda\rangle \eta_T^{1/2}$.
- Thus, we will work with the two basis vectors $\{|\mathbf{0}, \pm\lambda\rangle\}$ satisfying:

$$(\mathbf{P}, J^z, I_s) |\mathbf{0}, \pm\lambda\rangle = |\mathbf{0}, \pm\lambda\rangle (\mathbf{0}, \pm\lambda, \eta_p), \;\; R_2(\pi) I_t |\mathbf{0}, \pm\lambda\rangle = |\mathbf{0}, \mp\lambda\rangle \,. \tag{10}$$

- As usual, we consider the basis vectors generated using the Lorentz transformations

$$|\mathbf{p}\lambda\rangle = R(\mathbf{p})\,|p\hat{\mathbf{e}}_z,\lambda\rangle\,, \quad |p\hat{\mathbf{e}}_z,\lambda\rangle = L_3(\xi)\,|\mathbf{0}\lambda\rangle\,. \tag{11}$$

- Since $I_s\mathbf{P}I_s^{-1} = -\mathbf{P}$ and $I_s\mathbf{J}I_s^{-1} = \mathbf{J}$, then $I_s\,|\mathbf{p}\lambda\rangle \sim |-\mathbf{p},-\lambda\rangle$.
- Similarly, $I_t\mathbf{P}I_t^{-1} = -\mathbf{P}$ and $I_t\mathbf{J}I_t^{-1} = -\mathbf{J}$, so that $I_t\,|\mathbf{p}\lambda\rangle \sim |-\mathbf{p}\lambda\rangle$.
- To establish the relative phases, we consider the action of $R_2(\pi)I_{s,t}$ on $|p\hat{\mathbf{e}}_z,\lambda\rangle$:

$$\begin{aligned}
R_2(\pi)I_s\,|p\hat{\mathbf{e}}_z,\lambda\rangle &= L_3(\xi)R_2(\pi)I_s\,|\mathbf{0}\lambda\rangle = |p\hat{\mathbf{e}}_z,-\lambda\rangle\,\eta_p(-1)^{s-\lambda},\\
R_2(\pi)I_t\,|p\hat{\mathbf{e}}_z,\lambda\rangle &= L_3(\xi)R_2(\pi)I_t\,|\mathbf{0}\lambda\rangle = |p\hat{\mathbf{e}}_z,\lambda\rangle\,.
\end{aligned} \tag{12}$$

- On a general state, we have

$$\begin{aligned}
I_s\,|\mathbf{p}\lambda\rangle &= R(\mathbf{p})I_s\,|p\hat{\mathbf{e}}_z,\lambda\rangle = R(-\mathbf{p})R_3(\mp\pi)\,|p\hat{\mathbf{e}}_z,-\lambda\rangle\,\eta_p(-1)^{s-\lambda}\\
&= |-\mathbf{p},-\lambda\rangle\,\eta_p e^{\mp i\pi s},\\
I_t\,|\mathbf{p}\lambda\rangle &= R(\mathbf{p})I_t\,|p\hat{\mathbf{e}}_z,\lambda\rangle = R(-\mathbf{p})R_3(\mp\pi)\,|p\hat{\mathbf{e}}_z\lambda\rangle\,(-1)^{s-\lambda}\\
&= |-\mathbf{p}\lambda\rangle\,e^{\pm i\pi\lambda},
\end{aligned} \tag{13}$$

where we used $R(\mathbf{p})R_2(-\pi) = R(-\mathbf{p})R_3(\mp\pi)$, with upper/lower sign when $0 \le \phi < \pi$ ($\pi \le \phi < 2\pi$) and $(-1)^{s-\lambda} = e^{\mp i\pi(s-\lambda)}$.

Light-like case $c_1 = 0$

- In this case, $p_l^\mu = (\omega_0, 0, 0, \omega_0)$ and

$$J_3 \left|\mathbf{p}_l\lambda\right\rangle = \left|\mathbf{p}_l\lambda\right\rangle \lambda, \quad (W_1, W_2)\left|\mathbf{p}_l\lambda\right\rangle = 0. \tag{14}$$

- Furthermore, including $\lambda = \pm m$ with $m > 0$, we have

$$R_2(\pi)I_s \left|\mathbf{p}_l m\right\rangle \equiv \left|\mathbf{p}_l, -m\right\rangle, \quad R_2(\pi)I_t \left|\mathbf{p}_l\lambda\right\rangle = \left|\mathbf{p}_l\lambda\right\rangle \eta_T, \tag{15}$$

  with η_T arbitrary (set to $\eta_T = 1$ henceforth).
- Note that $R_2(\pi)I_s \left|\mathbf{p}_l, -m\right\rangle = \left|\mathbf{p}_l m\right\rangle (-1)^{2m}$.
- For an arbitrary vector $\left|\mathbf{p}\lambda\right\rangle = H(p)\left|\mathbf{p}_l\lambda\right\rangle$, we have

$$I_s \left|\mathbf{p}\lambda\right\rangle = \left|-\mathbf{p}, -\lambda\right\rangle e^{\mp i\pi|\lambda|}, \quad I_t \left|\mathbf{p}\lambda\right\rangle = \left|-\mathbf{p}\lambda\right\rangle e^{\pm i\pi\lambda}\eta_T, \tag{16}$$

  with the upper/lower signs corresponding to $0 < \phi < \pi$ $(\pi < \phi < 2\pi)$.

Exercises

1. The Dirac spinor $\psi = (\xi, \eta)^T$ is a collection of four complex numbers that can be arranged in two two-spinors, $\xi = (\xi_1, \xi_2)^T$ and $\eta = (\eta_1, \eta_2)$. Under Lorentz transformations, ψ behaves as

$$\psi(x) \to \psi'(x) = D[\Lambda]\psi(\Lambda^{-1}x), \quad D[\Lambda] = e^{-\frac{i}{2}\omega_{\alpha\beta} D[J^{\alpha\beta}]}, \tag{T1.1}$$

where the generators of the Lorentz transformations are

$$D[\mathbf{J}] = \frac{1}{2}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}, \quad D[\mathbf{K}] = \frac{i}{2}\begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \tag{T1.2}$$

with $\boldsymbol{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ being the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{T1.3}$$

   a) Construct $\mathbf{M} = \frac{1}{2}(\mathbf{J} + i\mathbf{K})$ and $\mathbf{N} = \frac{1}{2}(\mathbf{J} - i\mathbf{K})$.
   b) Compute $\mathbf{M}^2$ and $\mathbf{N}^2$. Find their eigenvalues (u, v) and simultaneous eigenvectors:

$$\mathbf{M}^2\psi_{u,v} = u(u+1)\psi_{u,v}, \quad \mathbf{N}^2\psi_{u,v} = v(v+1)\psi_{u,v}. \tag{T1.4}$$

   Therefore show that the Dirac field transforms as $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$.
   c) Knowing that $\psi(t, \mathbf{x}) \xrightarrow{I_s} \gamma^0\psi(t, -\mathbf{x})$, find explicitly the basis vectors $|kl\rangle_{u,v}$ satisfying $I_s\,|kl\rangle_{u,v} = |lk\rangle_{v,u}$.

Exercises

WKT11.7 Show that $I_s \left|\mathbf{p}_l \lambda\right\rangle$, with $p_l^\mu = (\omega_0, 0, 0, \omega_0)$, is an eigenvector of P^μ, J^z and $W^{1,2}$, and evaluate the eigenvalues.

WKT12.1 (i) Prove that η_T in Eq. (9) is independent of λ. (ii) From Eq. (9), construct an explicit basis $\left|\mathbf{0}, \lambda\right\rangle'$ s.t. $I_t \left|\mathbf{0}, \pm\lambda\right\rangle' = \left|\mathbf{0}, \mp\lambda\right\rangle' (-1)^{s+\lambda}$, keeping in mind the Lemma on slide 5.

WKT12.2 Prove that $I_t^2 = (-1)^{2s}$ by applying I_t on both sides of Eqs. (12) and (13).