Symmetries in Physics
Lecture 9

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Lecture contents

III.1. The Lorentz and Poincare groups

III.1.1. Definitions

- **Def:** An *event* is characterized by four numbers: $x^\mu = (x^0, \mathbf{x})$, with $x^0 = ct$ representing the "temporal length" (time multiplied by c) and $\mathbf{x}$ the usual spatial coordinates.
- **Def:** The difference $x^\mu = x_1^\mu - x_2^\mu$ between the coordinates of two events is a *coordinate four-vector*.
- **Obs:** The coordinates of an event can be considered as a four-vector when (implicitly) expressed as the difference with respect to the origin at $(ct, \mathbf{x}) = (0, \mathbf{0})$.
- **Def:** The *scalar product* of two four-vectors u^μ and v^μ can be defined in terms of the *Minkowski metric tensor* $g_{\mu\nu}$:

$$u \cdot v = g_{\mu\nu} u^\mu v^\nu = u^0 v^0 - \mathbf{u} \cdot \mathbf{v}, \quad g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1). \quad (1)$$

- **Def:** The *length* of a four-vector is $u^2 = g_{\mu\nu} u^\mu u^\nu = (u^0)^2 - \mathbf{u}^2$.
- **Def:** The 4D spate-time endowed with the Minkowski metric $g_{\mu\nu}$ is called the *Minkowski space-time* (ST).
- **Def:** The components u^μ are *contravariant* components.
- **Def:** $u_\mu = g_{\mu\nu} u^\nu = (u^0, -\mathbf{u})$ are *covariant* components.
- **Def:** $u^\mu = g^{\mu\nu} u_\nu$ uses the *inverse metric tensor*, $g^{\mu\nu} = g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.

III.1.2. Homogeneous Lorentz transformations

- **Def:** The *homogeneous Lorentz transformations* are continuous linear transformations which preserve the length of 4-vectors:

$$\hat{e}_\mu \to \hat{e}_{\mu'} = \hat{e}_\mu \Lambda^\mu{}_{\mu'}, \quad x^\mu \to x^{\mu'} = \Lambda^{\mu'}{}_\nu x^\nu, \quad x^2 = x'^2. \qquad (2)$$

- **Obs:** The preservation of norm implies $x'^2 = g_{\mu'\nu'} x^{\mu'} x^{\nu'} = g_{\mu\nu} x^\mu x^\nu$, i.e.

$$g_{\mu'\nu'} \Lambda^{\mu'}{}_\mu \Lambda^{\nu'}{}_\nu = g_{\mu\nu} \quad \Leftrightarrow \quad \Lambda^T g \Lambda = g, \quad \Lambda^{-1} = g \Lambda^T g^{-1}. \qquad (3)$$

- Since $\det g = -1 \Rightarrow \det \Lambda = \Lambda^0{}_\mu \Lambda^1{}_\nu \Lambda^2{}_\lambda \Lambda^3{}_\sigma \varepsilon^{\mu\nu\lambda\sigma} = \pm 1$, with $\varepsilon^{0123} = 1$.
- Moreover, $g_{\mu\nu} \Lambda^\mu{}_0 \Lambda^\nu{}_0 = (\Lambda^0{}_0)^2 - (\Lambda^i{}_0)^2 = 1 \Rightarrow |\Lambda^0{}_0| \geq 1$.
- **Def:** The homogeneous Lorentz transfs. are linear 4×4 matrices with $\Lambda^0{}_0 \geq 0$ that leave $g^{\mu\nu}$ and $\varepsilon^{\mu\nu\lambda\sigma}$ invariant (i.e., $\det \Lambda = 1$).
- Eq. (3) provides 10 relations $\Rightarrow \Lambda^\mu{}_\nu$ is characterized by $16 - 10 = 6$ indep. params.: 3 rotations and 3 Lorentz boosts:

$$R^\mu{}_\nu = \begin{pmatrix} 1 & 0 \\ 0 & R^i{}_j \end{pmatrix} \quad (L_1)^\mu{}_\nu = \begin{pmatrix} \cosh\xi & \sinh\xi & 0 & 0 \\ \sinh\xi & \cosh\xi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad (4)$$

with $\cosh\xi = \gamma$ and $\sinh\xi = \beta\gamma$, s.t. $\tanh\xi = \beta = v/c$.

III.1.3. Proper Lorentz group

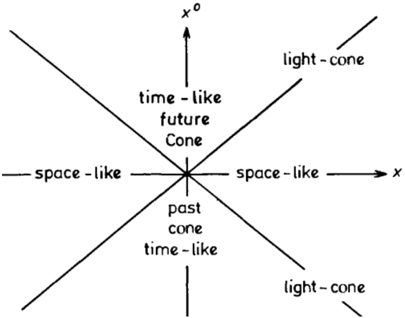
- The *proper Lorentz group* $\widetilde{L}_+$ is comprised of the homogeneous Lorentz transfs. that are continuously connected to E, i.e. having $\det\Lambda = 1$ and $\Lambda^0{}_0 \geq 1$.
- The group is called $SO(1,3)$, signaling the signature of the metric $g_{\mu\nu}$ with respect to which Λ are orthogonal.
- The pure boosts, such as L_1, are unbounded, as $-\infty < \xi < \infty \Rightarrow \widetilde{L}_+$ is non-compact.
- It can be seen that $(L_1^\dagger)^\mu{}_\nu = (L_1)^\mu{}_\nu \neq (L_1^{-1})^\mu{}_\nu \Rightarrow \Lambda$ are non-unitary.
- Covariant components transform with the inverse matrix:

$$u_{\mu'} = g_{\mu'\nu'}\Lambda^{\nu'}{}_\nu u^\nu = \Lambda_{\mu'}{}^\nu u_\nu = u_\nu(\Lambda^{-1})^\nu{}_{\mu'}. \qquad (5)$$

- Similarly, the four-derivative $\partial_\mu = (c^{-1}\partial_t, \boldsymbol{\nabla})$ transforms as a covariant vector:

$$\partial_{\mu'} = \frac{\partial x^\lambda}{\partial x^{\mu'}}\partial_\lambda = (\Lambda^{-1})^\lambda{}_{\mu'}\partial_\lambda = \Lambda_{\mu'}{}^\lambda\partial_\lambda. \qquad (6)$$

III.1.4. Causality structure



- When expressed w.r.t. $x_0^\mu = (0,0)$, events x^μ fall in one of three distinct ST regions, separated by the *light cone* (LC):

$$x^2 = c^2 t^2 - \mathbf{x}^2 = 0. \tag{7}$$

- *Future cone:* when $x^0 > 0$ and $x^2 = (x^0)^2 - \mathbf{x}^2 > 0$;
- *Past cone:* when $x^0 > 0$ and $x^2 = (x^0)^2 - \mathbf{x}^2 > 0$;
- Region *outside the light cone*: $x^2 < 0$.
- Intervals within the LC are *time-like* and $\exists \Lambda \in \widetilde{L}_+$ s.t. $x^{\mu'} = (ct', 0)$.
- Int. outside the LC are *space-like* and $\exists \Lambda \in \widetilde{L}_+$ s.t. $x^{\mu'} = (0, \mathbf{x}')$.
- Vectors are time-like ($u^2 > 0$), space-like ($u^2 < 0$), or null ($u^2 = 0$).

III.1.5. Decomposition of Lorentz transformations

- **Theorem:** A general $\Lambda \in \widetilde{L}_+$ can be uniquely factorized as

$$\Lambda = R(\alpha, \beta, 0) L_3(\xi) R^{-1}(\phi, \theta, \psi), \tag{8}$$

where $L_3(\xi)$ is a boost along the positive z axis: $0 \leq \xi < \infty$.

Proof: (i) Since $g_{\mu\nu}\Lambda^\mu{}_0\Lambda^\nu{}_0 = (\Lambda^0{}_0)^2 - (\Lambda^1{}_1)^2 - (\Lambda^2{}_2)^2 - (\Lambda^3{}_3)^2 = 1$, one can parametrize

$$\Lambda^0{}_0 \equiv A^0 = \cosh\xi, \quad (\Lambda^1{}_0, \Lambda^2{}_0, \Lambda^3{}_0) \equiv \mathbf{A} = \sinh\xi\, \mathbf{n}(\beta, \alpha), \tag{9}$$

with $\mathbf{n}(\beta, \alpha) = \sin\beta(\mathbf{i}\cos\alpha + \mathbf{j}\sin\alpha) + \mathbf{k}\cos\beta$.
(ii) Consider $\hat{t}^\mu = \delta^\mu_0 = (1, 0, 0, 0)$ a unit vector along the time axis. Then, all $SO(3)$ rotations form the *little group* of $\hat{t}^\mu$, since $R\hat{t} = \hat{t}$. Furthermore,

$$[\Lambda\hat{t}]^\mu = \Lambda^\mu{}_0 = A^\mu, \quad [R(\alpha, \beta, 0) L_3(\xi)\hat{t}]^\mu = A^\mu. \tag{10}$$

Hence, $\Lambda^{-1} R(\alpha, \beta, 0) L_3(\xi) \in SO(3)$ as it leaves $\hat{t}$ invariant:

$$\Lambda^{-1} R(\alpha, \beta, 0) L_3(\xi) = R(\phi, \theta, \psi) \Rightarrow \Lambda = R(\alpha, \beta, 0) L_3(\xi) R^{-1}(\phi, \theta, \psi).$$

- A pure rotation corresponds to $\xi = 0$;
- A pure boost along $\hat{\mathbf{n}}(\theta, \phi)$ corresponds to $\psi = 0$, $\alpha = \phi$ and $\beta = \theta$.

III.1.6. Relation to $SL(2,\mathbb{C})$

- Let $x^\mu \to X = x^\mu \sigma_\mu$, with $\sigma^\mu = (E, \boldsymbol{\sigma})$ and

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (11)$$

- Then:

$$X = \begin{pmatrix} x^0 - x^3 & -x^1 + ix^2 \\ -x^1 - ix^2 & x^0 + x^3 \end{pmatrix}, \quad \det X = (x^0)^2 - \mathbf{x}^2 = x^2. \quad (12)$$

- A Lorentz transf. acting on x^μ induces a transformation A acting as $X \to X' = AXA^\dagger$.
- Since $x^{\mu'} = \Lambda^{\mu'}{}_\nu x^\nu$ has the same length as x^μ, we have

$$\det X' = \det X |\det A|^2 = \det X. \quad (13)$$

- We fix the phase of A such that $\det A = 1$.
- **Def:** The *Special linear group* $SL(2,\mathbf{C})$ consists of the complex 2×2 matrices A having unit determinant: $\mathrm{det} A = 1$.
- $SL(2,\mathbb{C})$ is characterized by 6 independent parameters ($4 \times 2 = 8$ real entries -2×1 from determinant constraint).
- To each matrix Λ, we can associate two matrices $\pm A(\Lambda)$, as both $\det A = 1$ and $X' = AXA^\dagger$ are quadratic in the elements of A.
- $SL(2,\mathbb{C})$ is the universal cov. gr. of $SO(1,3)$, as $SU(2)$ is for $SO(3)$.

III.1.7. Poincaré group.

- **Def:** The space-time translations and proper Lorentz transformations, as well as their products, form the group $\widetilde{P}$, called the *Poincaré Group*, or the *inhomogeneous Lorentz group*.
- $g(b,\Lambda) \in \widetilde{P}$ acts as $x^\mu \to x^{\mu'} = b^{\mu'} + \Lambda^{\mu'}{}_\nu x^\nu$.
- The products is $g(b',\Lambda')g(b,\Lambda) = g(\Lambda' b + b', \Lambda'\Lambda)$, while $g^{-1}(b,\Lambda) = g(-\Lambda^{-1}b, \Lambda^{-1})$.
- The transformation can be represented using 5×5 matrices:

$$g(b,\Lambda) \to \begin{pmatrix} \Lambda^\mu{}_\nu & b^\mu \\ 0 & 1 \end{pmatrix}, \quad x^\mu \to \begin{pmatrix} x^\mu \\ 1 \end{pmatrix}. \tag{14}$$

- **Theorem:** A general element of the Poincaré group can be factorized as $g(b,\Lambda) = T(b)\Lambda$.
- **Theorem:** The translations form an *invariant subgroup*: $\Lambda T(b)\Lambda^{-1} = T(\Lambda b)$.

III.1.8. Lie algebra of the Poincaré group

- The Poincaré group has 10 generators: 4 for space-time translations; 3 for the $SO(3)$ rotations; and 3 for the Lorentz boosts.
- **Def:** The covariant *generators for translations* P_μ are defined by

$$T(\delta b) = E - i\delta b^\mu P_\mu \quad \Rightarrow \quad T(b) = e^{-ib^\mu P_\mu}. \tag{15}$$

- **Theorem:** Under the Lorentz group, P_μ transform as

$$\Lambda P_\mu \Lambda^{-1} = P_\nu \Lambda^\nu{}_\mu, \qquad \Lambda P^\mu \Lambda^{-1} = (\Lambda^{-1})^\mu{}_\nu P^\nu. \tag{16}$$

Proof: Applying $\Lambda T(b)\Lambda^{-1} = T(\Lambda b)$ infinitesimally gives

$$\Lambda(\delta b^\mu P_\mu)\Lambda^{-1} = (\Lambda^\nu{}_\mu \delta b^\mu) P_\nu \quad \Rightarrow \quad \Lambda P_\mu \Lambda^{-1} = P_\nu \Lambda^\nu{}_\mu. \tag{17}$$

Raising the index gives $\Lambda P^\mu \Lambda^{-1} = \Lambda_\nu{}^\mu P^\nu = (\Lambda^{-1})^\mu{}_\nu P^\nu$.

- **Def:** The covariant *generators for the Lorentz transformations* $J_{\mu\nu}$ are anti-symmetric tensors defined by $\Lambda(\delta\omega) = E - \frac{i}{2}\delta\omega^{\mu\nu}J_{\mu\nu}$.
- In the case of rotations, $R(\delta\theta) = E - i\delta\boldsymbol{\theta}\cdot\mathbf{J}$ gives the identification:

$$\begin{aligned} \delta\theta^1 &= \delta\omega^{23} = -\delta\omega^{32}, & J^1 &= J_{23} = -J_{32}, \\ \delta\theta^2 &= \delta\omega^{31} = -\delta\omega^{13}, & J^2 &= J_{31} = -J_{13}, \\ \delta\theta^3 &= \delta\omega^{12} = -\delta\omega^{21}, & J^3 &= J_{12} = -J_{21}. \end{aligned} \tag{18}$$

- More compactly, $J^k = \frac{1}{2}\varepsilon^{0kmn}J_{mn}$ and $J_{12} = -\varepsilon_{0123}J^3$, with $\varepsilon^{0123} = -\varepsilon_{0123} = 1$.
- In the case of the Lorentz boosts, $\Lambda(\delta\xi) = E - i\delta\boldsymbol{\xi}\cdot\mathbf{K}$, with

$$\delta\xi^m = \delta\omega^{0m} = -\delta\omega^{m0}, \qquad K^m = J^{0m}. \tag{19}$$

- In the usual representation, $(J_{\mu\nu})^{\alpha}{}_{\beta} = i(\delta^{\alpha}_{\mu}g_{\nu\beta} - \delta^{\alpha}_{\nu}g_{\mu\beta})$, e.g.:

$$J^x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad J^y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad J^z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$K^x = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K^y = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K^z = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}.$$

- **Theorem:** The *Lie algebra of the Poincaré group* is given by:

$$[P_\mu, P_\lambda] = 0, \quad [P_\mu, J_{\lambda\sigma}] = -i(P_\lambda g_{\mu\sigma} - P_\sigma g_{\mu\lambda}),$$
$$[J_{\mu\nu}, J_{\lambda\sigma}] = -i(J_{\mu\lambda} g_{\nu\sigma} - J_{\mu\sigma} g_{\nu\lambda} - J_{\nu\lambda} g_{\mu\sigma} + J_{\nu\sigma} g_{\mu\lambda}). \tag{20}$$

  Proof: By brute-force calculation, using the explicit form in the natural representation.

- **Theorem:** (i) The generators $J_{\mu\nu}$ transform under Ω as:

$$\Omega J_{\mu\nu} \Omega^{-1} = J_{\mu'\nu'} \Omega^{\mu'}{}_{\mu} \Omega^{\nu'}{}_{\nu}. \tag{21}$$

  (ii) Let $\Lambda(\omega)$ be a proper Lorentz transformation. Then:

$$\Omega \Lambda(\omega) \Omega^{-1} = \Lambda(\omega'), \qquad \omega^{\mu'\nu'} = \Omega^{\mu'}{}_{\lambda} \Omega^{\nu'}{}_{\sigma} \omega^{\lambda\sigma}. \tag{22}$$

  Proof: (i) For small $\delta\Omega^{\alpha\beta}$, LHS evaluates to:

$$\text{LHS} = J_{\mu\nu} - i\delta\Omega^{\alpha\beta}[J_{\alpha\beta}, J_{\mu\nu}] = J_{\mu\nu} - 2J_{\alpha\mu}\delta\Omega^{\alpha}{}_{\nu} + 2J_{\alpha\nu}\Omega^{\alpha}{}_{\mu}.$$

  RHS evaluates to:

$$\text{RHS} = J_{\mu'\nu'}(\delta^{\mu'}_{\mu} - i\Omega^{\alpha\beta}(J_{\alpha\beta})^{\mu'}{}_{\mu})(\delta^{\nu'}_{\nu} - i\Omega^{\alpha\beta}(J_{\alpha\beta})^{\nu'}{}_{\nu}),$$

  in agreement with LHS.
  (ii) Follows automatically from (i).

Exercises

1. Using the group multiplication rule shown on slide 9, show that $\Lambda T(b)\Lambda^{-1} = T(\Lambda b)$.
2. Compute $[J_{\mu\nu}, J_{\lambda\sigma}]$ using the explicit expression for the Lorentz transformations generators in the natural representation, shown on slide 11, and confirm Eq. (20).

WKT10.3 Verify the following commutation relations using (a) the 5×5 matrix representation; (b) using Eq. (20):

$$[P^0, P^m] = [P^n, P^m] = 0, \quad [P^0, J^n] = 0,$$
$$[P^m, J^n] = i\varepsilon^{mnl} P^l, \quad [P^m, K^n] = i\delta_{mn} P^0, \quad [P^0, K^n] = iP^n,$$
$$[J^m, J^n] = i\varepsilon^{mnl} J^l, \quad [K^m, J^n] = i\varepsilon^{mnl} K^l, \quad [K^m, K^n] = -i\varepsilon^{mnl} J^l, \tag{23}$$

with $\varepsilon^{mnl} = \varepsilon^{0mnl} = 1$.

3. Show that a) $RK_m R^{-1} = K_{m'} R^{m'}{}_m$; and b) $RL_{\hat{\mathbf{n}}}(\xi)R^{-1} = L_{R\hat{\mathbf{n}}}(\xi)$.

WKT10.4 Express $L_m(\xi) J_n L_m(\xi)^{-1}$ in terms of generators of the Lorentz group.