Symmetries in Physics
Lecture 8

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Lecture contents

Chapter 2. Continuous symmetry groups

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II.4. The Euclidean group E_n

II.4.1. Definition.

- **Def:** The *Euclidean group* E_n consists of all continuous linear transformations on the n-dimensional Euclidean space R_n which leave the length of all vectors invariant.
- E_n consists of two types of transformations: Translations $T(\mathbf{b})$ and rotations $R_{\mathbf{n}}(\psi)$.
- E_2 consists of 2 translations: $T(\mathbf{i}b^1 + \mathbf{j}b^2) = T_x(b^1) + T_y(b^2)$; and one rotation: $R_2(\psi)$, having the action $\mathbf{x} \to \mathbf{x}' = g(\mathbf{b}, \psi)\mathbf{x}$, with

$$x'^1 = x^1 \cos\psi - x^2 \sin\psi + b^1, \quad x'^2 = x^1 \sin\psi + x^2 \cos\psi + b^2. \quad (1)$$

- This can be put in matrix form with respect to 3-comp. vectors:

$$\mathbf{x}_3 = \begin{pmatrix} x^1 \\ x^2 \\ 1 \end{pmatrix}, \quad g(\mathbf{b}, \psi) = \begin{pmatrix} \cos\psi & -\sin\psi & b^1 \\ \sin\psi & \cos\psi & b^2 \\ 0 & 0 & 1 \end{pmatrix}. \quad (2)$$

- The same trick works for E_n:

$$\mathbf{x}_{n+1} = \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix}, \qquad g(\mathbf{b}; \mathbf{n}, \psi) = \begin{pmatrix} R_{\mathbf{n}}(\psi) & \mathbf{b} \\ 0 & 1 \end{pmatrix}. \quad (3)$$

II.4.2. E_n group structure

- **Theorem:** The general $g \in E_n$ can be *decomposed* as:

$$g(\mathbf{b}, R) = T(\mathbf{b})R. \tag{4}$$

- The translations form a conjugacy class with respect to E_n, since:

$$RT(\mathbf{b})R^{-1} = T(R\mathbf{b}), \qquad TT(\mathbf{b})T^{-1} = T(\mathbf{b}). \tag{5}$$

- The group multiplication law reads:

$$g_2 g_1 = g_3, \quad \mathbf{b}_3 = \mathbf{b}_2 + R_2\mathbf{b}_1, \quad R_3 = R_2 R_1. \tag{6}$$

- Clearly, $g^{-1}(\mathbf{b}, R) = g(-R^{-1}\mathbf{b}, R^{-1})$.
- For E_2, $\mathbf{b}_3 = \mathbf{b}_2 + R(\psi_2)\mathbf{b}_1$, $\psi_3 = \psi_1 + \psi_2$, and $g^{-1} = g(-R(-\psi)\mathbf{b}, -\psi)$.
- **Theorem:** The translations form an *invariant subgroup* T_n of E_n. The factor group E_n/T_n is isomorphic to $SO(n)$.
  Proof: Eq. (5) shows that $gT(\mathbf{b})g^{-1} = T(R\mathbf{b})$, hence T_n is an invariant subgroup. The elements of the factor group E_n/T_n are (right) cosests $\{Tg(\mathbf{b}, R)\} = \{g(\mathbf{b}, R)\} \Rightarrow$ distinct cosets are defined by one specific rotation and are in 1:1 corresp. with $SO(n) \Rightarrow E_n/T_n \simeq SO(n)$.

II.4.3. Lie algebra of E_2

- In the representation (2), the generators read:

$$J = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{pmatrix}. \tag{7}$$

- **Theorem:** The generators of E_2 satisfy the following commutation relations (Lie algebra):

$$[P_1, P_2] = 0, \quad [J, P_k] = i\varepsilon_{km} P_m \qquad (k = 1, 2). \tag{8}$$

- Since $e^{-i\psi J} P_k e^{i\psi J} = P_m R(\psi)^m{}_k$, it can be checked that

$$e^{-i\psi J} \mathbf{P} \cdot \mathbf{b} e^{i\psi J} = \mathbf{P} \cdot \mathbf{b}', \quad \mathbf{b}' = R(\psi)\mathbf{b}. \tag{9}$$

- Since $E_2 / T_2 \simeq SO(2)$, the irreps of $SO(2)$ are also irreps of E_2:

$$U(\mathbf{b}, \psi) \to U_m(\mathbf{b}, \psi) = e^{-im\psi}. \tag{10}$$

- It is easy to check that

$$U_m(\mathbf{b}, \psi) U_m(\mathbf{a}, \chi) = e^{-im(\psi+\chi)} = U_m[R(\psi)\mathbf{a} + \mathbf{b}, \psi + \chi]. \tag{11}$$

- These are the only finite-dimensional (indeed, $1D$) irreps of E_2.

II.4.4. Unitary irreps of E_2: angular momentum basis

- We introduce $P_\pm = P_1 \pm iP_2$ satisfying $[J, P_\pm] = \pm P_\pm$.
- It can be checked that $\mathbf{P}^2 = P_+P_- = P_-P_+$ is a Casimir operator:

$$[\mathbf{P}^2, J] = [\mathbf{P}^2, P_\pm] = 0. \tag{12}$$

- We take the basis $|pm\rangle$ with

$$\mathbf{P}^2 |pm\rangle = |pm\rangle\, p^2, \quad J\,|pm\rangle = |pm\rangle\, m, \tag{13}$$

  with $p^2 \geq 0$ and $m = 0, \pm 1, \dots$.
- Taking $\langle pm|pm'\rangle = \delta_{mm'}$, $P_\pm |pm\rangle = |pm \pm 1\rangle \mathcal{N}_m^\pm$, with

$$|\mathcal{N}_m^\pm|^2 = \langle pm|P_\pm^\dagger P_\pm|pm\rangle = p^2 \quad \Rightarrow \quad \mathcal{N}_m^\pm = \mp ip. \tag{14}$$

- When $p^2 = 0$, clearly $P_\pm |0m\rangle = 0$ and

$$J\,|0m\rangle = |0m\rangle\, m, \quad R(\psi)\,|0m\rangle = |0m\rangle\, e^{-im\psi}, \quad T(\mathbf{b})\,|0m\rangle = |0m\rangle\,, \tag{15}$$

  i.e. we uncover the degenerate irrep induced by $SO(2)$.

Theorem: The faithful unitary irreps of E_2 are characterized by $p > 0$; the matrix elements of the generators are given by

$$\langle pm'|J|pm\rangle = m\delta^{m'}_{m}, \quad \langle pm'|P_\pm|pm\rangle = \mp ip\delta^{m'}_{m\pm1}, \tag{16}$$

and the representation matrices for finite transformations are

$$D_p(\mathbf{b},\psi)^{m'}{}_{m} = e^{i(m-m')\phi} J_{m-m'}(pb)e^{-im\psi}, \tag{17}$$

where (b,ϕ) are the polar coordinates of $\mathbf{b}$ and $J_n(z)$ is the Bessel function of first kind.

Proof: (i) The matrix elements of the generators follow from Eqs. (13) and (14). (ii) Writing $U(\mathbf{b},\psi) = T(\mathbf{b})R(\psi)$, as well as $T(\mathbf{b}) = R(\phi)T_x(b)R(-\phi)$, we have

$$\langle pm'|U(\mathbf{b},\psi)|pm\rangle = e^{-i(m-m')\phi} \langle pm'|U(b\mathbf{i},0)|pm\rangle \, e^{-im\psi}. \tag{18}$$

Writing $P_x = (P_+ + P_-)/2$, we have

$$U(b\mathbf{i},0) = e^{-\frac{i}{2}bP_+}e^{-\frac{i}{2}bP_-} = \sum_{l,l'} \left(-\frac{ib}{2}\right)^{l+l'} \frac{P_+^{l'}P_-^{l}}{l'!l!}. \tag{19}$$

Noting that $\langle pm'|P_-^l P_+^{l'}|pm\rangle = \delta_{m+l',m'+l}(-ip)^{l'}(ip)^l$,

$$D_p(b\mathbf{i},0)^{m'}{}_{m} = (-1)^{m-m'}\left(\frac{pb}{2}\right)^{m-m'} \sum_{k=0}^{\infty}\left(\frac{pb}{2}\right)^{2k} \frac{(-1)^k}{l!(l+m-m')!}, \tag{20}$$

where we replaced $l' = l + m' - m$ and $k = l - m + m'$ when $m \geq m'$; and $l = l' + m - m'$ and $k = l' - m' + m$ when $m' > m$. Noting that $J_n(z) = (\frac{z}{2})^n \sum_{k=0}^{\infty}(-z^2/4)^k/[k!(k+n)!]$, we recover Eq. (17).

II.4.5. Induced irreps of E_2: plane wave basis

- The idea of induced reps. is to consider the algebra of the invariant (abelian) subgroup, T_2, and select a particular eigenvector:

$$P_1 |\mathbf{p}_0\rangle = |\mathbf{p}_0\rangle\, p, \quad P_2 |\mathbf{p}_0\rangle = 0, \quad \mathbf{P}^2 |\mathbf{p}_0\rangle = |\mathbf{p}_0\rangle\, p^2. \tag{21}$$

- Acting with $R(\theta)$ gives an eigenstate of P_k with eigenvalue:

$$P_k R(\theta) |\mathbf{p}_0\rangle = R(\theta) |\mathbf{p}_0\rangle\, p_k, \tag{22}$$

  with $p_k = p_{0l} R(-\theta)^l{}_k$ or $p^k = R(\theta)^k{}_l p_0^l$.
- $R(\theta) |\mathbf{p}_0\rangle = |\mathbf{p}\rangle$, where $\mathbf{p} = R(\theta)\mathbf{p}_0$ has polar coordinates (p, θ).
- $|\mathbf{p}\rangle$ with fixed p forms the basis of an irreducible vector space, invariant under E_2, with $\langle \mathbf{p}' | \mathbf{p} \rangle = \langle p, \theta' | p, \theta \rangle = 2\pi\delta(\theta' - \theta)$.

II.4.6. Connection between angular momentum and plane wave bases

- The vector $|m\rangle$ can be obtained using the projection method:

$$|\tilde{m}\rangle = \int_0^{2\pi} \frac{d\phi}{2\pi} R(\phi)\,|\mathbf{p}_0\rangle\, e^{im\phi} = \int \frac{d\phi}{2\pi}\,|\phi\rangle\, e^{im\phi}. \tag{23}$$

- Clearly, $R(\theta)\,|\tilde{m}\rangle = |\tilde{m}\rangle\, e^{-im\theta}$ and $J\,|\tilde{m}\rangle = |\tilde{m}\rangle\, m$.
- Since $\langle \tilde{m}' | \tilde{m}\rangle = \delta_{mm'} = \delta_{mm'}$, we have $\langle \tilde{m}\rangle = |m\rangle\, e^{i\psi_m}$.
- Applying $P_\pm$ gives

$$P_\pm\,|\tilde{m}\rangle = \int \frac{d\phi}{2\pi}\,|\phi\rangle\, e^{i(m\pm1)\phi} = |m \tilde{\pm} 1\rangle\, p. \tag{24}$$

- At the same time, $P_\pm\,|m\rangle = |m\pm1\rangle\,(\mp ip) \Rightarrow \psi_m = \psi_0 - m\pi/2$.
- Taking by convention $\psi_0 = 0$, we have $|m\rangle = |\tilde{m}\rangle\, i^m$ and

$$|m\rangle = \int \frac{d\phi}{2\pi}\,|\phi\rangle\, e^{im(\phi+\frac{\pi}{2})}, \quad |\phi\rangle = \sum_m |m\rangle\, e^{-im(\phi+\frac{\pi}{2})}, \tag{25}$$

with $\langle \phi | m\rangle = e^{im(\phi+\pi/2)}$.

II.4.7. Properties of Bessel functions

- Writing $T(\mathbf{b}) = e^{-i(b^- P_+ + b^+ P_-)}$, with $b^\pm = (b^1 \pm i b^2)/2$, we have

$$i \frac{\partial}{\partial b^\mp} T(\mathbf{b}) = i e^{\pm i\phi} \left[\frac{\partial}{\partial b} \pm \frac{i}{b} \frac{\partial}{\partial \phi} \right] T(\mathbf{b}) = T(\mathbf{b}) P_\pm, \tag{26}$$

  with $b^\pm = b e^{\pm i\phi}/2$.

- Multiplying by $|m\rangle \langle m'|$ and tracing leads to

$$e^{\pm i\phi} \left[\frac{\partial}{\partial b} \pm \frac{i}{b} \frac{\partial}{\partial \phi} \right] \langle m' | T(\mathbf{b}) | m \rangle = \mp p \langle m' | T(\mathbf{b}) | m+1 \rangle, \tag{27}$$

  in other words, $J_n'(z) \mp (n/z) J_n(z) = \mp J_{n+1}(z)$.

- **Theorem:** The Bessel functions $J_n(z)$ satisfy the *recursion formulas:*

$$2J_n'(z) = J_{n-1}(z) - J_{n+1}(z), \quad \frac{2n}{z} J_n(z) = J_{n-1}(z) + J_{n+1}(z). \tag{28}$$

- Applying Eq. (26) twice and using $\mathbf{P}^2 = P_+ P_-$ gives

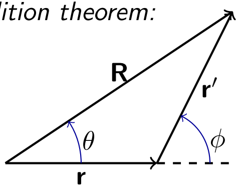
$$-\left(\frac{\partial^2}{\partial b^2} + \frac{1}{b^2}\frac{\partial}{\partial \phi} + \frac{1}{b^2}\frac{\partial^2}{\partial \phi^2}\right) T(\mathbf{b}) = T(\mathbf{b})\mathbf{P}^2. \tag{29}$$

- **Theorem:** The Bessel functions satisfy the *differential eq.*:

$$\left[\frac{d^2}{dz^2} + \frac{1}{z}\frac{d}{dz} + 1 - \frac{n^2}{z^2}\right] J_n(z) = 0. \tag{30}$$

- **Theorem:** The Bessel functions satisfy the *addition theorem:*

$$e^{in\theta} J_n(R) = \sum_k e^{ik\phi} J_k(r) J_{n-k}(r'),$$



where the notation is derived from $T(\mathbf{r})T(\mathbf{r}') = T(\mathbf{R})$, with $\mathbf{r} = (r, 0)$, $\mathbf{r}' = (r', \phi)$ and $\mathbf{R} = (R, \theta)$.

II.4.8. E_3: Lie algebra and group structure

- E_3 consists of 6 generators: **P** and **J**.
- **Theorem:** The *Lie algebra of E_3* is specified by

$$[P_k, P_l] = 0, \quad [J_k, J_l] = i\varepsilon_{klm} J_m, \quad [P_k, J_l] = i\varepsilon_{klm} P_m. \tag{31}$$

- **Theorem:** T_3 forms an invariant subgroup of E_3 and

$$RP_iR^{-1} = P_jR^j{}_i, \quad RT(\mathbf{b})R^{-1} = T(R\mathbf{b}). \tag{32}$$

- **Corollary:** The group elements can be written as $g = T(\mathbf{b})R(\alpha, \beta, \gamma)$ or $g = R(\phi, \theta, 0)T(b\mathbf{k})R(\alpha, \beta, \gamma)$.
- **Theorem:** The *Casimir operators* of E_3 are $\mathbf{P}^2$ and $\mathbf{J} \cdot \mathbf{P}$.
- Since $E_3/T_3 \simeq SO(3)$, the representations of $SO(3)$ represent degenerate representations of E_3, via $TR \to D_j(R)$, for which both Casimirs vanish: $\mathbf{P}^2 = \mathbf{J} \cdot \mathbf{P} = 0$.

II.4.9. E_3: Unitary irreps by induced rep. method

- We consider the eigenvectors of $\mathbf{P}^2$, $\mathbf{J}\cdot\mathbf{P}$ and $\mathbf{P}$:

$$\mathbf{P}^2 \left|\mathbf{p}\lambda\right\rangle = \left|\mathbf{p}\lambda\right\rangle p^2, \quad \mathbf{J}\cdot\mathbf{P} \left|\mathbf{p}\lambda\right\rangle = \left|\mathbf{p}\lambda\right\rangle \lambda p, \quad \mathbf{P} \left|\mathbf{p}\lambda\right\rangle = \left|\mathbf{p}\lambda\right\rangle \mathbf{p}. \quad (33)$$

- Consider the subspace characterized by $\mathbf{p}_0 = p\mathbf{k}$.
- **Def:** All group elements in the factor group leaving the subpsace corresponding to $\mathbf{p}_0$ invariant form the *little group* of $\mathbf{p}_0$.
- In the case of E_3, the little group consists of rotations about the z axis, $R_3(\psi)$, whose irreps are labeled by λ.
- The Casimir operators are $(\boldsymbol{P}^2, \mathbf{J}\cdot\mathbf{P}) \left|\mathbf{p}_0\lambda\right\rangle = \left|\mathbf{p}_0\lambda\right\rangle (p\lambda, p^2)$, while

$$R_3(\psi) \left|\mathbf{p}_0\lambda\right\rangle = \left|\mathbf{p}_0\lambda\right\rangle e^{-i\psi\lambda}, \quad T(\mathbf{b}) \left|\mathbf{p}_0\lambda\right\rangle = \left|\mathbf{p}_0\lambda\right\rangle e^{-i\mathbf{b}\cdot\mathbf{p}_0}. \quad (34)$$

- The full irrep is generated using rotations that are not in the little group:

$$\left|\mathbf{p}\lambda\right\rangle = R(\phi, \theta, 0) \left|\mathbf{p}_0\lambda\right\rangle, \quad (35)$$

where $\mathbf{p} \equiv (p, \theta, \phi)$.

- **Theorem:** The basis vectors (35) satisfy Eqs. (33). The effect of the group operations is:

$$T(\mathbf{b})\,|\mathbf{p}\lambda\rangle = |\mathbf{p}\lambda\rangle\, e^{-i\mathbf{b}\cdot\mathbf{p}}, \quad R(\alpha,\beta,\gamma)\,|\mathbf{p}\lambda\rangle = |\mathbf{p}'\lambda\rangle\, e^{-i\lambda\psi}, \qquad (36)$$

where $\mathbf{p} = (p,\theta,\phi)$, $\mathbf{p}' = R(\alpha,\beta,\gamma)\mathbf{p} = (p,\theta',\phi')$, with ψ defined by

$$R(0,0,\psi) = R(\phi',\theta',0)^{-1}R(\alpha,\beta,\gamma)R(\phi,\theta,0). \qquad (37)$$

Proof: (i) Let $\mathbf{p} = R\mathbf{p}_0$. Then

$$T(\mathbf{b})\,|\mathbf{p}\lambda\rangle = R[R^{-1}T(\mathbf{b})R]\,|\mathbf{p}_0\lambda\rangle = RT(R^{-1}\mathbf{b})\,|\mathbf{p}_0\lambda\rangle = |\mathbf{p}\lambda\rangle\, e^{-i\mathbf{b}\cdot\mathbf{p}}, \qquad (38)$$

where we used $(R^{-1}\mathbf{b})\cdot\mathbf{p}_0 = \mathbf{b}\cdot\mathbf{p}$, since $\mathbf{p}_0 = R^{-1}\mathbf{p}$.
(ii) Let $R \equiv R(\alpha,\beta,\gamma)$, $R_{\mathbf{p}} = R(\phi,\theta,0)$ and $R_{\mathbf{p}'} = R(\phi',\theta',0)$. We know $R\mathbf{p} = RR_{\mathbf{p}}\mathbf{p}_0 = \mathbf{p}'$. On the other hand, $\mathbf{p}' = R_{\mathbf{p}'}\mathbf{p}_0 = R_{\mathbf{p}'}R_3(\psi)\mathbf{p}_0$, for any value of ψ [Since $R_3(\psi)$ is in the little group of $\mathbf{p}_0$]. This angle can be found via:

$$RR_{\mathbf{p}} = R(\phi',\theta',\psi) \Rightarrow R_3(\psi) = R_{\mathbf{p}'}^{-1}RR_{\mathbf{p}}.$$

Then:

$$R\,|\mathbf{p}\lambda\rangle = R_{\mathbf{p}'}R_{\mathbf{p}'}^{-1}RR_{\mathbf{p}}\,|\mathbf{p}_0\lambda\rangle = R_{\mathbf{p}'}R_3(\psi)\,|\mathbf{p}_0\lambda\rangle = |\mathbf{p}'\lambda\rangle\, e^{-i\lambda\psi}. \qquad (39)$$

- The subspace thus created is invariant under E_3 and irreducible, as it is spanned by applying E_3 on $|\mathbf{p}_0\lambda\rangle$.
- The vectors are normalized according to:

$$\langle\mathbf{p}'|\mathbf{p}\rangle = 4\pi\delta(\cos\theta' - \cos\theta)\delta(\phi' - \phi), \qquad (40)$$

where the structure of delta functions is inspired by the invariant integration measure on $SO(3) = E_3/T_3$.

II.4.10. E_3: Angular momentum basis

- We seek basis vectors $|p,\lambda; jm\rangle \equiv |jm\rangle$, satisfying $(\mathbf{P}^2, \mathbf{J}\cdot\mathbf{p})\,|jm\rangle = |jm\rangle\,(p^2, \lambda p)$, as well as:

$$\mathbf{P}^2\,|jm\rangle = |jm\rangle\,p^2,\;\; \mathbf{J}\cdot\mathbf{p}\,|jm\rangle = |jm\rangle\,p\lambda,\;\; \mathbf{J}^2\,|jm\rangle = |jm\rangle\,j(j+1),$$
$$J_3\,|jm\rangle = |jm\rangle\,m,\;\; J_\pm\,|jm\rangle = |jm\pm 1\rangle\,\sqrt{j(j+1)-m(m\pm 1)}. \tag{41}$$

- The relation between $|jm\rangle$ and $|\mathbf{p}\rangle$ can be obtained using the projection method:

$$|jm\rangle = \int d\Omega_{\mathbf{p}}\,|\mathbf{p}\rangle\,D_j^\dagger(\phi,\theta,0)^\lambda{}_m\,\frac{2j+1}{4\pi},\quad |\mathbf{p}\rangle = \sum_{jm}|jm\rangle\,D_j(\phi,\theta,0)^m{}_\lambda. \tag{42}$$

- **Theorem:** The operators $R_{p\lambda}(\alpha,\beta,\gamma)$ and $T_{p\lambda}(\mathbf{b})$ on the subspace of the (p,λ) irrep satisfy:

$$\langle l'm'|R_{p\lambda}|lm\rangle = \delta_l^{l'}\,D_l(R)^{m'}{}_m, \tag{43a}$$
$$\langle l'm'|T_{p\lambda}(\mathbf{b})|lm\rangle = \sum_n \langle l'n|T_{p\lambda}(b\mathbf{k})|ln\rangle\,D_{l'}(\mathbf{b})^{m'}{}_n D_l^*(\mathbf{b})^m{}_n, \tag{43b}$$
$$\langle l'n|T_{p\lambda}(b\mathbf{k})|ln\rangle = \sum_L (2L+1)(-i)^L j_L(pb)\,\langle n0(l'L)ln\rangle\,\langle l\lambda(l'L)\lambda 0\rangle\,, \tag{43c}$$

where $j_l(z)$ is the spherical Bessel function of order l.

Proof: (i) Since $|lm\rangle$ is an irrep of $SO(3)$, we have $R\,|lm\rangle = |lm'\rangle\, D_l[R]^{m'}{}_{m}$, leading automatically to Eq. (43a).
(ii) Consider $R(\mathbf{b})$ such that $\mathbf{b} = R(\mathbf{b})(b\mathbf{k})$. Then, $T(\mathbf{b}) = R(\mathbf{b})T(b\mathbf{k})R^{-1}(\mathbf{b})$ and

$$\langle l'\, m' \,|\, T(\mathbf{b}) | lm\rangle = \sum_{n,n'} D_l(\mathbf{b})^{m'}{}_{n'}\, \langle l'\, n' \,|\, T(b\mathbf{k}) | ln\rangle\, D_l^*(\mathbf{b})^{m}{}_{n}. \tag{44}$$

Since $[J_z, P_z] = 0$, $\langle l'\, n' \,|\, T(b\mathbf{k}) | ln\rangle = \langle l'\, n | \, T(b\mathbf{k}) | ln\rangle\, \delta^{n'}{}_{n}$, thereby establishing Eq. (43b).
(iii) Applying $T(b\mathbf{k})$ on $|ln\rangle$ using Eq. (42) gives

$$\langle l'\, n |\, T(b\mathbf{k}) | ln\rangle = \frac{2l+1}{4\pi} \int d\Omega_{\mathbf{p}} D_{l'}(\hat{\mathbf{p}})^{n'}{}_{\lambda} D_l^{\dagger}(\hat{\mathbf{p}})^{\lambda}{}_{n} e^{-ipb\cos\theta_p}. \tag{45}$$

The exponential factor can be expanded w.r.t. the Legendre polynomials and spherical Bessel functions:

$$e^{-ipb\cos\theta_p} = \sum_{L=0}^{\infty} (2L+1)(-i)^L j_L(pb) P_L(\cos\theta_p). \tag{46}$$

Noting that $P_L(\cos\theta_p) = D_L(\hat{\mathbf{p}})^0{}_0$, and using

$$D_{l'}(\hat{\mathbf{p}})^{n'}{}_{\lambda} D_L(\hat{\mathbf{p}})^0{}_0 = \sum_J \langle n'0(l'L)Jn'\rangle\, D_J(\hat{\mathbf{p}})^{n'}{}_{\lambda}\, \langle J\lambda(l'L)\lambda 0\rangle\,, \tag{47}$$

as well as the orthogonality relation:

$$\frac{2J+1}{4\pi} \int d\Omega_{\mathbf{p}} D_J(\hat{\mathbf{p}})^{n'}{}_{\lambda} D_l^{\dagger}(\hat{\mathbf{p}})^{\lambda}{}_{n} = \delta^J_l\, \delta^{n'}_n\,, \tag{48}$$

we arrive at

$$\langle l'\, n' \,|\, T_{p\lambda}(b\mathbf{k}) | ln\rangle = \delta^{n'}_n \sum_L (2L+1)(-i)^L j_L(pb)\, \langle n0(l'L)ln\rangle\, \langle l\lambda(l'L)\lambda 0\rangle\,. \tag{49}$$

Exercises

1. Use Eq. (23) to show that $\langle \tilde{m}' | \tilde{m} \rangle = \delta_{mm'}$.
2. Evaluate $\langle m' | T(\mathbf{b}) | m \rangle$ using Eq. (25) to obtain the integral representation of the Bessel functions:

$$J_n(z) = \int_0^{2\pi} \frac{d\psi}{2\pi} e^{in\psi - iz \sin\psi}. \tag{50}$$

3. **Group contraction – $SO(3)$ to E_2.** Consider the mapping $(J_x/R, J_y/R, J_z) \to (-P_y, P_x, J_z)$ between the generators of $SO(3)$ and those of E_2.
   a) Compute the Lie algebra in the $R \to \infty$ limit and show that it coincides with Eq. (8).
   b) Consider the irreps of $SO(3)$ with basis vectors $|jm\rangle$. Show that $j = pR$, with p finite while $R \to \infty$, reproduces the Lie algebra of E_2 in Eq. (16).
   c) Consider the matrix element $d_j(\theta)^{m'}{}_m = \langle jm' | R_2(\theta) | jm \rangle$, with $\theta = b/R = bp/j$. Show that

$$\lim_{j \to \inf} d_j(\theta = z/j)^{-n}{}_0 = \langle -n | T(z, 0) | 0 \rangle_{p=1} = J_n(z). \tag{51}$$

Exercises

4. Using the definition of the Clebsch-Gordan coefficients, $\langle mm'(jj')JM\rangle = \langle jm; j'm'|JM\rangle$, use Eq. (43c) to show that

$$\langle 00|T_{p0}(b\mathbf{k})|l0\rangle = (-i)^l(2l+1)j_l(pb). \tag{52}$$

WKT9.5 Using group-theoretical methods, derive the recursion formulas for spherical Bessel functions:

$$\begin{aligned} \frac{2l+1}{x}j_l(x) &= j_{l-1}(x) + j_{l+1}(x), \\ \frac{d}{dx}j_l(x) &= \frac{1}{2l+1}[lj_{l-1}(x) - (l+1)j_{l+1}(x)]. \end{aligned} \tag{53}$$

Exercises

WKT9.4 Using the definition of the Clebsch-Gordan coefficients, $\langle mm'(jj')JM\rangle = \langle jm; j'm'|JM\rangle$, use Eqs. (43b) and (43c) to show that

$$\langle l'm'|T_{p\lambda}(\mathbf{b})|lm\rangle = (-1)^{m+m'}\sum_L (2L+1)(-i)^L j_L(pb) D_L(\mathbf{b})^{m'-m}{}_0 \times \langle m', m-m'(l'L)lm\rangle \langle l\lambda(l'L)\lambda 0\rangle\,. \quad (54)$$

Hint: Use the symmetry relations of the C-G coeffs:

$$\begin{aligned}\langle mm'(jj')JM\rangle &= (-1)^{j+j'-J}\langle m'm(j'j)JM\rangle \\ &= (-1)^{j+j'-J}\langle -m, -m'(jj')J, -M\rangle \\ &= (-1)^{j-J+m'}\langle M, -m'(Jj')jm\rangle\sqrt{\frac{2J+1}{2j+1}},\end{aligned} \quad (55)$$

the completeness and orthogonality of the C-G coeffs:

$$\begin{aligned}\sum_{mm'}\langle JM(jj')mm'\rangle\langle mm'(jj')J'M'\rangle &= \delta^J_{J'}\delta^M_{M'}, \\ \sum_{JM}\langle mm'(jj')JM\rangle\langle JM(jj')nn'\rangle &= \delta^m_n\delta^{m'}_{n'},\end{aligned} \quad (56)$$

the relation

$$\delta^J_{J'} D_J(R)^M{}_{M'} = \sum_{mm'nn'}\langle JM(jj')mm'\rangle D_j(R)^m_n D_{j'}(R)^{m'}_{n'}\langle nn'(jj')J'M'\rangle\,, \quad (57)$$

as well as $D_j^*(R)^m{}_n = (-1)^{m-n}D_j(R)^{-m}{}_{-n}$.