Symmetries in Physics
Lecture 10

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Lecture contents

Chapter 3. The Lorentz and Poincare groups

III.2. Representations of the Poincaré group

III.2.1. Equivalence of the Lorentz group to $SU(2) \times SU(2)$

- Consider the new basis

$$M^m = \frac{1}{2}(J^m + iK^m), \qquad N^m = \frac{1}{2}(J^m - iK^m), \tag{1}$$

  or inversely,

$$J^m = M^m + N^m, \qquad K^m = -i(M^m - N^m). \tag{2}$$

- Their commutators read:

$$[M^m, M^n] = i\varepsilon^{mnk} M^k, \quad [N^m, N^n] = i\varepsilon^{mnk} N^k, \quad [M^m, N^n] = 0. \tag{3}$$

- The Lie algebra corresponds to $SU(2)_M \times SU(2)_N$.
- The irreps of $SU(2)$ can be used to generate irreps of $\widetilde{L}_+$.
- Since $SU(2)$ is compact, its unitary irreps are finite-dimensional; however, $\widetilde{L}_+$ is non-compact.
- M_m, N_m, J_m, K_m cannot be simultaneously Hermitian $\Rightarrow$ the corresponding irreps will be non-unitary.
- $\widetilde{L}_+$ and $SU(2)_M \times SU(2)_N$ are equivalent only at Lie algebra level; the groups are isomorphic only close to E.

III.2.2. Finite-dimensional representations

- We impose $\mathbf{M}^\dagger = \mathbf{M}$ and $\mathbf{N}^\dagger = \mathbf{N}$.
- Let $u(u+1)$ and $v(v+1)$ be the eigenvalues of the Casimir operators $\mathbf{M}^2$ and $\mathbf{N}^2$ of $SU(2)_M$ and $SU(2)_N$.
- The basis vectors can be labeled as $|kl\rangle \equiv |u,k;v,l\rangle$:

$$\begin{aligned} J^3\,|kl\rangle &= |kl\rangle\,(l+k), \qquad K^3\,|kl\rangle = |kl\rangle\, i(l-k), \\ J^\pm\,|kl\rangle &= |k\pm 1, l\rangle\,\sqrt{u(u+1)-k(k\pm 1)} \\ &\quad + |k, l\pm 1\rangle\,\sqrt{v(v+1)-l(l\pm 1)}, \\ K^\pm\,|kl\rangle &= |k, l\pm 1\rangle\, i\sqrt{v(v+1)-l(l\pm 1)} \\ &\quad - |k\pm 1, l\rangle\, i\sqrt{u(u+1)-k(k\pm 1)}. \end{aligned} \tag{4}$$

- $K_3^\dagger = -K_3$ is anti-Hermitian and the corresponding representations are non-unitary.
- W.r.t. the $\mathbf{J}^2$, J^3 basis, the $|kl\rangle$ rep. can be decomposed into irreps $|j, m_j\rangle$, labeled (j_0, j_1), with $j_0 = |u-v|$ and $j_1 = u+v$.
- 4-vectors a^μ transform with $(u,v) = (1/2, 1/2)$: a^0 is invariant under rotations ($j_0 = 0$); a^i transforms as a 3-vector ($j_1 = 1$).
- Second-rank tensors $t^{\mu\nu}$ transform as $(0,0)$ (trace), $(1,0)\oplus(0,1)$ (antisymmetric part) and $(1,1)$ irreps (see homework).

III.2.3. Unitary representations of $\widetilde{L}_+$

- We now use the $|jm\rangle \equiv |j_0, j_1; j, m\rangle$ basis and impose unitarity, $J_m^\dagger = J_m$ and $K_m^\dagger = K_m$.
- The matrix elements of $\mathbf{K} \to K^1_\sigma = \{-K^+/\sqrt{2}, K^3, K^-/\sqrt{2}\}$ are

$$\begin{aligned}\langle j'm'|K^3|jm\rangle &= A^{j'}{}_j \, \langle j'm'(1,j)0m\rangle \,, \\ \langle j'm'|K^\pm|jm\rangle &= \mp\sqrt{2}A^{j'}{}_j \, \langle j'm'(1,j)\pm 1m\rangle \,, \end{aligned} \tag{5}$$

  where we used the Wigner-Eckart theorem and $A^{j'}{}_j$ is the reduced matrix element.
- The selection rules impose $|j-1| \le j' \le j+1$.
- Using $[K^\pm, K^3] = \pm J^\pm$ and $[K^+, K^-] = -2J^3$ eventually leads to

$$\begin{aligned} A^j{}_j &= \frac{i\nu j_0}{\sqrt{j(j+1)}}, \qquad & A^j{}_{j-1} &= -\sqrt{j(2j-1)}B_j\xi_j, \\ A^{j-1}{}_j &= \sqrt{j(2j+1)}\frac{B_j}{\xi_j}, \qquad & B_j^2 &= \frac{(j^2-j_0^2)(j^2-\nu^2)}{j^2(4j^2-1)}, \end{aligned} \tag{6}$$

  where ν and ξ_j are arbitrary complex numbers.

- Imposing $K_3^\dagger = K_3$ and $K_\pm^\dagger = K_\mp$ leads to 2 irreps:
- **Theorem:** There are two classes of unitary irreps of $\widetilde{L}_+$: a) the *principal series*, with $v = -iw$ and $w \in \mathbb{R}$, while $j_0 = 0, 1/2, 1, \ldots$; b) the *complementary series*, when $-1 \leq \nu \leq 1$ and $j_0 = 0$. In both cases, $\xi_j = 1$.
  Proof: Will be discussed later.
- In terms of $\mathbf{M}^2$ and $\mathbf{N}^2$, we have $j_0 = |u - v|$ and $\nu = u + v$.

III.2.4. Casimir operators of the Poincaré group $\widetilde{P}$

- $\widetilde{P}$ has the Casimir operator $C_1 = P_\mu P^\mu = P_0^2 - \mathbf{P}^2$.
  Proof: Clearly, $[P^\mu, C_1] = 0$ since $T(b)$ is abelian; Then, using $[J_{\alpha\beta}, P_\mu] = i(P_\alpha g_{\beta\mu} - P_\beta g_{\alpha\mu})$, we have

$$[J_{\alpha\beta}, C_1] = P^\mu [J_{\alpha\beta}, P_\mu] + [J_{\alpha\beta}, P_\mu] P^\mu = 0. \tag{7}$$

- Its eigenvalue $c_1 \in \mathbb{R}$ distinguishes between: timelike ($c_1 > 0$); null ($c_1 = 0$); and spacelike ($c_1 < 0$) states.
- **Theorem:** The *Pauli-Lubanski vector* $W^\mu = \frac{1}{2}\varepsilon^{\mu\alpha\beta\lambda} J_{\alpha\beta} P_\lambda$ satisfies:

$$W^\mu P_\mu = 0, \quad [W^\mu, P^\lambda] = 0, \quad [W^\mu, J^{\lambda\sigma}] = i(W^\lambda g^{\mu\sigma} - W^\sigma g^{\mu\lambda}),$$
$$[W^\mu, W^\nu] = i\varepsilon^{\mu\nu\lambda\sigma} W_\lambda P_\sigma. \tag{8}$$

  Proof: (i) is trivial; (ii) follows from $[J_{\alpha\beta}, P_\mu] = i(P_\alpha g_{\beta\mu} - P_\beta g_{\alpha\mu})$:

$$[W^\mu, P^\lambda] = \frac{i}{2}\varepsilon^{\mu\nu\alpha\beta} P_\nu (P_\alpha \delta_\beta^\lambda - P_\beta \delta_\alpha^\lambda) = 0. \tag{9}$$

  (iii) follows from the definition of W^μ as a four-vector; (iv) HW!
- **Theorem:** $C_2 = -W^\mu W_\mu$ is a Casimir operator of $\widetilde{P}$.
  Proof: $[C_2, P_\lambda] = 0$ trivially; $[C_2, J_{\alpha\beta}] = 2i(W^\beta W^\alpha - W^\alpha W^\beta) = 0$.

III.2.5 Induced unitary irreps of $\widetilde{P}$
Trivial case

- We consider momentum eigenstates: $P^\mu |p\rangle = |p\rangle\, p^\mu$.
- When $p^\mu = 0$, both $c_1 = c_2 = 0$ and the little group is $\widetilde{L}_+$.
- The irreps of $\widetilde{P}$ for this case are those of the Lorentz group:

$$T(b)\,|0jm\rangle = |0jm\rangle\,, \quad \Lambda\,|0jm\rangle = |0j'm'\rangle\, D_{j_0,\nu}[\Lambda]^{j'm'}{}_{jm}. \qquad (10)$$

Time-like case ($c_1 = M^2 > 0$)

- The standard vector is taken as $p_t^\mu = (M, \mathbf{0})$, corresponding to a mass M at rest.
- The little group of p_t^μ is $SO(3)$ and every unitary irrep of $SO(3)$ induces a unitary irrep of $\widetilde{P}$.
- Consider the vectors $|\mathbf{0}\lambda\rangle$, satisfying

$$(P^\mu, \mathbf{J}^2, J^3)\,|\mathbf{0}\lambda\rangle = |\mathbf{0}\lambda\rangle\,(p_t^\mu, s(s+1), \lambda), \tag{11}$$

  with $\lambda = -s, -s+1, \ldots s$, generating an inv. subsp. w.r.t. $SO(3)$.
- The second Casimir evaluates to $c_2 = M^2\mathbf{J}^2 = s(s+1)M^2$.
- The other vectors can be generated via

$$|\mathbf{p}\lambda\rangle = H(p)\,|\mathbf{0}\lambda\rangle\,, \qquad H(p) = R(\phi, \theta, 0)L_3(\xi), \tag{12}$$

  with $\cosh\xi = p^0/M$.
- The eigenvalue λ can be linked to the helicity operator, $h = \mathbf{J}\cdot\mathbf{P}/|\mathbf{P}|$:

$$\frac{\mathbf{J}\cdot\mathbf{P}}{|\mathbf{P}|}\,|\mathbf{p}\lambda\rangle = R\frac{\mathbf{J}\cdot\mathbf{P}}{|\mathbf{P}|}\,|p\mathbf{k}, \lambda\rangle = RJ^3L_3(\xi)\,|\mathbf{0}\lambda\rangle = |\mathbf{p}\lambda\rangle\,\lambda. \tag{13}$$

Time-like case

- **Theorem:** (i) The vector space spanned by $|\mathbf{p}\lambda\rangle$ is invariant under the Poincaré group transformations, that act as follows:

$$T(b)\,|\mathbf{p}\lambda\rangle = \mathbf{p}\lambda e^{-ib^\mu p_\mu}, \quad \Lambda\,|\mathbf{p}\lambda\rangle = |\mathbf{p}'\lambda'\rangle\, D_s[R(\Lambda,p)]^{\lambda'}{}_{\lambda}, \qquad (14)$$

with $p'^\mu = \Lambda^\mu{}_\nu p^\nu$ and $R(\Lambda,p) = H^{-1}(p')\Lambda H(p)$.
(ii) The resulting representation, labelled (M,s), is unitary and irreducible.
Proof: $P^\mu\,|\mathbf{p}\lambda\rangle = P^\mu H(p)\,|\mathbf{0}\lambda\rangle = H(p)P^\nu\,|\mathbf{0}\lambda\rangle\, H(p)^\mu{}_\nu = |\mathbf{p}\lambda\rangle\, p^\mu$.
Acting with Λ on $|\mathbf{p}\lambda\rangle = H(p)\,|\mathbf{0}\lambda\rangle$ gives:

$$\Lambda\,|\mathbf{p}\lambda\rangle = H(p')[H^{-1}(p')\Lambda H(p)]\,|0\lambda\rangle = |\mathbf{p}'\lambda'\rangle\, D_s[R(\Lambda,p)]^{\lambda'}{}_{\lambda},$$

where $p' = \Lambda p$, such that $H^{-1}(p')\Lambda H(p) = R(\Lambda,p)$ is an element of the little group of p_t.

- **Theorem:** (i) The independent components of W^μ form the Lie algebra of the little group of p^μ. (ii) To every irrep of the little group corresponds an *induced rep.* of the full Poincaré group by successive application of $\widetilde{L}_+$. (iii) The unitary irreps are characterized by $C_1 = P^2$ and $C_2 = -W^2$.

Light-like case ($c_1 = 0$)

- Since $p^2 = p_0^2 - \mathbf{p}^2 = 0 \Rightarrow \mathbf{p}^2/p_0^2 = v^2/c^2 = 1$.
- Light-like 4-vectors have no rest frame $\Rightarrow$ the standard momentum is taken as $p_l^\mu = (\omega_0, 0, 0, \omega_0)$.
- Let $p^\mu = (\omega, \mathbf{p})$, where $\mathbf{p} = \omega\hat{\mathbf{p}}$ and $\hat{\mathbf{p}} \equiv \hat{\mathbf{p}}(\theta, \phi)$ a unit vector. Then:

$$p^\mu = H(p)^\mu{}_\nu p_l^\nu, \quad H(p) = R(\phi, \theta, 0) L_3(\xi), \tag{15}$$

  with $\omega = (\cosh\xi + \sinh\xi)\omega_0 = e^\xi \omega_0$.
- The little group of p_l^μ can be found by computing:

$$W^\mu \to \frac{\omega_0}{2} J_{\alpha\beta}(\varepsilon^{\mu\alpha\beta 0} - \varepsilon^{\mu\alpha\beta 3}) = \omega^0(J^3, J^1 + K^2, J^2 - K^1, J^3). \tag{16}$$

- The Lie algbera is:

$$[W^1, W^2] = 0, \quad [W^2, J^3] = iW^1, \quad [W^1, J^3] = -iW^2, \tag{17}$$

  corresponding to E_2 (when $W^{1,2} \to P^{1,2}$).

Light-like caes

- The irreps of E_2 are labelled by $|w\lambda\rangle$, with,

$$J^3 \, |w\lambda\rangle = |w\lambda\rangle \, \lambda, \quad C_2 \, |w\lambda\rangle = (W_1^2 + W_2^2) \, |w\lambda\rangle = |w\lambda\rangle \, w^2. \quad (18)$$

- In physics, only the $w = 0$ states are relevant: neutrinos ($\lambda = -1/2$), anti-neutrinos ($\lambda = 1/2$), photons ($\lambda = \pm 1$), etc.
- We consider the states $|\mathbf{p}_l \lambda\rangle$ satisfying:

$$(P^\mu, J_3) \, |\mathbf{p}_l \lambda\rangle = |\mathbf{p}_l \lambda\rangle \, (p_l^\mu, \lambda), \quad (W_1, W_2) \, |\mathbf{p}_l \lambda\rangle = (0, 0). \quad (19)$$

- The induced subspace is $|\mathbf{p}\lambda\rangle = H(p) \, |\mathbf{p}_l \lambda\rangle$, with $H = R(\phi, \theta, 0) L_3(\xi)$ and $p = \omega_0 e^\xi$.
- **Theorem:** $|\mathbf{p}\lambda\rangle$ span a vector space invariant under $\widetilde{P}_+$; the resulting rep., labelled $(M = 0, \lambda)$, is unitary and irreducible and

$$T(b) \, |\mathbf{p}\lambda\rangle = |\mathbf{p}\lambda\rangle \, e^{-ib^\mu p_\mu}, \quad \Lambda \, |\mathbf{p}\lambda\rangle = |\mathbf{p}'\lambda\rangle \, e^{-i\lambda\theta(\Lambda, p)}, \quad (20)$$

where $p'^\mu = \Lambda^\mu{}_\nu p^\nu$ and

$$\langle \mathbf{p}_l \lambda' | H^{-1}(\Lambda p) \Lambda H(p) | \mathbf{p}_l \lambda\rangle = e^{-i\lambda\theta(\Lambda, p)} \delta^{\lambda'}{}_\lambda. \quad (21)$$

- Important difference: the helicity λ is invariant under Λ for $M = 0$, while for $M > 0$, it is transformed among all $2s + 1$ possible values.

Space-like case: $c_1 = -Q^2 < 0$

- The standard momentum is $p_s^\mu = (0,0,0,Q)$ and the little group is:

$$W^\mu \to -\frac{Q}{2}\varepsilon^{\mu\alpha\beta 3} = Q(-J^3, K^2, -K^1, 0). \quad (22)$$

- The second Casimir operator is $C_2 = -W^2 = K_1^2 + K_2^2 - J_3^2$.
- The Lie algebra consists of

$$[K^2, J^3] = iK^1, \quad [J^3, K^1] = iK^2, \quad [K^1, K^2] = -iJ^3, \quad (23)$$

corresp. to the (non-compact) $SO(2,1)$ group $\Rightarrow$ infinite-dim. unitary irreps.

- $w = c_2/Q^2$ can take either continuous, positive values: $0 < w < \infty$; or discrete negative values: $w = -j(j+1)$, with $j = 0, 1/2, 1, \ldots$.
- The states $|\mathbf{p}\lambda\rangle$ can be generated as follows:

$$|\mathbf{p}\lambda\rangle = H(p)\,|p_s\lambda\rangle\,, \quad H(p) = R_3(\phi)L_1(\zeta)L_3(\xi), \quad (24)$$

where

$$\begin{aligned}(0,0,0,Q) &\xrightarrow{L_3(\xi)} Q(\sinh\xi, 0, 0, \cosh\xi)\\ &\xrightarrow{L_1(\zeta)} Q(\sinh\xi\cosh\zeta, \sinh\xi\sinh\zeta, 0, \cosh\xi)\\ &\xrightarrow{R_3(\phi)} (Q\sinh\xi\cosh\zeta, \mathbf{p}_\perp, \cosh\xi), \end{aligned} \quad (25)$$

with $\mathbf{p}_\perp = Q\sinh\xi\sinh\zeta(\mathbf{i}\cos\phi + \mathbf{j}\sin\phi)$.

Exercises

1. Compute the Clebsch-Gordan coefficients $\langle JM(j,1)mm'\rangle$ by directly decomposing the direct product representation $|jm;j'm'\rangle$, with $j'=1$. Writing $m = M - m'$, express the result as the table below:

$$\begin{array}{c|ccc} j'=1 & m'=1 & m'=0 & m'=-1 \\ \hline J=j+1 & \sqrt{\frac{(j+M)(j+M+1)}{(2j+1)(2j+2)}} & \sqrt{\frac{(j-M+1)(j+M+1)}{(2j+1)(j+1)}} & \sqrt{\frac{(j-M)(j-M+1)}{(2j+1)(2j+2)}} \\ J=j & -\sqrt{\frac{(j+M)(j-M+1)}{2j(j+1)}} & \frac{M}{\sqrt{j(j+1)}} & \sqrt{\frac{(j-M)(j+M+1)}{2j(j+1)}} \\ J=j-1 & \sqrt{\frac{(j-M)(j-M+1)}{2j(j+1)}} & -\sqrt{\frac{(j-M)(j+M)}{j(2j+1)}} & \sqrt{\frac{(j+M+1)(j+M)}{2j(2j+1)}} \end{array} \tag{26}$$

2. Follow Appendix VII in Wu-Ki Tung to derive Eq. (6) and to establish the Theorem on slide 6.

WKT10.5 Show that an anti-symmetric second-rank tensor in Minkowski space transforms as the $(1,0)\oplus(0,1)$ representation of the Lorentz group.

WTK10.6 (i) Show that the trace of a second-rank tensor, $t^{\mu}{}_{\mu}$, is invariant under all Lorentz transformations, so that it transforms as the $(0,0)$ representation. (ii) Show that the traceless symmetric tensor $\tilde{t}^{\mu\nu} = \frac{1}{2}(t^{\mu\nu}+t^{\nu\mu}) - \frac{1}{4}g^{\mu\nu}t^{\lambda}{}_{\lambda}$ transforms under Lorentz transformations as the $(1,1)$ representation.

Exercises

3. Consider the Pauli-Lubanski vector, $W^\mu = \frac{1}{2}\varepsilon^{\mu\alpha\beta\lambda} J_{\alpha\beta} P_\lambda$.
   a) Consider an index $\xi \neq \mu$ with fixed value. Show that

   $$W^\mu = \varepsilon^{\mu\alpha\beta\xi}\left(\tfrac{1}{2} J_{\alpha\beta} P_\xi + J_{\xi\alpha} P_\beta\right) \qquad \text{(no summation over } \xi\text{)}. \quad (27)$$

   b) Compute $[W^\mu, J^{\lambda\sigma}]$, knowing $[P_\mu, J_{\lambda\sigma}] = -i(P_\lambda g_{\mu\sigma} - P_\sigma g_{\mu\lambda})$ and $[J_{\mu\nu}, J_{\lambda\sigma}] = -i(J_{\mu\lambda} g_{\nu\sigma} - J_{\mu\sigma} g_{\nu\lambda} - J_{\nu\lambda} g_{\mu\sigma} + J_{\nu\sigma} g_{\mu\lambda})$.

   c) Clearly, $\lambda \neq \sigma$. Consider now simultaneously that $\mu \neq \lambda$ and $\mu \neq \sigma$, and let ξ be a fourth index such that, without loss of generality, $(\mu, \lambda, \sigma, \xi)$ form an even permutation of $(0, 1, 2, 3)$ (i.e., $\varepsilon^{\mu\lambda\sigma\xi} = 1$). Show that, in this case, $[W^\mu, J^{\lambda\sigma}] = 0$.

   d) Consider now $\mu = \lambda$ but $\mu \neq \sigma$. Considering the result from part a), show that $[W^\mu, J^{\lambda\sigma}] = -i(W^\lambda g^{\mu\sigma} - W^\sigma g^{\mu\lambda})$.

WKT10.9 Derive the unitary irreps of the group $SO(2,1)$, which has as its generators (K_1, K_2, J_3).

WKT10.11 Show that if p is a time-like or light-like 4-vector, then the sign of its time component and that of Λp is the same, for all proper homogeneous Lorentz transformations Λ.