Symmetries in Physics
Lecture 6

Victor E. Ambruș

Universitatea de Vest din Timișoara

Lecture contents

Chapter 2. Continuous symmetry groups

II.2.5. Irreps of the $SO(3)$ Lie algebra

- **Theorem:** The *irreducible representations of the Lie algebra of* $SO(3)$ are each characterized by an angular momentum eigenvalue j from the set of non-negative integers and half-integers. The orthonormal basis vectors can be specified by the following equations:

$$\mathbf{J}^2 |jm\rangle = |jm\rangle\, j(j+1), \qquad J_3 |jm\rangle = |jm\rangle\, m,$$
$$J_\pm |jm\rangle = |jm \pm 1\rangle\, [j(j+1) - m(m \pm 1)]^{1/2}. \tag{1}$$

Proof: The first two relations were already proved. The third relation can be proved by writing $J_\pm |jm\rangle = N_m^\pm |jm \pm 1\rangle$, with the convention $\langle jm|j'm'\rangle = \delta_{jj'}\delta_{mm'}$ and evaluating:

$$|N_m^-|^2 = \langle jm|J_+J_-|jm\rangle = \langle jm|\mathbf{J}^2 - J_3^2 + J_3|jm\rangle = j(j+1) - m(m-1),$$
$$|N_m^+|^2 = \langle jm|J_-J_+|jm\rangle = \langle jm|\mathbf{J}^2 - J_3^2 - J_3|jm\rangle = j(j+1) - m(m+1).$$

The phase convention $N_m^\pm = [j(j+1) - m(m \pm 1)]^{1/2}$ is called the *Condon-Shortley convention*.

II.2.6. Irreps of the $SO(3)$ group

- Since J_k only modify m, the angular momentum number j identifies the inequivalent irreps of $SO(3)$.
- The action of a group element $R(\alpha, \beta\gamma)$ is

$$U(\alpha,\beta,\gamma)\,|jm\rangle = |jm'\rangle\, D_j(\alpha,\beta,\gamma)^{m'}{}_m. \quad (2)$$

- Since $R(\alpha,\beta,\gamma) = R_3(\alpha)R_3(\beta)R_3(\gamma)$ and keeping in mind that $R_3(\psi)\,|jm\rangle = |jm\rangle\, e^{-i\psi m}$, we have

$$\begin{aligned} U(\alpha,\beta,\gamma)\,|jm\rangle &= U_3(\alpha)U_2(\beta)\,|jm\rangle\, e^{-i\gamma m} = U_3(\alpha)\,|jm'\rangle\, d_j(\beta)^{m'}{}_m e^{-i\gamma m} \\ &= |jm'\rangle\, e^{-i\alpha m'} d_j(\beta)^{m'}{}_m e^{-i\gamma m}, \end{aligned} \quad (3)$$

  with $d_j(\beta)^{m'}{}_m = \langle jm'|e^{-i\beta J_2}|jm\rangle$.
- In the C-S convention, $iJ_2 = \frac{1}{2}(J_+ - J_-)$ is a real matrix and $d_j(\beta)^{m'}{}_m$ is a real, orthogonal matrix:

$$d_j^\dagger(\beta) = d_j^T(\beta) = d_j^{-1}(\beta). \quad (4)$$

Example: $j = 1/2$

- When $j = 1/2$, the invariant subspace has only two states: $|\pm 1/2\rangle$.
- The representation matrices are the Pauli matrices, $J_i = \frac{1}{2}\sigma_i$, such that

$$J_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad J_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{5}$$

- The reduced representation matrix can be obtained using $\sigma_2^2 = 1$:

$$d_{1/2}(\beta) = e^{-\frac{i}{2}\beta\sigma_2} = E\cos\frac{\beta}{2} - i\sigma_2 \sin\frac{\beta}{2} = \begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}. \tag{6}$$

- The full representation matrix reads

$$D_{1/2}(\alpha,\beta,\gamma) = \begin{pmatrix} e^{-\frac{i}{2}(\alpha+\gamma)}\cos\frac{\beta}{2} & -e^{-\frac{i}{2}(\alpha-\gamma)}\sin\frac{\beta}{2} \\ e^{\frac{i}{2}(\alpha-\gamma)}\sin\frac{\beta}{2} & e^{\frac{i}{2}(\alpha+\gamma)}\cos\frac{\beta}{2} \end{pmatrix}. \tag{7}$$

- Within the $j = 1/2$ representation, the 2π rotation gives

$$D[R_{\mathbf{n}}(2\pi)] = D[R]e^{-\frac{i}{2}2\pi\sigma_2}D[R]^{-1} = -E, \tag{8}$$

hence $D[R_{\mathbf{n}}(4\pi)] = E$ and the representation is double-valued.

Example: $j = 1$

- The generators for $j = 1$ have the following matrix representations:

$$J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad J_+ = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}, \quad J_- = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}. \tag{9}$$

- The reduced representation matrix can be obtained as

$$\begin{aligned} d_1(\beta) &= 1 - J_2^2(1-\cos\beta) - iJ_2\sin\beta \\ &= \begin{pmatrix} \frac{1}{2}(1+\cos\beta) & -\frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1-\cos\beta) \\ \frac{1}{\sqrt{2}}\sin\beta & \cos\beta & -\frac{1}{\sqrt{2}}\sin\beta \\ \frac{1}{2}(1-\cos\beta) & \frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1+\cos\beta) \end{pmatrix}. \end{aligned} \tag{10}$$

- The full representation matrix reads

$$D_1(\alpha,\beta,\gamma) = \begin{pmatrix} \frac{e^{-i(\alpha+\gamma)}}{2}(1+\cos\beta) & -\frac{e^{-i\alpha}}{\sqrt{2}}\sin\beta & \frac{e^{-i(\alpha-\gamma)}}{2}(1-\cos\beta) \\ \frac{e^{-i\gamma}}{\sqrt{2}}\sin\beta & \cos\beta & -\frac{e^{i\gamma}}{\sqrt{2}}\sin\beta \\ \frac{e^{i(\alpha-\gamma)}}{2}(1-\cos\beta) & \frac{e^{i\alpha}}{\sqrt{2}}\sin\beta & \frac{e^{i(\alpha+\gamma)}}{2}(1+\cos\beta) \end{pmatrix}. \tag{11}$$

- **Theorem:** The irreps of the $so(3)$ Lie algebra give rise to $SO(3)$ irreps belonging to 2 categories: (i) for j a non-negative integer: single-valued reps.; (ii) for j an odd half-integer: double-valued reps.
  Proof: Consider the rep. matrix for $R_3(2\pi)$:

$$\begin{aligned} D_j[R_3(2\pi)]^{m'}{}_m &= D_j[e^{-2\pi i J_3}]^{m'}{}_m = \delta_{m',m} e^{-2\pi i m} \\ &= \delta_{m',m} e^{2\pi i j} = (-1)^{2j} \delta_{m',m}. \end{aligned} \tag{12}$$

- The existence of the double-valued representations are a consequence of the double-connectedness of the group manifold.
- Since the Lie algebra is related to the group properties in the vicinity of E, there is no *a priori* control over $D[R(2\pi)]$, $D[R(4\pi)]$, etc.
- All $SO(3)$ reps. become single-valued reps. of the covering group, $SU(2)$.
- In nature, fermion wave functions correspond to 2v reps. reps;
- 1v reps. describe boson systems.

II.2.7. Characters

- For fixed ψ, $R_{\mathbf{n}}(\psi) = RR_3(\psi)R^{-1}$ belong to the same conj. class.
- The character $\chi_j(\psi)$ can be evaluated using the $R_3(\psi)$ rotation:

$$\psi_j(\psi) = \sum_m D_j[R_3(\psi)]^m{}_m = \sum_{m=-j}^{j} e^{-im\psi} = \frac{\sin[(j+\frac{1}{2})\psi]}{\sin(\psi/2)}. \qquad (13)$$

- In particular, $\chi_{1/2}(\psi) = 2\cos(\psi/2)$ and $\chi_1(\psi) = 1 + 2\cos\psi$.

II.2.8. Properties of $D_j(\alpha, \beta, \gamma)$

- **Unitarity:** $D^\dagger(\alpha, \beta, \gamma) = D^{-1}(\alpha, \beta, \gamma) = D(-\gamma, -\beta, -\alpha)$.
- **Unit determinant (special):** In the angle-axis parametrization,

$$\det D[R_{\mathbf{n}}(\psi)] = \det D[RR_3(\psi)R^{-1}] = \prod_{m=-j}^{j} e^{-im\psi} = 1. \qquad (14)$$

- **Reality of $d(\beta)$:** True in the Condon-Shortley convention, where $d^{-1}(\beta) = d^T(\beta) = d(-\beta)$.
- **Complex conjugation:** $D^*(\alpha, \beta, \gamma) = YD(\alpha, \beta, \gamma)Y^{-1}$, where $Y = D[R_2(\pi)]$, since

$$YD(\alpha, \beta, \gamma) = [YD_3(\alpha)Y^{-1}][YD_2(\beta)Y^{-1}][YD_3(\gamma)Y^{-1}], \qquad (15)$$

and $YD_3(\psi)Y^{-1} = D_3(-\psi) = D_3^*(\psi)$, while $YD_2(\psi)Y^{-1} = D_2(\psi) = D_2^*(\psi)$.
- **Symmetry relations:** The reduced matrices $d_j(\beta)$ satisfy:

$$\begin{aligned} d_j(\beta)^{m'}{}_m = d_j(-\beta)^m{}_{m'} &= (-1)^{j-m'} d_j(\pi - \beta)^{-m'}{}_m \\ &= (-1)^{m'-m} d_j(\beta)^{-m'}{}_{-m}. \end{aligned} \qquad (16)$$

II.2.9. Relation to spherical harmonics $Y_{lm}(\theta, \phi)$

- Consider a unit vector $|\hat{\mathbf{r}}\rangle = |\theta, \phi\rangle$.
- When $j \to l \in \mathbb{N}$, $\langle \theta, \phi | lm \rangle = Y_{lm}(\theta, \phi)$ is *the spherical harmonic*.
- Noting that $|\theta, \phi\rangle = U(\phi, \theta, 0)\,|0, 0\rangle$, we have

$$Y_{lm}(\theta, \phi) = \langle 0, 0 | U^\dagger(\phi, \theta, 0) | lm \rangle = \langle 0, 0 | lm' \rangle \, [D_l^\dagger(\phi, \theta, 0)]^{m'}{}_m. \quad (17)$$

- $R_3(\psi)\,|0, 0\rangle = |0, 0\rangle \Rightarrow J_3\,|0, 0\rangle = 0 \Rightarrow \langle 0, 0 | lm' \rangle = \delta_{m',0} Y_{l0}(0, 0)$.
- Finally, we have $Y_{lm}(\theta, \phi) = Y_{l0}(0, 0)[D_l^*(\phi, \theta, 0)]^m{}_0$.
- Noting that $Y_{l0}(0, 0) = \sqrt{(2l + 1)/4\pi}$, the above discussion provides a connection between the representation matrices, $Y_{lm}(\theta, \phi)$, associated Legendre functions $P_{lm}(\cos\theta)$ and Legendre polynomials $P_l(\cos\theta)$:

$$Y_{lm}(\theta, \phi) = \sqrt{\frac{2l + 1}{4\pi}} [D_l(\phi, \theta, 0)^m{}_0]^* = \sqrt{\frac{2l + 1}{4\pi}} e^{im\phi} d_l(\theta)^m{}_0,$$

$$P_{lm}(\cos\theta) = (-1)^m \sqrt{\frac{(l + m)!}{(l - m)!}} d^l(\theta)^m{}_0,$$

$$P_l(\cos\theta) = P_{l0}(\cos\theta) = d_l(\theta)^0{}_0 = \sqrt{\frac{4\pi}{2l + 1}} Y_{l0}(\theta, \phi). \quad (18)$$

II.2.10. Transformation of wave functions and operators

- **Theorem:** The wave function of an arbitrary state transforms under rotations as: $\psi(\mathbf{x}) \to \psi'(\mathbf{x}) = \psi(R^{-1}\mathbf{x})$.
  Proof: $\psi'(\mathbf{x}) = \langle \mathbf{x}|U(R)|\psi\rangle = \langle R^{-1}\mathbf{x}|\psi\rangle = \psi(R^{-1}\mathbf{x})$.
- **Def:** A set of multi-component functions $\{\phi^m(\mathbf{x}), m = -j, \ldots j\}$ forms an *irreducible wave function* or *irreducible field* of spin j if they transform under rotations as

$$\phi^m(\mathbf{x}) \xrightarrow{R} \phi'^m(\mathbf{x}) = D_j[R]^m{}_n \phi^n(R^{-1}\mathbf{x}), \tag{19}$$

  where $D_j[R]^m{}_n$ is the angular momentum j irrep matrix of $SO(3)$.
- **Ex:** The two-component Pauli wave function $\psi^\sigma(\mathbf{x})$ of a particle with spin 1/2 transforms as $\psi'^\sigma(\mathbf{x}) = \langle \mathbf{x}|U(R)|\psi^\sigma\rangle = D_{1/2}[R]^\sigma{}_\lambda \psi^\lambda(R^{-1}\mathbf{x})$.
- **Theorem:** The components of the coordinate vector operator $\mathbf{X}$ transform under rotations as $X_i' = U[R]X_iU[R]^{-1} = X_jR^j{}_i$.
  Proof: $X_i'\,|\mathbf{x}\rangle = U[R]X_i\,|R^{-1}\mathbf{x}\rangle = |\mathbf{x}\rangle\,(R^{-1})_i{}^j x_j = (X_jR^j{}_i)\,|\mathbf{x}\rangle$.
- **Def:** Any set of operators transforming under rotations as the components of the coordinate operator constitute a *vector operator.*
- **Ex:** Consider the field operator $\langle 0|\Psi^\sigma(\mathbf{x})|\psi\rangle = \psi^\sigma(\mathbf{x})$, where $|0\rangle = U(R)\,|0\rangle$ is the rotationally-invariant vacuum state. Then, $\psi^\sigma(\mathbf{x}) = \langle 0|U[R]\Psi^\sigma(\mathbf{x})U[R^{-1}]|\psi'\rangle = D_j[R^{-1}]^\sigma{}_\lambda \psi'^\lambda(R\mathbf{x})$, by which

$$U[R]\Psi^\sigma(\mathbf{x})U[R]^{-1} = D_j[R^{-1}]^\sigma{}_\lambda \Psi^\lambda(R\mathbf{x}). \tag{20}$$

II.2.11. Direct product representations

- The vector $|jm; j'm'\rangle = |jm\rangle \otimes |j'm'\rangle$ transforms with the direct product representation:

$$U(R)\,|jm; j'm'\rangle = |jn; j'n'\rangle\, D_j(R)^n{}_m D_{j'}(R)^{n'}{}_{m'}\,. \qquad (21)$$

- **Theorem:** The generators of a direct product representation are the sums of the corresponding generators of its constituent representations: $J_n = J^j_n \otimes E^{j'} + E^j \otimes J^{j'}_n$.
- The resulting representation is in general reducible to $|JM\rangle$, with $\mathbf{J}^2\,|JM\rangle = |JM\rangle\, J(J+1)$ and $J_3\,|JM\rangle = |JM\rangle\, M$, where

$$J_3 = J^j_3 \otimes E_{j'} + E_j \otimes J^{j'}_3\,, \quad \mathbf{J}^2 = \mathbf{J}^2_j \otimes E_{j'} + E_j \otimes \mathbf{J}^2_{j'} + 2\mathbf{J}_j \cdot \mathbf{J}_{j'}\,. \quad (22)$$

- The maximum value $j + j'$ of J_3 corresponds to $|jj; j'j'\rangle$:

$$J_3\,|jj; j'j'\rangle = |jj; j'j'\rangle\,(j+j'), \quad \mathbf{J}^2\,|jj; j'j'\rangle = |jj; j'j'\rangle\,(j+j')(j+j'+1), \quad (23)$$

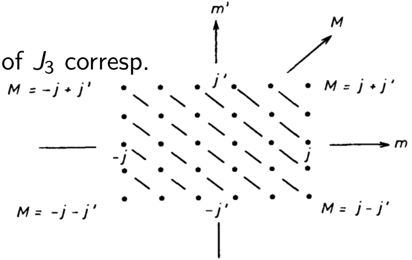
where we used $\mathbf{J}_j \cdot \mathbf{J}_{j'} = \frac{1}{2}(J^+_j \otimes J^-_{j'} + J^-_j \otimes J^+_{j'}) + J^3_j \otimes J^3_{j'}$.
- The states $|j+j', M\rangle$ can be generated by repeated application of $J_- = J^j_- \otimes E^{j'} + E^j \otimes J^{j'}_-$, e.g.

$$|j+j', j+j'-1\rangle = |jj-1; j'j'\rangle \sqrt{\frac{j}{j+j'}} + |jj; j'j'-1\rangle \sqrt{\frac{j'}{j+j'}}.$$

Decomposition into irreps

- There is one more eigenvector of J_3 corresp. to $M = j + j' - 1$:

$$|j+j'-1, j+j'-1\rangle = |jj; j'j'-1\rangle \sqrt{\frac{j}{j+j'}} - |jj-1; j'j'\rangle \sqrt{\frac{j'}{j+j'}}.$$



- This is the highest-M vector for the $J = j + j' - 1$ rep.
- The inequivalent irreps continue down to $J = |j - j'|$. Each irrep appears exactly once. The total number of basis elements is:

$$\sum_{J=|j-j'|}^{j+j'} (2J+1) = 4jj' + 2j + 2j' = (2j+1)(2j'+1). \tag{24}$$

- In general, the irreps basis vectors can be obtained from the direct product basis vectors via the C-G coeffs:

$$|JM\rangle = \sum_{m,m'} |jm; j'm'\rangle \langle mm'(jj')JM\rangle\,, \;\; |jm; j'm'\rangle = \sum_{J,M} |J, M\rangle \langle JM(jj')mm'\rangle\,,$$

with $\langle JM(jj')mm'\rangle = \langle mm'(jj')JM\rangle^*$.

- In the Condon-Shortley convention, $\langle mm'(jj')JM\rangle \in \mathbb{R}$ and $\langle j, J-j(jj')JJ\rangle = \langle jj; j'J-j|JJ\rangle > 0$.

Exercises

1. **Particle in a central potential.** Consider $H = T + V$ with $T = \mathbf{P}^2/2m$ and $V \equiv V(r)$.
   a) Show that $[H, U(R)] = 0, \forall R \in SO(3) \Rightarrow [H, J_i] = 0$.
   b) Show that $|\mathbf{x}\rangle = |r, \theta, \phi\rangle = \sum_{lm} |rlm\rangle\, Y^*_{lm}(\theta, \phi)$.
   c) Consider $|Elm\rangle$ a simultaneous eigenstate of $\{H, \mathbf{J}^2, J_3\}$:

   $$H\,|Elm\rangle = |Elm\rangle\, E,\;\; \mathbf{J}^2\,|Elm\rangle = |Elm\rangle\, l(l+1),\;\; J_3\,|Elm\rangle = |Elm\rangle\, m.$$

   Show that $\psi_{Elm}(\mathbf{x}) = \langle \mathbf{x}|Elm\rangle = \psi_{El}(r)Y_{lm}(\theta, \phi)$, where the radial wavefunction $\psi_{El}(r)$ depends only on r.
2. Show that $Y_{lm}(R^{-1}\hat{\mathbf{x}}) = Y_{lm'}(\hat{\mathbf{x}})D_l[R]^{m'}{}_m$.
3. Consider $|\psi\rangle = |\mathbf{p}\rangle$. Using $\langle \mathbf{x}|\mathbf{p}\rangle = e^{\frac{i}{\hbar}\mathbf{x}\cdot\mathbf{p}}/(2\pi\hbar)^{3/2}$, show that under a rotation, $\psi'(\mathbf{x}) = U[R]\psi(\mathbf{x}) = \psi(R^{-1}\mathbf{x})$, where $\psi(\mathbf{x}) = \langle \mathbf{x}|\psi\rangle$.

Exercises

4. **Partial wave decomposition.** Consider a particle with definite initial momentum $|\mathbf{p}_i\rangle = |p, 0, 0\rangle$, scattering off a central potential $V(r)$ into a state $|\mathbf{p}_f\rangle = |p, \theta, \phi\rangle$.
   a) Show that $|p, \theta, \phi\rangle = \sum_{lm} |plm\rangle\, Y^*_{lm}(\theta, \varphi)$.
   b) Show that the scattering amplitude $\langle \mathbf{p}_f | T | \mathbf{p}_i \rangle$ reduces to
   $$\langle \mathbf{p}_f | T | \mathbf{p}_i \rangle = \sum_{l,l',m} \sqrt{\frac{2l'+1}{4\pi}}\, Y_{lm}(\theta, \phi)\, \langle plm | T | pl'0 \rangle\,.$$
   c) In the case of a rotationally-invariant interaction, $U(R)TU^\dagger(R) = T$, show that $\langle plm | T | pl'm' \rangle = T_l(p)\delta_{ll'}\delta_{mm'}$.
   d) Thus prove the partial wave decomposition formula:
   $$\langle \mathbf{p}_f | T | \mathbf{p}_i \rangle = \sum_l \frac{2l+1}{4\pi}\, T_l(E). \qquad (25)$$

5. Let $|\psi\rangle = |Elm\rangle$, with $\langle \mathbf{x} | \psi \rangle = \psi_{El}(r) Y_{lm}(\theta, \phi)$. Show that
   $$\psi'(\mathbf{x}) = \psi_{El}(r) Y_{lm}(R^{-1}\mathbf{x}) = \psi_{El}(r) Y_{lm'}(\mathbf{x}) D_l[R]^{m'}{}_m. \qquad (26)$$

Exercises

WKT7.7 (i) From the definition of the canonical basis vectors and the Lie algebra of $SO(3)$, show that $U[R_2(\pi)]\,|jm\rangle = |j,-m\rangle\,\eta^j_m$, where $|\eta^j_m|^2 = 1$ and $\eta^j_{m\pm1} = -\eta^j_m$.
(ii) Using $\eta^{1/2}_{1/2} = 1$, prove that $\eta^j_j = 1, \forall j$, by math. induction.
(iii) Combine (i) and (ii) to show that $D_j[R_2(\pi)]^{m'}{}_m = (-1)^{j-m}\delta^{m'}_{-m}$.
(iv) Derive the explicit expression for $D_j[R_1(\pi)]$.

WKT7.8 Verify that $|jj;j'j'\rangle$ is an eigenvector of $\mathbf{J}^2$ with eigenvalue $(j+j')(j+j'+1)$.

6. Explicitly construct the decomposition into irreps of the direct product representation $D_{1/2}\otimes D_{1/2} = D_1 \oplus D_0$. Use the result to express the relevant Clebsch-Gordan coefficients.
7. Explicitly construct the decomposition into irreps of the direct product representation $D_{1/2}\otimes D_1 = D_{3/2}\oplus D_{1/2}$ and evaluate the relevant Clebsch-Gordan coefficients.
8. Use the results from 7 and 8 above to construct the decomposition into irreps of $D_{1/2}\otimes D_{1/2}\otimes D_{1/2} = D_{3/2}\oplus D_{1/2}\oplus D_{1/2}$.