Symmetries in Physics
Lecture 7

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Lecture contents

Chapter 2. Continuous symmetry groups

Properties of the Clebsch-Gordan coefficients

- *Angular momentum selection rule:* $\langle mm'(jj')JM\rangle = 0$ unless $m + m' = M$ and $|j - j'| \le J \le j + j'$.
- *Orthogonality and completeness*:

$$\sum_{m,m'} \langle JM(jj')mm'\rangle \langle mm'(jj')J'M'\rangle = \delta^J_{J'}\delta^M_{M'}\,,$$

$$\sum_{JM} \langle mm'(jj')JM\rangle \langle JM(jj')nn'\rangle = \delta^m_n \delta^{m'}_{n'}\,. \qquad (1)$$

- Symmetry relations:

$$\begin{aligned}\langle mm'(jj')JM\rangle &= (-1)^{j+j'-J}\langle m'm(j'j)JM\rangle \\ &= (-1)^{j+j'-J}\langle -m,-m'(jj')J,-M\rangle \\ &= (-1)^{j-J+m'}\langle M,-m'(Jj')jm\rangle\sqrt{\frac{2J+1}{2j+1}}.\end{aligned} \qquad (2)$$

- Explicitly, the direct product and irreducible representations are related through:

$$D_j(R)^m{}_n D_{j'}(R)^{m'}{}_{n'} = \sum_{J,M,N} \langle mm'(jj')JM\rangle\, D_J(R)^M{}_N \,\langle JN(jj')nn'\rangle\,,$$

$$\delta^J{}_{J'} D_J(R)^M{}_{M'} = \sum_{m,m',n,n'} \langle JM(jj')(mm')\rangle\, D_j(R)^m{}_n D_{j'}(R)^{m'}{}_{n'} \,\langle nn'(jj')J'M'\rangle\,. \qquad (3)$$

II.2.12. Direct product decomposition using Young tableaux

- The decomposition into irreps of the direct product of $j = 1/2$ representations of $SO(3)$ [more specifically, of $SU(2)$] can be obtained using Young tableaux, using the following rules:
  - $\square$: $j = 1/2$ irrep (of dimension 2);
  - $\underbrace{\square\square\cdots\square}_{n \text{ times}}$: $j = n/2$ irrep (of dimension $2j+1 = n+1$);
  - $\begin{array}{c}\square\\\square\end{array} = 1$: $j = 0$ irrep (dimension 1);
  - $\underbrace{\square\square\cdots\square}_{n \text{ times}} \times \square = \underbrace{\square\square\cdots\square}_{n+1 \text{ times}} + \underbrace{\square\square\cdots\square}_{n-1 \text{ times}}$: product rule.
- Example:

$$\underbrace{\square \times \square}_{2\times2=4} = \underbrace{\square\square + 1}_{3+1=4}, \quad \underbrace{\square \times \square \times \square}_{2\times2\times2=8} = \underbrace{\square\square\square + 2 \times \square}_{4+2\times2=8},$$
$$\underbrace{\square \times \square \times \square \times \square}_{2\times2\times2\times2=16} = \underbrace{\square\square\square\square + 3 \times \square\square + 2}_{5+3\times3+2=16}. \qquad (4)$$

- The exact decomposition of the direct product basis into constitutive irreps must be performed explicitly, e.g. by explicit construction or using projector operators.

II.2.13. Irreducible tensors

- **Def:** A set of operators $\{O^s_\lambda, \lambda = -s, -s+1, \dots s\}$ form the components of an *irreducible spherical tensor* of angular momentum s if $U(R) O^s_\lambda U(R^{-1}) = \sum_{\lambda'} O^s_{\lambda'} D_s(R)^{\lambda'}{}_\lambda$.
- **Theorem:** The comps. of an irreducible sph. tensor satisfy:

$$[\mathbf{J}^2, O^s_\lambda] = s(s+1) O^s_\lambda, \qquad [J_3, O^s_\lambda] = \lambda O^s_\lambda,$$
$$[J_\pm, O^s_\lambda] = \sqrt{s(s+1) - \lambda(\lambda \pm 1)}\, O^s_{\lambda \pm 1}. \tag{5}$$

Proof: For infinitesimal $R_l(\delta\psi)$, $U_l O^s_\lambda U_l^{-1} \simeq O^s_\lambda - i\delta\psi [J_l, O^s_\lambda]$ and $D_s(R)^{\lambda'}{}_\lambda \delta^{\lambda'}_\lambda - i\delta\psi (J^s_l)^{\lambda'}{}_\lambda$, hence $[J_l, O^s_\lambda] = O^s_{\lambda'} (J^s_l)^{\lambda'}{}_\lambda$.

- **Def:** $\{A_l, l = 1, 2, 3\}$ are the *Cartesian components of a vector* if $[J_k, A_l] = i\varepsilon_{klm} A_m$.
- **Def:** $\{T_{l_1 \dots l_n}; l_i = 1, 2, 3\}$ are the components of an *n*-th rank tensor if $[J_k, T_{l_1 \dots l_n}] = i(\varepsilon_{kl_1m} T_{ml_2 \dots l_n} + \dots + \varepsilon^{kl_nm} T_{l_1 \dots l_{n-1} m})$.

II.2.14. Wigner-Eckart theorem

- According to the W-E theorem,

$$\langle j' m' | O^s_\lambda | jm \rangle = \langle j' m' (s,j) \lambda m \rangle \langle j' || O^s || j \rangle \,, \tag{6}$$

  where $\langle j' || O^s || j \rangle$ is independent of m, m' and λ.
- The *selection rules* imply that the matrix elements vanish unless $|j - s| \leq j' \leq j + s$ and $m' = \lambda + m$.
- The branching ratios are determined completely by C-G coefficients:

$$\frac{\langle j' m' | O^s_\lambda | jm \rangle}{\langle j' n' | O^s_\sigma | jn \rangle} = \frac{\langle j' m' (s,j) \lambda m \rangle}{\langle j' n' (s,j) \sigma n \rangle} . \tag{7}$$

Example: Dipole selection rules

- A particle in a weak external electromagnetic field is described by

$$H = \frac{\boldsymbol{\pi}^2}{2m} + V(r) + qU, \qquad \boldsymbol{\pi} = \mathbf{p} - q\mathbf{A}, \quad \mathbf{p} = -i\hbar\boldsymbol{\nabla}. \qquad (8)$$

- Taking $U = 0$ and the Coulomb gauge, $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$, we have

$$H \simeq H_0 + H_I, \quad H_I = -\frac{q}{m}\mathbf{A} \cdot \mathbf{p} = \frac{iq}{\hbar}\mathbf{A} \cdot [\mathbf{x}, H_0]. \qquad (9)$$

- Writing $\mathbf{A} = \mathbf{A}_0 e^{-i(\omega t - \mathbf{k}\cdot\mathbf{x})} \simeq \mathbf{A}_0 e^{-i\omega t}$ (when $\mathbf{k} \cdot \mathbf{r} \ll 1 \equiv$ dipole approximation, the transition matrix $T_{i\to f} = \langle f|H_I|i\rangle$ reduces to

$$\langle f|H_I|i\rangle = -\frac{q\omega_{fi}}{\omega}\mathbf{E} \cdot \langle f|\mathbf{x}|i\rangle\,, \quad \mathbf{E} = -\partial_t \mathbf{A} = i\omega\mathbf{A}, \quad \hbar\omega_{fi} = E_f - E_i.$$

- Taking now $|i\rangle = |nlm\rangle$ and $|f\rangle = |n'l'm'\rangle$, W-E gives

$$\langle n'l'm'|X^1_\lambda|nlm\rangle = \langle l'm'(1l)\lambda m\rangle\,\langle n'l'||X^1||nl\rangle\,. \qquad (10)$$

- If $\mathbf{E} = E\mathbf{k}$, then $\mathbf{E} \cdot \mathbf{x} = EX_0$ and $m' = m$.
- If $\mathbf{E}$ is in the $x - y$ plane, then $m' = m \pm 1$.
- From the CG series, $l' - l = 0$ or ± 1.

- Since $\mathbf{L} = \mathbf{x} \times \mathbf{p}$, $\mathbf{x} \cdot \mathbf{L} = 0$ and

$$\langle lm|\mathbf{x} \cdot \mathbf{L}|lm\rangle = \sum_{l',m'} \langle lm|\mathbf{x}|l'm'\rangle \cdot \langle l'm'|\mathbf{L}|lm\rangle = 0. \tag{11}$$

- Considering that $\mathbf{x} \cdot \mathbf{L} = \sum_{M=-1}^{1} (-1)^M X_M^1 L_{-M}^1$, with

$$X_{-1}^1 = \frac{x - iy}{\sqrt{2}}, \quad X_0^1 = z, \quad X_1^1 = -\frac{x + iy}{\sqrt{2}}, \tag{12}$$

and similarly for L_λ^1, we have

$$0 = \sum_{l',m',M} (-1)^M \langle lm(1l')Mm'\rangle \langle l'm'(1l) - M, m\rangle \langle l||X^1||l'\rangle \langle l'||L^1||l\rangle . \tag{13}$$

- Since $\langle l'm'|L_\lambda^1|lm\rangle \simeq \delta_{ll'}\delta_{m',m+\lambda}$, we have

$$0 = \sum_{M=-1}^{1} (-1)^M \langle lm(1l)M, m + M\rangle \langle l, m + M(1l) - M, m\rangle \times \langle l||X^1||l\rangle \langle l||L^1||l\rangle . \tag{14}$$

- The sum over the CG coefficients is not zero and $\langle l||L^1||l\rangle \neq 0 \Rightarrow \langle l||X^1||l\rangle = 0$, leading to ($\mathbf{E} = E\mathbf{k}$):

$$\langle n'l'm'|H_I|nlm\rangle = -\frac{qE\omega_{fi}}{\omega}[\delta_{l',l+1} \langle l + 1m(1l)0m\rangle \langle n'l + 1||X^1||nl\rangle + \delta_{l',l-1} \langle l - 1m(1l)0m\rangle \langle n'l - 1||X^1||nl\rangle].$$

II.3. The group $SU(2)$

II.3.1. Parametrization of $SU(2)$ matrices

- $SU(2)$ is the group of complex, unitary matrices of size 2.
- **Theorem:** An arbitrary 2×2 unitary matrix can be parametrized as

$$U = e^{i\lambda} \begin{pmatrix} \cos\theta e^{i\zeta} & -\sin\theta e^{i\eta} \\ \sin\theta e^{-i\eta} & \cos\theta e^{-i\zeta} \end{pmatrix}, \qquad \begin{array}{l} 0 \leq \zeta, \eta < 2\pi, \\ 0 \leq \lambda < \pi, \\ 0 \leq \theta \leq \pi/2. \end{array} \tag{15}$$

  Proof: Can be verified by explicit construction. Additionally, a 2×2 complex matrix has a total of 8 real dofs, 4 of which are fixed by $U^\dagger U = E$. The remaining 4 parametrize Eq. (15) and their ranges is established as follows:

  - ζ and η cover the entire circle;
  - θ covers the first quadrant, as the other quadrants can be obtained by changing the sign of either $\cos\theta$ or $\sin\theta$ by suitable choices of the phases ζ and η;
  - The overall phase λ covers the upper two quadrants, since $e^{i\pi} = -1$ can be absorbed in the phases $e^{i\zeta}$ and $e^{i\eta}$.
- Imposing now $\det U = 1$ gives $\lambda = 0 \Rightarrow$ an arbitrary $SU(2)$ matrix A can be parametrized using 3 real parameters: θ, η and ζ.

II.3.2. Relationship to $SO(3)$

- The $SU(2)$ matrices and the $j = 1/2$ representation of $SO(3)$ matrices,

$$A = \begin{pmatrix} \cos\theta e^{i\zeta} & -\sin\theta e^{i\eta} \\ \sin\theta e^{-i\eta} & \cos\theta e^{-i\zeta} \end{pmatrix}, \; D^{1/2}(\alpha,\beta,\gamma) = \begin{pmatrix} e^{-\frac{i}{2}(\alpha+\gamma)}\cos\frac{\beta}{2} & -e^{-\frac{i}{2}(\alpha-\gamma)}\sin\frac{\beta}{2} \\ e^{\frac{i}{2}(\alpha-\gamma)}\sin\frac{\beta}{2} & e^{\frac{i}{2}(\alpha+\gamma)}\cos\frac{\beta}{2} \end{pmatrix}, \tag{16}$$

  are identical provided

$$\theta = \frac{\beta}{2}, \quad \zeta = -\frac{\alpha+\gamma}{2}, \quad \eta = -\frac{\alpha-\gamma}{2}. \tag{17}$$

- Clearly, $0 \le \beta < \pi$.
- To fully cover $SU(2)$, $0 \le \zeta, \eta < 2\pi$, $0 \le \alpha < 2\pi$ and $0 \le \gamma < 4\pi$.
- Since γ covers the circle twice, the $SU(2)$ matrices form a double-valued rep. of $SO(3)$.
- Any $SU(2)$ matrix can be represented in the angle-axis p.:

$$A_{\mathbf{n}}(\psi) = e^{-\frac{i}{2}\psi \mathbf{n}\cdot\boldsymbol{\sigma}} = \begin{pmatrix} \cos\frac{\psi}{2} - i\sin\frac{\psi}{2}\cos\theta & -i\sin\frac{\psi}{2}\sin\theta e^{-i\varphi} \\ -i\sin\frac{\psi}{2}\sin\theta e^{i\varphi} & \cos\frac{\psi}{2} + i\sin\frac{\psi}{2}\cos\theta \end{pmatrix}.$$

- Using Euler angles, $A(\alpha,\beta,\gamma) = e^{-\frac{i}{2}\alpha\sigma_3} e^{-\frac{i}{2}\beta\sigma_2} e^{-\frac{i}{2}\gamma\sigma_3}$, where

$$A_3(\psi) = \begin{pmatrix} e^{-\frac{i}{2}\psi} & 0 \\ 0 & e^{\frac{i}{2}\psi} \end{pmatrix}, \qquad A_2(\psi) \equiv d_{1/2}(\psi) = \begin{pmatrix} \cos\frac{\psi}{2} & -\sin\frac{\psi}{2} \\ \sin\frac{\psi}{2} & \cos\frac{\psi}{2} \end{pmatrix}.$$

II.3.3 Invariant integration measure

- Integrations over the group elements must be compatible with the rearrangement lemma:

$$\int f(A)d\tau_A = \int f(B^{-1}A')d\tau_{B^{-1}A'} = \int f(B^{-1}A')d\tau_{A'}, \qquad (18)$$

  with $A' = BA$.

- Using generic $\{\xi, \eta, \zeta\}$ to parametrize de group elements, we seek a weight function ρ such that

$$d\tau_A = \rho(\xi, \eta, \zeta)d\xi d\eta d\zeta = \rho(\xi', \eta', \zeta')d\xi' d\eta' d\zeta', \qquad (19)$$

  where $\{\xi', \eta', \zeta'\}$ corresponds to $A' = BA$, and

$$\frac{\rho(\xi, \eta, \zeta)}{\rho(\xi', \eta', \zeta')} = \frac{\partial(\xi', \eta', \zeta')}{\partial(\xi, \eta, \zeta)}. \qquad (20)$$

- The above equation becomes easy when (ξ', η', ζ') are linear functions of (ξ, η, ζ).
- The overall normalization is chosen such that $\int d\tau_A = 1$.

- Consider the parametrization of an $SU(2)$ matrix via

$$A = \begin{pmatrix} r_0 - ir_3 & -r_2 - ir_1 \\ r_2 - ir_1 & r_0 + ir_3 \end{pmatrix}, \quad \det A = r_0^2 + \mathbf{r}^2 = 1. \qquad (21)$$

- The real parameters $r_0, \dots r_3$ are connected to θ, ζ, η and an extra parameter r via

$$\begin{pmatrix} r_0 \\ r_3 \end{pmatrix} = r\cos\theta \begin{pmatrix} \cos\zeta \\ -\sin\zeta \end{pmatrix}, \qquad \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = r\sin\theta \begin{pmatrix} \sin\eta \\ -\cos\eta \end{pmatrix}, \qquad (22)$$

such that $\det A = r^2 = 1$.

- Consider now r_i and r_i', parametrizing A and A', and s_i, parametrizing B, such that $A' = BA$. The Jacobian of the transformation is:

$$\frac{\partial(r_0', r_1', r_2', r_3')}{\partial(r_0, r_1, r_2, r_3)} = \begin{vmatrix} s_0 & -s_1 & -s_2 & -s_3 \\ s_1 & s_0 & -s_3 & s_2 \\ s_2 & s_3 & s_0 & -s_1 \\ s_3 & -s_2 & s_1 & s_0 \end{vmatrix} = (s_0^2 + \mathbf{s}^2)^2 = 1, \qquad (23)$$

which means $dr_0 dr_1 dr_2 dr_3 = dr_0' dr_1' dr_2' dr_3'$.

- Imposing $\det A = 1$, the integration measure can be taken as

$$V_G d\tau_A = d^4 r\, \delta(1 - r_0^2 - \mathbf{r}^2) = \frac{d^4 r}{2\sqrt{1-\mathbf{r}^2}} \sum_{\varsigma=\pm 1} \delta(r_0 - \varsigma\sqrt{1-\mathbf{r}^2}), \quad (24)$$

where $d^4 r = dr_0 dr_1 dr_2 dr_3$, while V_G is a normalization constant:

$$V_G = \int d^4 r \delta(1 - r_0^2 - \mathbf{r}^2) = 4\pi \int_0^1 \frac{dr\, r^2}{\sqrt{1-r^2}} = \pi^2. \quad (25)$$

- With respect to (r, θ, ζ, η), we have

$$d\tau_A = \frac{\delta(1-r^2)}{\pi^2} \frac{\partial(r_0, r_1, r_2, r_3)}{\partial(r, \theta, \zeta, \eta)} dr d\theta d\zeta d\eta \to \frac{\sin(2\theta)}{4\pi^2} d\theta d\zeta d\eta. \quad (26)$$

- W.r.t. the Euler angle and angle-axis parametrizations,

$$SU(2): \quad d\tau_A = \frac{d\alpha d\cos\beta d\gamma}{16\pi^2} = \sin^2\frac{\psi}{2} \frac{d\psi d\cos\theta d\phi}{4\pi^2},$$
$$SO(3): \quad d\tau_R = \frac{d\alpha d\cos\beta d\gamma}{8\pi^2} = \sin^2\frac{\psi}{2} \frac{d\psi d\cos\theta d\phi}{2\pi^2}, \quad (27)$$

where $0 \le \gamma < 4\pi$ for $SU(2)$ and $0 \le \gamma < 2\pi$ for $SO(3)$; For angle-axis, we use $0 \le \phi < 2\pi$, $0 \le \theta \le \pi$ and $0 \le \psi < 2\pi$ for $SU(2)$ and $0 \le \psi < \pi$ for $SO(3)$.

II.3.4. Orthonormality of the representation matrices

- In the case of discrete groups, $\frac{n_\mu}{n_G}\sum_g D^\dagger_\mu(g)^k{}_i D_\nu(g)^j{}_l = \delta_{\mu\nu}\delta^k_l\delta^j_i$.
- For $SU(2)$, irrep j has size $n_\mu \to n_j = 2j+1$ and

$$(2j+1)\int d\tau_A D^\dagger_j(A)^m{}_n D_{j'}(A)^{n'}{}_{m'} = \delta_{jj'}\delta^m{}_{m'}\delta^{n'}{}_n. \qquad (28)$$

- Using the Euler angle parametrization, the α and γ integrations can be performed, giving (no summation over n, m):

$$\frac{2j+1}{2}\int_{-1}^{1} d(\cos\beta) d_j(\beta)^n{}_m d_{j'}(\beta)^n{}_m = \delta_{jj'}. \qquad (29)$$

- Multiplying Eq. (28) by $\delta^n_m\delta^{m'}_{n'}$ gives

$$\int d\tau_A \chi^\dagger_j(A)\chi_{j'}(A) = \delta_{jj'}. \qquad (30)$$

- Using $\chi_j(A) = \sin[(j+\frac{1}{2})\psi]/\sin\frac{\psi}{2}$, we have

$$\int d\tau_A \chi^\dagger_j(A)\chi_{j'}(A) = \int_0^{2\pi}\frac{d\psi}{2\pi}\left(\cos[(j-j')\psi] - \cos[(j+j'+1)\psi]\right) = \delta_{jj'}.$$

II.3.5. Completeness: Peter-Weyl theorem

- **Theorem (Peter-Weyl):** The irrep functions $D_j(A)^m{}_n$ form a complete basis in the space of (Lebesgue) square-integrable functions defined on the group manifold.
- Let $f(A)$ be such a function, then:

$$f(A) = \sum_{jmn} f^n_{jm} D_j(A)^m{}_n, \quad f^n_{jm} = (2j+1) \int d\tau_A D^\dagger_j(A)^n{}_m f(A), \qquad (31)$$

which implies

$$\sum_{jmn} (2j+1) D_j(A)^m{}_n D^\dagger_j(A')^n{}_m = \delta(A - A'), \qquad (32)$$

where $\delta(A - A') = 16\pi^2 \delta(\alpha - \alpha')\delta(\cos\beta - \cos\beta')\delta(\gamma - \gamma')$ in the Euler angle parametrization.
- Bose-Einstein: $f(\alpha, \beta, \gamma + 2\pi) = f(\alpha, \beta, \gamma)$ and $f^n_{jm} = 0$ when $j = l + \frac{1}{2}$.
- Fermi-Dirac: $f(\alpha, \beta, \gamma + 2\pi) = -f(\alpha, \beta, \gamma)$ and $f^n_{jm} = 0$ when $j = l$.
- In the case when $f(\alpha, \beta, \gamma)$ is independent of γ, denoting $(\alpha, \beta) \to (\phi, \theta)$, we write

$$f(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_{lm} Y_{lm}(\theta, \phi), \quad Y_{lm}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}} D^*_l(\phi, \theta, 0)^m{}_0,$$

and $f_{lm} = \int d\cos\theta d\phi Y^*_{lm}(\theta, \phi) f(\theta, \phi)$.

II.3.6. Projection operators

- The invariant integration measure allows the projection operators to be constructed:

$$P^{n}_{jm} = (2j+1) \int d\tau_A D^{\dagger}_{j}(A)^{n}{}_{m} U(A), \tag{33}$$

by which $P^{n}_{jm} \, |j' m'\rangle = |j' m'\rangle \, \delta^{j'}_{j} \, \delta^{n}_{m'}$.

II.3.7. Differential equation for D_j

- For $R(\alpha, \beta, \gamma) = e^{-i\alpha J_3} e^{-i\beta J_2} e^{-i\gamma J_3}$, we have

$$i\frac{\partial R}{\partial \alpha} = R[R^{-1} J_3 R], \quad i\frac{\partial R}{\partial \beta} = R[e^{i\gamma J_3} J_2 e^{-i\gamma J_3}], \quad i\frac{\partial R}{\partial \gamma} = RJ_3. \quad (34)$$

- Using $R^{-1} J_i R = R_i{}^j J_j$, the square brackets evaluate to

$$R^{-1} J_3 R = -\frac{\sin\beta}{2}(J_+ e^{i\gamma} + J_- e^{-i\gamma}) + J_3 \cos\beta,$$
$$e^{i\gamma J_3} J_2 e^{-i\gamma J_3} = \frac{i}{2}(-J_+ e^{i\gamma} + J_- e^{-i\gamma}). \quad (35)$$

- Plugging Eq. (35) into Eq. (34) gives

$$e^{-i\gamma}\left[-\frac{\partial}{\partial\beta} - \frac{i}{\sin\beta}\left(\frac{\partial}{\partial\alpha} - \cos\beta\frac{\partial}{\partial\gamma}\right)\right] R = RJ_+,$$
$$e^{i\gamma}\left[\frac{\partial}{\partial\beta} - \frac{i}{\sin\beta}\left(\frac{\partial}{\partial\alpha} - \cos\beta\frac{\partial}{\partial\gamma}\right)\right] R = RJ_-, \quad i\frac{\partial R}{\partial \gamma} = RJ_3. \quad (36)$$

- Evaluating $\langle jm'|RJ_\pm|jm\rangle$ gives

$$\left(-\frac{d}{d\beta}-\frac{m'-m\cos\beta}{\sin\beta}\right)d_j(\beta)^{m'}{}_m = d_j(\beta)^{m'}{}_{m+1}\sqrt{j(j+1)-m(m+1)},$$
$$\left(\frac{d}{d\beta}-\frac{m'-m\cos\beta}{\sin\beta}\right)d_j(\beta)^{m'}{}_m = d_j(\beta)^{m'}{}_{m-1}\sqrt{j(j+1)-m(m-1)}. \tag{37}$$

- The above relations can be used as recurrence relations: Knowing $d_j(\beta)^{m'}{}_m$ for a combination of m and m', it is possible to obtain it for all other m.
- At the same time, a differential eq. can be obtained for $d_j(\beta)^{m'}{}_m$ by applying $\mathbf{J}^2 = J_3^2 - J_3 + J_+J_-$, since:

$$R\mathbf{J}^2 = \left\{e^{-i\gamma}\left[-\frac{d}{d\beta}-\frac{i}{\sin\beta}\left(\frac{\partial}{\partial\alpha}-\cos\beta\frac{\partial}{\partial\gamma}\right)\right]\right.$$
$$\left.\times e^{i\gamma}\left[\frac{\partial}{\partial\beta}-\frac{i}{\sin\beta}\left(\frac{\partial}{\partial\alpha}-\cos\beta\frac{\partial}{\partial\gamma}\right)\right]-\frac{\partial^2}{\partial\gamma^2}-i\frac{\partial}{\partial\gamma}\right\}R. \tag{38}$$

- Using $\langle jm'|R\mathbf{J}^2|jm\rangle = j(j+1)e^{-i\alpha m'-i\gamma m}d_j(\beta)^{m'}{}_m$ leads to

$$\left(\frac{1}{\sin\beta}\frac{d}{d\beta}\sin\beta\frac{d}{d\beta}-\frac{m^2+m'^2-2mm'\cos\beta}{\sin^2\beta}+j(j+1)\right)d_j(\beta)^{m'}{}_m = 0. \tag{39}$$

II.3.8. Relation to spherical harmonics

- When $m = 0$, setting $(m', j, \beta) \to (m, l, \theta)$ and restoring $D_l(\phi, \theta, 0)^m{}_0 = e^{-im\phi} d_l(\theta)^m{}_0$ gives

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + l(l+1)\right] D_l(\phi,\theta,0)^m{}_0 = 0, \tag{40}$$

which is solved by the spherical harmonics:

$$[D_l(\phi,\theta,0)^m{}_0]^* = \sqrt{\frac{4\pi}{2l+1}}\, Y_{lm}(\theta,\phi), \quad d_l(\theta)^m{}_0 = \sqrt{\frac{(l-m)!}{(l+m)!}}(-1)^m P_{lm}(\theta). \tag{41}$$

- For general (m, m'), the differential eq. can be put in the form for the Jacobi polynomials,

$$\left\{(1-z^2)\frac{d^2}{dz^2} + [\beta - \alpha - (2+\alpha+\beta)z]\frac{d}{dz} + l(l+\alpha+\beta+1)\right\} P_l^{(\alpha,\beta)}(z) = 0, \tag{42}$$

such that

$$d_j(\beta)^{m'}{}_m = \sqrt{\frac{(j+m')!(j-m')!}{(j+m)!(j-m)!}} \left(\cos\frac{\beta}{2}\right)^{m+m'} \left(\sin\frac{\beta}{2}\right)^{m-m'} P_{j-m'}^{m'-m,m'+m}(\cos\beta). \tag{43}$$

- This reproduces the $m = 0$ result since

$$P_{l-m}^{m,m}(z) = (-2)^m \frac{l!}{(l-m)!}(1-z^2)^{-m/2} P_{lm}(z), \tag{44}$$

as well as $P_l(\cos\theta) = P_{l0}(\cos\theta) = P_l^{0,0}(\cos\theta) = d_l(\theta)^0{}_0$.

II.3.9. Properties of spherical harmonics

- **Transformation under rotations:** Consider $|\xi,\psi\rangle = U(\alpha,\beta,\gamma)\,|\theta,\phi\rangle$. Then:
$$Y_{lm}(\theta,\phi) = \langle\xi,\psi|U(\alpha,\beta,\gamma)|lm\rangle = Y_{lm'}(\xi,\psi)D_l(\alpha,\beta,\gamma)^{m'}{}_{m}. \qquad (45)$$
- **Addition theorem:** Taking $m=0$ and using $Y_{l0}(\theta,\phi) = P_l(\cos\theta)\sqrt{(2l+1)/4\pi}$, such that
$$\sum_{m'} Y_{lm'}(\xi,\psi)Y_{lm'}(\beta,\alpha) = \frac{2l+1}{4\pi}P_l(\cos\theta), \qquad (46)$$
where $\cos\theta = \mathbf{n}(\xi,\psi)\cdot\mathbf{n}(\beta,\alpha)$.
- **Decomposition** of products of Y_{lm} with the same arguments:
$$Y_{lm}(\theta,\phi)Y_{l'm'}(\theta,\phi) = \sum_L \langle mm'(ll')Lm+m'\rangle\, Y_{L,m+m'}(\theta,\phi) \times \langle L0(ll')00\rangle \sqrt{\frac{(2l+1)(2l'+1)}{4\pi(2L+1)}}. \qquad (47)$$
- **Symmetry in** m**:** $Y_{l,-m}(\theta,\phi) = (-1)^m Y^*_{lm}(\theta,\phi)$.

- Eqs. (37) provide **recurrence relations at fixed l:**

$$\sqrt{l(l+1)-m(m+1)}\,Y_{l,m+1}(\theta,\phi)=e^{i\phi}\left(\frac{d}{d\theta}-m\cot\theta\right)Y_{lm}(\theta,\phi),$$
$$\sqrt{l(l+1)-m(m-1)}\,Y_{l,m-1}(\theta,\phi)=e^{-i\phi}\left(-\frac{d}{d\theta}-m\cot\theta\right)Y_{lm}(\theta,\phi). \quad (48)$$

- **Recurrence relations at fixed m** can be obtained by multiplying $Y_{lm}Y_{10}$, with $Y_{10}(\theta,\phi)=\cos\theta\sqrt{3/4\pi}$:

$$\cos\theta\, Y_{lm}(\theta,\phi)=\sqrt{\frac{(l+m)(l-m)}{(2l-1)(2l+1)}}\,Y_{l-1,m}(\theta,\phi)$$
$$+\sqrt{\frac{(l+m+1)(l-m+1)}{(2l+1)(2l+3)}}\,Y_{l+1,m}(\theta,\phi), \quad (49)$$

where we used:

$$\langle m0(l1)l+1m\rangle=\sqrt{\frac{(l-m+1)(l+m+1)}{(2l+1)(l+1)}},\quad \langle m0(l1)lm\rangle=\frac{m}{\sqrt{l(l+1)}},$$
$$\langle m0(l1)l-1m\rangle=-\sqrt{\frac{(l-m)(l+m)}{l(2l+1)}}. \quad (50)$$

- **Orthogonality:** $\int d\Omega\, Y^*_{lm}(\theta,\phi)Y_{l'm'}(\theta,\phi)=\delta_{ll'}\delta_{mm'}$.
- **Completeness:** $\sum_{l,m}Y_{lm}(\theta,\phi)Y^*_{lm}(\theta',\varphi')=\delta(\cos\theta-\cos\theta')\delta(\phi-\phi')$.

Exercises

1. Starting from the commutation relations $[J_i, X_j] = i\varepsilon_{ijk}X_k$, show that the components of the irreducible vector operator X^1_λ are:

$$X^1_0 = X_3, \qquad X^1_{\pm 1} = \frac{1}{\sqrt{2}}(\mp X_1 - iX_2).$$

WKT7.9 If $\{T_{ij}, i, j = 1, 2, 3\}$ are the components of a second-rank tensor, show that:
(i) $\mathrm{tr}\, T = \delta^{ij} T_{ij}$ is invariant under $SO(3)$;
(ii) $\hat{T}_{ij} = (T_{ij} - T_{ji})/2$ remains antisymmetric after an $SO(3)$ transformation, and $\hat{T}_k = \frac{1}{2}\varepsilon_{kij} T_{ij}$ transforms as a vector;
(iii) $\widetilde{T}_{ij} = (T_{ij} + T_{ji})/2 - \frac{1}{3}\delta_{ij}\mathrm{tr}\, T$ remains symmetric under $SO(3)$ and the 5 independent components of $\widetilde{T}$ transform with the $j = 2$ representation.

2. Derive the invariant integration measure for $SU(2)$ with respect to the parameters θ, ζ, η, shown in Eq. (26). [Hint: multiply A in Eq. (16) by a factor r and impose at the end $r = 1$ via a Dirac delta function]

Exercises

3. Derive the invariant integration measure with respect to the angle-axis parameters, using the convention $0 \le \phi < 2\pi$, $0 \le \theta \le \pi/2$ and $0 \le \psi < 4\pi$ for $SU(2)$. Use the explicit relation between the angle-axis parameters and the Euler angles:

$$\phi = \frac{\pi + \alpha - \gamma}{2}, \quad \tan\theta = \frac{\tan(\beta/2)}{\sin[(\gamma+\alpha)/2]}, \quad \cos\frac{\psi}{2} = \cos\left(\frac{\beta}{2}\right)\cos\left(\frac{\alpha+\gamma}{2}\right).$$

4. **Completeness of characters.**
a) Use Schur's lemma to show that $A^j_\psi = \int \frac{d\Omega_{\mathbf{n}}}{4\pi} D_j[R_{\mathbf{n}}(\psi)] = c^j_\psi E$, where E is the identity matrix.
b) Using the angle-axis parametrization of the $SU(2)$ matrices A and A', integrate Eq. (32) with respect to $d\Omega_{\mathbf{n}'} = d(\cos\theta')d\phi'$ to show that:

$$\sin^2\frac{\psi}{2}\sum_j \chi_j(\psi)\chi_j(\psi') = \pi\delta(\psi - \psi'). \tag{51}$$

c) Therefore deduce that $\sum_{j=0}^{\infty}\frac{1}{2j+1}\sin[\psi(j+\frac{1}{2})] = \frac{\pi}{4}$.

Exercises

5. **Invariant integration measure.** Consider the parametrization (θ, ζ, η) of the $SU(2)$ matrices in Eq. (16).
   a) Calculate $\partial A/\partial_{\xi_i}$, with $\xi_i = \{\theta, \zeta, \eta\}$.
   b) Express the product $A^{-1}(\xi)(\partial A/\partial \xi_i)$ with respect to the generators J_α, using the matrix $\widetilde{A}$ defined below:
   $$A^{-1}\frac{\partial A}{\partial \xi_i} = \sum_\alpha J_\alpha \widetilde{A}(\xi)^\alpha{}_i. \tag{52}$$
   c) Compute $d\tau_A = \rho_A(\xi)\prod_i d\xi_i$, with $\rho_A(\xi) = \det \widetilde{A}(\xi)^\alpha{}_i$.
   d) Repeat the above for the angle-axis parametrization.
6. **Reduced representation matrix $d_j(\beta)$.** Consider the direct product of n irreducible representations with $j = 1/2$. Of these, consider the basis $|jm\rangle$ corresponding to the $j = n/2$ (maximal) irrep.
   a) Show that $|jj\rangle = |\underbrace{+ + \cdots +}_{n \text{ elements}}\rangle$.
   b) By repeated application of J_-, show that
   $$|jm\rangle = \sqrt{\frac{(j-m)!(j+m)!}{(2j)!}}\,(\underbrace{|\overbrace{+ + \cdots +}^{j+m \text{ elements}}\overbrace{- - \cdots -}^{j-m \text{ el.}}\rangle + \cdots + |\overbrace{- - \cdots -}^{j-m \text{ el.}}\overbrace{+ + \cdots +}^{j+m \text{ el.}}\rangle}_{(2j)!/[(j+m)!(j-m)!] \text{ terms}}).$$
   c) Show that $U_2(\beta)\,|\pm\rangle = |\pm\rangle\, c \pm |\mp\rangle\, s$, where $c = \cos(\beta/2)$ and $s = \sin(\beta/2)$.
   d) Taking into account that $|jm\rangle$ and $U_2(\beta)\,|jm\rangle$ are totally symmetric, show that
   $$\langle jm'|U_2(\beta)|jm\rangle = \sum_{k=0}^{j-m}(-1)^k \frac{\sqrt{(j+m)!(j-m)!(j+m')!(j-m')!}}{k!(j+m'-k)!(m-m'+k)!(j-m-k)!} \times c^{2j-2k-m+m'} s^{2k+m-m'}. \tag{53}$$