Symmetries in Physics
Lecture 11

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Lecture contents

III.3. Finite-dimensional vs. unitary representations
III.3.1. Covariant normalization

- In the induced representation method, we defined $|\mathbf{p}\lambda\rangle = H(p)\,|\tilde{\mathbf{p}}\lambda\rangle$, with $\tilde{p} \in \{p_t, p_l, p_s\}$ being the representative momentum.
- Let us consider a normalization: $\langle \mathbf{p}'\lambda'|\mathbf{p}\lambda\rangle = N(p)\delta^3(\mathbf{p}-\mathbf{p}')\delta^{\lambda'}{}_{\lambda}$.
- For unitary representations, $\Lambda^\dagger\Lambda = E$ and

$$\langle \mathbf{p}'\lambda'|\mathbf{p}\lambda\rangle = \langle \mathbf{p}'\lambda'|\Lambda^\dagger\Lambda|\mathbf{p}\lambda\rangle = D_s^\dagger[R']^{\lambda'}{}_{\sigma'}\,\langle \Lambda\mathbf{p}'\sigma'|\Lambda\mathbf{p}\sigma\rangle\,D_s[R]^{\sigma}{}_{\lambda}, \quad (1)$$

  where $R' \equiv R(\Lambda, p')$ and $R \equiv R(\Lambda, p)$, $\forall \Lambda \in \widetilde{L}_+$.
- Imposing the same normalization, we have

$$N(p)\delta^3(\mathbf{p}-\mathbf{p}') = N(\Lambda p)\delta^3(\Lambda\mathbf{p}-\Lambda\mathbf{p}'), \quad (2)$$

  where we used that $D_s^\dagger[R]^{\lambda'}{}_{\sigma'}\delta_{\sigma}^{\sigma'}D_s[R]^{\sigma}{}_{\lambda} = D_s[R^{-1}R]^{\lambda'}{}_{\lambda} = \delta_{\lambda}^{\lambda'}$.
- Eq. (2) represents the condition for a covariant normalization.

III.3.2. Covariant integration measure

- To find $N(p)$, consider the expansion of a state vector $|\psi\rangle$:

$$|\psi\rangle = \sum_\lambda \int |\mathbf{p}\lambda\rangle\, \psi_\lambda(p)\widetilde{dp}, \tag{3}$$

where $\widetilde{dp}$ is the yet-to-be-determined integration measure.

- The components $\psi_{\lambda'}(p') = \langle \mathbf{p}'\lambda'|\psi\rangle$ can be obtained as

$$\psi_{\lambda'}(p') = \int \psi_{\lambda'}(p)N(p)\delta^3(\mathbf{p}-\mathbf{p}')\widetilde{dp} \quad \Rightarrow \quad \widetilde{dp} = \frac{d^3p}{N(p)}. \tag{4}$$

- If $N(p)$ is covariant, then $\widetilde{dp} = \widetilde{d\Lambda p}$ and the integration measure becomes Lorentz-invariant.
- For a 4-vector with arbitrary components $p^\mu = (p^0, \mathbf{p})$, we have

$$d^4(\Lambda p) = (\mathrm{det}\Lambda)d^4p = d^4p, \qquad \forall \Lambda \in \widetilde{L}_+. \tag{5}$$

- Imposing that $p^0 > 0$ and moreover, $p^2 = p_0^2 - \mathbf{p}^2 = c_1$, we have

$$\widetilde{dp} = \frac{2}{N_0}\theta(p^0)\delta(p^2 - c_1)d^4p = \frac{d^3p}{(2\pi)^3p^0}, \tag{6}$$

where $p^0 = \sqrt{\mathbf{p}^2 + c_1}$ in the last expression and $N_0 = (2\pi)^3$ is a conventional momentum-independent constant.

III.3.3. Relativistic wave functions

- **Def:** A c-number *relativistic wave function* is a set of n space-time functions $\psi_\sigma(x)$ which transform under $\Lambda \in \widetilde{L}_+$ as

$$\psi \xrightarrow{\Lambda} \psi', \quad \psi'^{\alpha}(x) = D[\Lambda]^{\alpha}{}_{\beta}\psi^{\beta}(\Lambda^{-1}x). \tag{7}$$

- Examples are shown below:
  - The Klein-Gordon wave function, $\phi(x)$, for which $n = 1$, transforming with the $(u, v) = (0, 0)$ irrep, describing spin-0 particles:

$$\phi'(x) = \phi(\Lambda^{-1}x). \tag{8}$$

  - The antisymmetric field strength tensor $F^{\mu\nu}(x)$, with $F^{0i} = E^i$ and $F^{ij} = \varepsilon^{ijk}B^k$, transforming as $(1, 0) \oplus (0, 1)$:

$$F'^{\mu\nu}(x) = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}F^{\alpha\beta}(\Lambda^{-1}x). \tag{9}$$

  - The four-potential $A^\mu(x)$, describing spin-1 particles (vector bosons), transforming as $(1/2, 1/2)$:

$$A'^{\mu}(x) = \Lambda^{\mu}{}_{\nu}A^{\nu}(\Lambda^{-1}x). \tag{10}$$

  - The Dirac wave function $\psi^\alpha(x)$, with $n = 4$, transforming as $(0, 1/2) \oplus (1/2, 0)$, describing spin-1/2 particles:

$$\psi'(x) = D[\Lambda]\psi(\Lambda^{-1}x),\ D[\Lambda] = e^{-\frac{i}{2}\omega_{\alpha\beta}S^{\alpha\beta}},\ S^{\alpha\beta} = \frac{i}{4}[\gamma^\alpha, \gamma^\beta]. \tag{11}$$

III.3.4. Relativistic field operators

- **Def:** For a given matrix representation $D[\Lambda]$ corresponding to the internal indices of a relativistic wave function, the *relativistic field operator* is a set of n operator-valued space-time functions $\{\Psi^\alpha(x)\}$ which transform as

$$U[\Lambda]\Psi^\alpha(x)U[\Lambda^{-1}] = D[\Lambda^{-1}]^\alpha{}_\beta \Psi^\beta(\Lambda x), \tag{12}$$

  where $U[\Lambda]$ acts on the Hilbert space where Ψ is defined.
- The relativistic field operators (quantum field theory) are the second-quantized versions of the relativistic wave functions (relativistic quantum mechanics).
- A free field $\Psi^\alpha(x)$ obeys the field equation $\Pi(m, i\partial)^\alpha{}_\beta \Psi^\beta(x) = 0$, where Π is a linear differential operator of first or second order w.r.t. ∂_μ, that acts as an $n \times n$ matrix w.r.t. the internal structure α, β.
- Example: the Klein-Gordon equation: $(\partial^\mu \partial_\mu - m^2)\Phi(x) = 0$, for a single-component operator Φ.
- Example: the Dirac equation: $(i\gamma^\mu \partial_\mu - m)\Psi = 0$, with γ^μ being the 4×4 gamma matrices.

III.3.5. Lorentz-covariant wave equations

- Under the Fourier transform $\Psi^{\alpha}(x) = \int \frac{d^4p}{(2\pi)^4} \widetilde{\Phi}^{\alpha}(p) e^{-ip\cdot x}$, the differential equation is converted into an algebraic one:

$$\Pi(m, p)\widetilde{\Phi}(p) = 0. \tag{13}$$

- Under a Lorentz transformation, we have

$$\int \frac{d^4p}{(2\pi)^4} U[\Lambda]\widetilde{\Phi}(p)U[\Lambda^{-1}]e^{-ip\cdot x} = \int \frac{d^4p}{(2\pi)^4} D[\Lambda^{-1}]\widetilde{\Phi}(p)e^{-ip\cdot \Lambda x}. \tag{14}$$

- Since $p \cdot (\Lambda x) = (\Lambda^{-1}p) \cdot x$ and $d^4p = d^4(\Lambda^{-1}p)$, we must have

$$U[\Lambda]\widetilde{\Phi}(p)U[\Lambda^{-1}] = D[\Lambda^{-1}]\widetilde{\Phi}(\Lambda p). \tag{15}$$

- Imposing that $\Pi(m, \Lambda p)\widetilde{\Phi}(\Lambda p) = 0$, we have

$$D[\Lambda^{-1}]\Pi(m, \Lambda p)D[\Lambda] = \Pi(m, p), \tag{16}$$

or $\Pi(m, \Lambda p) = D[\Lambda]\Pi(m, p)D[\Lambda^{-1}]$.

III.3.6. Plane-wave expansion

- Field equations must contain the mass shell condition,

$$(p^2 - m^2)\widetilde{\Phi}(p) = 0. \tag{17}$$

- This can be enforced by writing the solution as

$$\widetilde{\Phi}(p) = 2(2\pi)\delta(p^2 - m^2)\Phi(p), \tag{18}$$

where $p^0 = \pm E_{\mathbf{p}}$, with $E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$.

- In QFT, the two branches corresponding to positive- and negative-energy solutions represent particle and anti-particle states, begin separated as

$$\Psi(x) = \int \widetilde{dp} \left[\Phi_+(\mathbf{p})e^{-ip\cdot x} + \Phi_-(\mathbf{p})e^{ip\cdot x}\right], \tag{19}$$

where $\Phi_\pm(\mathbf{p}) = \Phi(\pm E_{\mathbf{p}}, \pm\mathbf{p})$ satisfies $\Pi(m; \pm E_{\mathbf{p}}, \pm\mathbf{p})\Phi_\pm(\mathbf{p}) = 0$.

- Consider now the case $c_1 = m^2 > 0$. The wavefunction $u^\alpha(\mathbf{p} = 0, \lambda)$ corresponding to the reference momentum satisfies

$$\Pi(m, p_t)^\alpha{}_\beta u^\beta(\mathbf{0}, \lambda) = 0, \quad \lambda = -s, -s+1, \dots s. \tag{20}$$

- Then, $u^\alpha(\mathbf{p}, \lambda) = D[H(p)]^\alpha{}_\beta u^\beta(\mathbf{0}, \lambda)$ satisfies

$$\Pi(m, p) u(\mathbf{p}, \lambda) = D[H(p)] \Pi(m, p_t) u(\mathbf{0}, \lambda) = 0, \tag{21}$$

  when $p^\mu = H(p)^\mu{}_\nu p_t^\nu$.
- The elementary plane wave solutions $u^\alpha(\mathbf{p}, \lambda)$ represent the concrete realisation of the momentum basis states $|\mathbf{p}, \lambda\rangle$.
- The general solution of the field equation is then

$$\Psi^\alpha(x) = \sum_\lambda \int \widetilde{dp} [e^{-ip\cdot x} u^\alpha(\mathbf{p}, \lambda) b(\mathbf{p}, \lambda) + \text{negative energy terms}]. \tag{22}$$

- In QFT, the expansion coefficients $b(\mathbf{p}, \lambda)$ are one-particle annihilation operators and their Hermitian conjugates, $b^\dagger(\mathbf{p}, \lambda)$, are one-particle creation operators.

III.3.7. Lorentz-Poincaré connection

- In QFT, particle states with definite momentum can be created from the vacuum $|0\rangle$ using the creation operators:

$$|\mathbf{p}, \lambda\rangle = b^\dagger(\mathbf{p}, \lambda)\,|0\rangle\,. \tag{23}$$

- Starting from the reference state $|\mathbf{0}, \lambda\rangle = b^\dagger(\mathbf{0}, \lambda)\,|0\rangle$, we have

$$|\mathbf{p}, \lambda\rangle = H(p)\,|\mathbf{0}\lambda\rangle = H(p) b^\dagger(\mathbf{0}, \lambda) H^{-1}(p) H(p)\,|0\rangle\,, \tag{24}$$

which shows that $b^\dagger(\mathbf{p}, \lambda) = H(p) b^\dagger(\mathbf{0}, \lambda) H^{-1}(p)$, while

$$\begin{aligned} U[\Lambda] b^\dagger(\mathbf{p}, \lambda) U[\Lambda^{-1}] &= b^\dagger(\Lambda\mathbf{p}, \lambda') D_s[R(\Lambda, p)]^{\lambda'}{}_{\lambda}, \\ U[\Lambda] b(\mathbf{p}, \lambda) U[\Lambda^{-1}] &= D_s[R^{-1}(\Lambda, p)]^{\lambda}{}_{\lambda'}\, b(\Lambda\mathbf{p}, \lambda'). \end{aligned} \tag{25}$$

- The wavefunction of a state $|\phi\rangle$ is $\phi^\alpha(x) = \langle 0|\Psi^\alpha(x)|\phi\rangle$.
- The plane wave solutions can be obtained as

$$\begin{aligned} u^\alpha(\mathbf{p}, \lambda) e^{-ip\cdot x} = \langle 0|\Psi^\alpha(x)|\mathbf{p}, \lambda\rangle = \sum_{\lambda'} \int \widetilde{dp'}\, e^{-ip'\cdot x} u^\alpha(\mathbf{p}', \lambda') \\ \times \langle 0|b(\mathbf{p}', \lambda') b^\dagger(\mathbf{p}, \lambda)|0\rangle\,, \end{aligned} \tag{26}$$

which implies

$$\langle 0|b(\mathbf{p}', \lambda') b^\dagger(\mathbf{p}, \lambda)|0\rangle = (2\pi)^3 E_{\mathbf{p}} \delta^3(\mathbf{p} - \mathbf{p}'). \tag{27}$$

- **Theorem:** The c-number wave functions $u^\alpha(\mathbf{p},\lambda)e^{-ip\cdot x}$ are the coefficient functions which connect the operators $b(\mathbf{p},\lambda)$, transforming as unitary irreps (m,s) of the Poincaré group, to the set of field operators $\Psi^\alpha(x)$, transforming as finite-dimensional non-unitary representations of the Lorentz group.
- Applying Λ on the plane-wave expansion in Eq. (22) gives:

$$U(\Lambda)\Psi^\alpha(x)U(\Lambda^{-1}) = D[\Lambda^{-1}]^\alpha{}_{\alpha'}\sum_\lambda \int \tilde{d}p \times [b(\Lambda\mathbf{p},\lambda)u^{\alpha'}(\Lambda\mathbf{p},\lambda)e^{-ip\cdot x} + \text{a.p.}], \quad (28)$$

where a.p. denotes the antiparticle sector.
- Using Eq. (25), we have

$$U(\Lambda)\Psi^\alpha(x)U(\Lambda^{-1}) = \sum_{\lambda,\lambda'} D_s[R^{-1}(\Lambda,p)]^{\lambda'}{}_\lambda \int \tilde{d}p \times [b(\Lambda\mathbf{p},\lambda)u^{\alpha'}(\mathbf{p},\lambda')e^{-ip\cdot x} + \text{a.p.}]. \quad (29)$$

- Comparing the two expressions above, it follows that

$$D[\Lambda]^\alpha{}_{\alpha'}u^{\alpha'}(\mathbf{p},\lambda) = u^\alpha(\Lambda\mathbf{p},\lambda')D_s[R(\Lambda,p)]^{\lambda'}{}_\lambda. \quad (30)$$

III.3.8. Relativistic wave equations

- Relativistic wave equations have as solutions the plane-wave basis vectors, $u(\mathbf{p}, \lambda)$, satisfying $\Pi(m,p)^{\alpha}{}_{\beta} u^{\beta}(\mathbf{p}, \lambda) = 0$.
- In general, $\Pi(m,p)$ acts on the Lorentz structure $\alpha, \beta, \ldots$ under a finite-dimensional (non-unitary) representation, e.g. $(u,v) = (\frac{1}{2}, \frac{1}{2})$ for the vector field A^{μ} and $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ for the Dirac field $\psi(x)$.
- A priori, $u^{\alpha}(\mathbf{p}, \lambda)$ can be decomposed into $|u - v| \le j \le u + v$. If the field operator $\Psi(x)$ characterizes particles with a definite spin s,
- Then, $\Pi(m,p)$ must project out all $j \neq s$. Focussing on the characteristic momentum state, p_t, we have

$$\Pi^{j'\lambda'}{}_{j\lambda}(p_t) \sim \delta_s^{j'} \delta_s^{j} \delta_{\lambda}^{\lambda'} . \tag{31}$$

- W.r.t. the direct product representation basis $|kl\rangle$ of (u,v), we have

$$\Pi^{k'l'}{}_{kl}(p_t) \sim \sum_{\lambda} \langle k'l'(uv)s\lambda\rangle \langle s\lambda(uv)kl\rangle . \tag{32}$$

- At arbitrary $\mathbf{p}$, we have

$$\Pi(m,p) = D[H(p)]\Pi(m,p_t)D[H(p)^{-1}], \tag{33}$$

or explicitly,

$$\Pi^{j'\lambda'}{}_{j\lambda}(m,p) \sim \sum_{\sigma} D[H(p)]^{j'\lambda'}{}_{s\sigma} D[H(p)^{-1}]^{s\sigma}{}_{j\lambda} . \tag{34}$$

III.4. Discrete transformations
III.4.1. Space inversion

- Space inversion is a transformation that changes the sign of one or several spatial coordinates.
- In 2D, we have $I_\alpha \hat{\mathbf{e}}_i = \hat{\mathbf{e}}_j (I_\alpha)^j{}_i$, with $\alpha \in \{1, 2, s\}$:

$$I_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad I_s = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{35}$$

- Since $\det I_s = 1$, $I_s \in SO(2)$, corresponding to a rotation by π.
- $\det I_1 = \det I_2 = -1 \Rightarrow I_1, I_2 \notin SO(2)$.
- Taking I_2 as reference, $I_\theta = R(\theta) I_2 R^{-1}(\theta)$ represents the reflection into the line inclined at θ w.r.t. the x axis.
- In general, $I R(\theta) I^{-1} = R(-\theta), \forall R(\theta) \in SO(2)$.
- Clearly, $I J I^{-1} = -J$.
- The orthogonal group $O(2)$ consists of all matrices satisfying $A^T A = E$ and is comprised of R and $I_2 R$, with $R \in SO(2)$.

III.4.2. Irreps of $O(2)$

- Considering the irrep given by $|m\rangle$ with $J|m\rangle = |m\rangle\, m$, the spatial inversion corresponds to $m \to -m$:

$$JI\,|m\rangle = I\,|m\rangle\,(-m) \quad \Rightarrow \quad I\,|m\rangle = |-m\rangle\, e^{i\alpha_m}. \tag{36}$$

- Since $I^2 = E$, we have $\alpha_{-m} = -\alpha_m$.
- The phase ambiguity can be absorbed in the definition of the basis vectors. Defining $|m\rangle' = |m\rangle$ and $|-m\rangle' = |-m\rangle\, e^{i\alpha_m}$, we have $I\,|m\rangle' = |-m\rangle'$ with $\alpha'_m = -\alpha'_m = 0$.
- **Theorem:** $O(2)$ has two types of inequiv. unitary irreps: (i) two degenerate $1D$ irreps:

$$R(\theta)\,|0\eta\rangle = |0\eta\rangle\,, \qquad I\,|0\eta\rangle = |0\eta\rangle\,\eta, \qquad \eta = \pm 1; \tag{37}$$

and (ii) the faithful angular momentum basis $\{|\pm m\rangle\}$, for each $m = 1, 2, \ldots,$ with

$$R(\theta)\,|\pm m\rangle = |\pm m\rangle\, e^{\mp i m\theta}, \qquad I\,|\pm m\rangle = |\mp m\rangle\,. \tag{38}$$

III.4.3. Extended Euclidean group $\widetilde{E}_2$

- **Def:** The *extended Euclidean group* $\widetilde{E}_2$ represents the full symmetry group of the $2D$ Euclidean space and consists of translations, rotations and space reflections.
- The action of a spatial inversion I on translations is

$$IT(\mathbf{b})I^{-1} = T(I\mathbf{b}), \quad IP_iI^{-1} = P_jI^j{}_i. \tag{39}$$

- Translations make an invariant subgroup $\Rightarrow$ the factor group $\widetilde{E}_2/T_2 \sim O(2)$.
- As in the case of E_2, we have the degenerate irrep corresponding to $p^2 = 0$, induced by the factor group $O(2)$.
- The faithful irreps are characterized by P^2 and η (inversion eigenvalue).
- Starting from the standard $\mathbf{p}_0 = p\mathbf{i}$, we have

$$\mathbf{P}\left|\mathbf{p}_0\right\rangle = \left|\mathbf{p}_0\right\rangle \mathbf{p}_0, \quad I_2\left|\mathbf{p}_0\right\rangle = \left|\mathbf{p}_0\right\rangle \eta, \tag{40}$$

with $\eta = \pm 1$.

III.4.4. The $O(3)$ group

- In $3D$, the spatial inversion is $I_s = \mathrm{diag}(-1,-1,-1)$, flipping simultaneously the signs of all spatial coordinates.
- **Theorem:** The elements A of the $O(3)$ group, satisfying $A^T A = E$, are divided into two classes: R and $I_s R$, $\forall R \in SO(3)$.
- **Theorem:** The space inversion commutes with all rotations and their generators:

$$I_s R I_s^{-1} = R, \qquad [I_s, J_i] = 0. \tag{41}$$

Proof: Straightforward, since $I_s = -E$ in the natural representation.
- Therefore, I_s is a Casimir operator. Together with j, its eigenvalue $\eta = \pm 1$ distinguishes between irreps:

$$(J_3, \mathbf{J}^2, I_s)\,|m\rangle = |m\rangle\,(m, j(j+1), \eta). \tag{42}$$

- In the case of spherical harmonics, $Y_{lm}(\hat{\mathbf{u}}) = \langle \hat{\mathbf{u}} | lm \rangle$ changes under space inversion to

$$Y_{lm}(-\hat{\mathbf{u}}) = \langle \hat{\mathbf{u}} | I_s | lm \rangle = \eta Y_{lm}(\hat{\mathbf{u}}). \tag{43}$$

- As a Casimir operator, η can be evaluated for $\hat{\mathbf{u}} = \hat{\mathbf{e}}_z$ and $m = 0$:

$$\langle \hat{\mathbf{e}}_z | I_s | l0 \rangle = \langle -\hat{\mathbf{e}}_z | l0 \rangle = \langle \hat{\mathbf{e}}_z | R_2(\pi) | l0 \rangle = (-1)^l Y_{l0}(\hat{\mathbf{e}}_z). \tag{44}$$

- Thus, $\eta = (-1)^l$.

III.4.5. The extended Euclidean group $\widetilde{E}_3$

- Spatial inversion acts on translations by flipping the sign of their argument:

$$I_s T(\mathbf{b}) I_s^{-1} = T(-\mathbf{b}), \qquad I_s \mathbf{P} I_s^{-1} = -\mathbf{P}. \tag{45}$$

- While $\mathbf{P}^2$ remains a Casimir operator, $\mathbf{J} \cdot \mathbf{P}$ flips sign under $I_s \Rightarrow$ the second Casimir is $(\mathbf{J} \cdot \mathbf{P})^2$.
- The degenerate representations, corresponding to $\mathbf{P} = 0$, are induced by the factor group $\widetilde{E}_3 / T_3 \sim O(3)$.
- The non-degenerate irreps are induced by $O(2)$, the little group of $\mathbf{p}_0 = p\hat{\mathbf{e}}_z$. The irrep is characterized by (p, η) and

$$(\mathbf{P}, J_3) \left|\mathbf{p}_0, \pm\right\rangle = \left|\mathbf{p}_0, \pm\right\rangle (\mathbf{p}_0, \pm\lambda), \quad I_2 \left|\mathbf{p}_0, \pm\right\rangle = \left|\mathbf{p}_0, \mp\right\rangle. \tag{46}$$

- The general basis vector is obtained via $\left|\mathbf{p}, \pm\right\rangle = R(\mathbf{p}) \left|\mathbf{p}_0, \pm\right\rangle$.
- The effect of $I_s = R_2(\pi) I_2$ is

$$\begin{aligned} I_s \left|\mathbf{p}, \pm\right\rangle &= I_s R(\phi, \theta, 0) \left|\mathbf{p}_0, \pm\right\rangle = R(\phi, \theta, 0) R_2(\pi) \left|\mathbf{p}, \mp\right\rangle \\ &= R(\pi + \phi, \pi) \left|-\mathbf{p} \mp\right\rangle (-1)^{\lambda}. \end{aligned} \tag{47}$$

Exercises

1. Show that a four-vector A^μ transforms under the Lorentz group as the $(u, v) = (\frac{1}{2}, \frac{1}{2})$ representation.

WKT11.2 Write down the 3×3 matrix representation of the momentum operators (P_1, P_2) and the inversions (I_1, I_2). Verify Eq. (39) explicitly.

WKT11.4 Derive de representation matrices for the operators $T(\mathbf{b})$, $R(\alpha, \beta, \gamma)$ and I_s in the irrep corresponding to $\mathbf{P}^2 = p^2 \neq 0$, $(\mathbf{J} \cdot \mathbf{P})^2 = 0$ for $\widetilde{E}_3$.

WKT11.5 Define the angular momentum basis of the (p, η) representation of $\widetilde{E}_3$ in terms of the linear momentum basis given by Eq. (**??**). Derive the representation matrices of the elements of the subgroup $O(3)$.

WKT11.6 Find eigenstates of the operator $I_\theta = R(\theta) I_2 R^{-1}(\theta)$ in the vector space of the m-irrep $(m = 1, 2, \dots)$ of the $O(2)$ group and the associated eigenvalues.