Symmetries in Physics
Lecture 5

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Lecture contents

Chapter 2. Continuous symmetry groups

- **II.1. Abelian groups:** $SO(2)$ **and** $T(3)$
- **II.2. The rotation group** $SO(3)$
- II.3. The group $SU(2)$
- II.4. The Euclidean group E_3

II.1.1. The rotation group $SO(2)$

- The rotations about the z axis of angle ϕ act on the basis vectors $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ of the xOy plane as follows:

$$\begin{aligned} R(\phi)\hat{\mathbf{e}}_1 &= \hat{\mathbf{e}}_1 \cos\phi + \hat{\mathbf{e}}_2 \sin\phi, \\ R(\phi)\hat{\mathbf{e}}_2 &= -\hat{\mathbf{e}}_1 \sin\phi + \hat{\mathbf{e}}_2 \cos\phi, \end{aligned} \quad \Rightarrow \quad R(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}. \tag{1}$$

- The basis vectors transform as $\hat{\mathbf{e}}_i \to \hat{\mathbf{e}}'_i = \hat{\mathbf{e}}_j R(\phi)^j{}_i$;
- The components of $\mathbf{x} = x^i \hat{\mathbf{e}}_i$ transform as $x^i \to x'^i = R(\phi)^i{}_j x^j$.
- Rotations preserve lengths: $|\mathbf{x}|^2 = |\mathbf{x}'|^2$, such that

$$R(\phi)^j{}_i R(\phi)_j{}^k = [R^T(\phi)R(\phi)]^k{}_i = \delta^k_i. \tag{2}$$

- Since $\det R(\phi) = 1$ and $R^T R = E$, they are *special orthogonal matrices* of rank $2 \equiv SO(2)$ matrices.
- **Theorem:** There is a $1:1$ correspondence between rotations in a plane and $SO(2)$ matrices.
- **Theorem:** Given: $R(\phi_2)R(\phi_1) = R(\phi_1 + \phi_2)$; $R(\phi = 0) = E$; and $R^{-1}(\phi) = R(-\phi) = R(2\pi - \phi) \Rightarrow \{R(\phi)\}$ form a group: R_2 or $SO(2)$.
- Since $R(\phi_1)R(\phi_2) = R(\phi_2)R(\phi_1)$, the group is abelian.

II.1.2. The generator of $SO(2)$

- By definition, $R(0) = E$. Close to E, we define the *generator* J of the group via

$$R(\delta\phi) = E - iJ\delta\phi \quad \Rightarrow \quad J = i\left.\frac{dR}{d\phi}\right|_{\phi=0} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \qquad (3)$$

Theorem: All $2D$ rotations can be expressed as $R(\phi) = e^{-i\phi J}$.

- For unitary representations, $R^{-1}(\phi) = R^{\dagger}(\phi) \Rightarrow J^{\dagger} = J$.
- Imposing $\det R(\phi) = 1$ and using $\det\exp(A) = \exp\mathrm{tr}(A) \Rightarrow \mathrm{tr}J = 0$.
- The generators J are hermitian (unitary representation) and traceless (unit determinant) matrices.
- The eigenvalues $\lambda_{\pm} = \pm 1$ of J are real numbers.
- The eigenvectors of J, satisfying $J\,|\hat{\mathbf{e}}_{\pm}\rangle = \pm\hat{\mathbf{e}}_{\pm}$, are also eigenvectors of $R(\phi)$:

$$R(\phi)\,|\hat{\mathbf{e}}_{\pm}\rangle = |\hat{\mathbf{e}}_{\pm}\rangle\, e^{\pm i\phi}. \qquad (4)$$

- Since the subspaces spanned by $|\hat{\mathbf{e}}_{\pm}\rangle = \frac{1}{\sqrt{2}}(\mathbf{e}_1 \pm i\mathbf{e}_2)$ are invariant under $R(\phi)$, they correspond to two irreps of $SO(2)$.

II.1.3. Irreps of $SO(2)$

- Let $U(\phi)$ be a representation of $R(\phi)$ on V and $J = i(dU/d\phi)_{\phi=0}$.
- Since $U(0) = E$, we have $U(\phi) = e^{-i\phi J}$, acting as an operator on V.
- Since $SO(2)$ is abelian, all its irreps are $1D$.
- Consider $|\alpha\rangle$ a vector in a minimal invariant subspace:

$$J\,|\alpha\rangle = |\alpha\rangle\,\alpha, \qquad U(\phi)\,|\alpha\rangle = |\alpha\rangle\,e^{-i\phi\alpha}. \tag{5}$$

- Due to the global (topological) constraint $U(\phi + 2\pi) = U(\phi)$, we have $e^{\mp 2\pi i\alpha} = 1$ and hence $\alpha \to m \in \mathbb{Z}$.
- **Theorem:** The single-valued irreps of $SO(2)$ are given by $J = m \in \mathbb{Z}$ and $U_m(\phi) = e^{-im\phi}$.
- **Obs:** Only the $m = \pm 1$ irreps are faithful representations.
- **Obs:** $|\hat{\mathbf{e}}_\pm\rangle$ correspond to the representations $m = \pm 1$.
- **Def:** Relaxing the constraint to $U_{m/n}(\phi + 2n\phi) = U_{m/n}(\phi)$ gives the n-valued representation of $SO(2)$:

$$R(\phi) \to U_{m/n}(\phi) = e^{-im\phi/n}, \tag{6}$$

  where (n, m) are coprime numbers (i.e., with no common factors).
- In physics, only single-valued (classical; quantum, bosonic) and double-valued (quantum, fermion) representations are relevant.

II.1.4. Invariant integration measure

- In analogy to finite groups, compact groups have finite "volume".
- Integration over group elements must be compatible with the rearrangement lemma:

$$\int d\tau_R f[R] = \int d\tau_R f[S^{-1}R] = \int d\tau_{SR} f[R]. \tag{7}$$

- For $SO(2)$, this is achieved via $d\tau_R = d\phi$.
- **Theorem:** The $SO(2)$ representation functions $U^n(\phi)$ satisfy:

$$\begin{aligned} \int_0^{2\pi} \frac{d\phi}{2\pi} U_n^\dagger(\phi) U_m(\phi) &= \delta_{mn}, && \text{(orthogonality)} \\ \sum_n U_n(\phi) U_n^\dagger(\phi;) &= \delta(\phi - \phi'), && \text{(completeness).} \end{aligned} \tag{8}$$

II.1.5. Translations

- Consider the translation $f(\mathbf{x}_0) \to T(\mathbf{x})f(\mathbf{x}_0) = f(\mathbf{x}_0 + \mathbf{x})$.
- The generators of translations satisfy

$$i\boldsymbol{\nabla}[T(\mathbf{x})f(\mathbf{x}_0)]_{\mathbf{x}=0} = [i\boldsymbol{\nabla} f]_{\mathbf{x}_0} \quad \Rightarrow \quad \mathbf{P} = i\boldsymbol{\nabla}. \tag{9}$$

- Since $T(\mathbf{x})T(\mathbf{x}') = T(\mathbf{x}')T(\mathbf{x})$, the translation group is abelian and its irreps are $1D$.
- Consider $\mathbf{P}\,|\mathbf{p}\rangle = |\mathbf{p}\rangle\,\mathbf{p} \Rightarrow U^{\mathbf{p}}(x)\,|\mathbf{p}\rangle = |\mathbf{p}\rangle\,e^{-i\mathbf{p}\cdot\mathbf{x}}$.
- **Theorem:** The irreps of $T(\mathbf{p})$ satisfy:

$$\begin{aligned}
\int_{-\infty}^{\infty} d^3x\, U^{\dagger}_{\mathbf{p}}(\mathbf{x})U^{\mathbf{q}}(\mathbf{x}) &= (2\pi)^3\delta^3(\mathbf{p}-\mathbf{q}), \quad \text{(orthogonality)} \\
\int_{-\infty}^{\infty} d^3p\, U^{\mathbf{p}}(\mathbf{x})U^{\dagger}_{\mathbf{p}}(\mathbf{y}) &= (2\pi)^3\delta^3(\mathbf{x}-\mathbf{y}), \quad \text{(completeness).}
\end{aligned} \tag{10}$$

II.1.6. Conjugate basis vectors: rotations

- Consider a state $|\psi\rangle$ and the coordinate basis $|r, \phi\rangle \equiv |\phi\rangle$ (r is unchanged by rotations).
- Its Fourier and inverse Fourier transforms read:

$$\langle\phi|\psi\rangle = \sum_{m=-\infty}^{\infty} \frac{e^{im\phi}}{\sqrt{2\pi}} \langle m|\psi\rangle\,, \quad \langle m|\psi\rangle = \int_0^{2\pi} \frac{d\phi}{\sqrt{2\pi}} e^{-im\phi} \langle\phi|\psi\rangle\,. \quad (11)$$

- Using $E = \int_0^{2\pi} d\phi \; |\phi\rangle \langle\phi| = \sum_{m=-\infty}^{\infty} |m\rangle \langle m|$, we find

$$\langle m|\phi\rangle = \langle\phi|m\rangle^* = \frac{1}{\sqrt{2\pi}} e^{-im\phi}. \quad (12)$$

- Since $J\,|m\rangle = |m\rangle\, m$, we have

$$J\,|\phi\rangle = \sum_m |m\rangle\, m e^{-im\phi} = i\frac{d}{d\phi}\,|\phi\rangle\,, \quad (13)$$

  such that $\langle\phi|J|\psi\rangle = -i d\psi/d\phi$.
- $J = -i\partial_\phi$ is just the (dimensionless) angular momentum operator along z!

II.1.7. Conjugate basis vectors: translations

- Consider a state $|\psi\rangle$ and the coordinate basis $|\mathbf{x}\rangle$.
- Its Fourier and inverse Fourier transforms read:

$$\langle \mathbf{x}|\psi\rangle = \int \frac{d^3p}{(2\pi\hbar)^{3/2}} e^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{x}} \langle \mathbf{p}|\psi\rangle \,,$$

$$\langle \mathbf{p}|\psi\rangle = \int \frac{d^3p}{(2\pi\hbar)^{3/2}} e^{-\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{x}} \langle \mathbf{x}|\psi\rangle \,. \tag{14}$$

- Using $E = \int d^3x \, |\mathbf{x}\rangle \langle \mathbf{x}| = \int d^3p \, |\mathbf{p}\rangle \langle \mathbf{p}|$, we find

$$\langle \mathbf{p}|\mathbf{x}\rangle = \langle \mathbf{x}|\mathbf{p}\rangle^* = \frac{1}{(2\pi\hbar)^{3/2}} e^{-\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{x}}. \tag{15}$$

- Since $\mathbf{P}\,|\mathbf{p}\rangle = |\mathbf{p}\rangle\,\mathbf{p}$, we have

$$\mathbf{P}\,|\mathbf{x}\rangle = \int d^3p \,|\mathbf{p}\rangle\, \mathbf{p} e^{-\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{x}} = i\hbar\boldsymbol{\nabla}\,|\mathbf{x}\rangle \,, \tag{16}$$

  such that $\langle \mathbf{x}|\mathbf{P}|\psi\rangle = -i\hbar\boldsymbol{\nabla}\psi$.
- $\mathbf{P} = -i\hbar\boldsymbol{\nabla}$ is just the momentum operator!

II.2. The rotation group.
II.2.1. Description of $SO(3)$

- **Def:** The $SO(3)$ *group* consists of all continuous linear transformations in $3D$ Euclidean space which leave the length of coordinate vectors invariant.
- Under a rotation $R \in SO(3)$, $\hat{\mathbf{e}}_i \to \hat{\mathbf{e}}'_i = \hat{\mathbf{e}}_j R^j{}_i$ and $x^i \to x'^i = R^i{}_j x^j$.
- The requirement $|\mathbf{x}| = |\mathbf{x}'|$ imposes $R^T R = R R^T = E$.
- The above implies $\det R = \pm 1$ [the $O(3)$ group].
- Matrices continuously connected to E have $\det R = 1$ [$SO(3)$ group].
- Any matrix with $\det M = -1$ can be written as $M = I_s R$, with $I_s = \mathrm{diag}(-1, -1, -1)$ the spatial reflection.
- The properties $R^T R = E$ and $\det R = 1$ can be written as

$$R_l{}^i R^l{}_j = \delta^i{}_j, \quad R^i{}_l R^j{}_m R^k{}_n \varepsilon^{lmn} = \varepsilon^{ijk} \det R = \varepsilon^{ijk}. \tag{17}$$

- The matrices describing rotations about the coordinate axes are:

$$R_1(\psi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{pmatrix}, \quad R_2(\psi) = \begin{pmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{pmatrix}, \quad R_3(\psi) = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

II.2.2. Angle-axis and Euler angle parametrizations

- Any rotation can be characterized by an angle $\psi \in [0, \pi]$ and a direction $\hat{\mathbf{n}} \equiv \hat{\mathbf{n}}(\theta, \varphi)$.
- Graphically, $R_{\mathbf{n}}(\psi)$ can be seen as a vector with polar coordinates (θ, φ) and magnitude ψ, covering the sphere of radius $0 \le \psi \le \pi$.
- Since $R_{\mathbf{n}}(\pi) = R_{-\mathbf{n}}(\pi)$, the sphere is *doubly-connected*.
- If $\hat{\mathbf{n}}^i = R(\mathbf{n})^i{}_j \hat{\mathbf{e}}_z^j$, then $R_{\mathbf{n}}(\psi) = R_{\mathbf{n}} R_3(\psi) R_{\mathbf{n}}^{-1}$.
- **Theorem:** All rotations by the same angle ψ belong to a single class of the group $SO(3)$.
  Proof: Let $R^i{}_j \hat{\mathbf{n}}^j = \hat{\mathbf{n}}'^i = R(\mathbf{n}')^i{}_j \hat{\mathbf{e}}_z^j$. Since $R(\mathbf{n}') = RR(\mathbf{n})$, we have

$$R_{\mathbf{n}'}(\psi) = R(\mathbf{n}') R_3(\psi) R^{-1}(\mathbf{n}') = R[R(\mathbf{n}) R_3(\psi) R^{-1}(\mathbf{n})] R^{-1}$$
$$= R R_{\mathbf{n}}(\psi) R^{-1}.$$

- Any rotation can be parametrized using the Euler angles as $R(\alpha, \beta, \gamma) = R_3(\alpha) R_2(\beta) R_3(\gamma)$.

II.2.3. The Lie algebra of $SO(3)$

- The generators J_i of the rotations along the coordinate axes i are:

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \qquad (18)$$

- The components of the generators satisfy $(J_i)^j{}_k = -i\varepsilon^{ijk}$.
- **Lemma:** Writing $R_{\mathbf{n}}(\psi) = e^{-i\psi J_{\mathbf{n}}}$, we have $RJ_{\mathbf{n}}R^{-1} = J_{R\mathbf{n}}$.
  Proof: Follows by noting that $Re^{-i\psi J}R^{-1} = e^{-i\psi RJR^{-1}}$.
- **Theorem:** Under rotations, J_k behave as the basis vectors: $RJ_kR^{-1} = J_lR^l{}_k$.
  Proof: Follows after multiplying Eq. (17) by $-iR_i{}^l$ and using $\varepsilon^{lmn} = i(J_l)^m{}_n$.
- **Theorem:** The generator of rotations around $\hat{\mathbf{n}}$ is $J_{\mathbf{n}} = \hat{\mathbf{n}} \cdot \mathbf{J}$.
  Proof: The generator around $\hat{\mathbf{e}}_z$ is $J_z = \mathbf{J} \cdot \hat{\mathbf{e}}_z$. Let $\hat{\mathbf{n}} = R\hat{\mathbf{e}}_z$. Then, $J_{\mathbf{n}} = RJ_zR^{-1} = J_kR^k{}_3 = \hat{\mathbf{n}} \cdot \mathbf{J}$.
- **Theorem:** The generators J_k satisfy the following *Lie algebra:*

$$[J_k, J_l] = J_kJ_l - J_lJ_k = i\varepsilon_{klm}J_m. \qquad (19)$$

Proof: By direct computation or by using $RJ_kR^{-1} = J_lR^l{}_k$.

II.2.4. Casimir operator: $\mathbf{J}^2$

- **Def:** An operator which commutes with all elements of a Lie group is a *Casimir operator*.
- $\mathbf{J}^2$ is a Casimir operator of the $SO(3)$ group, since $[J_k, \mathbf{J}^2] = 0$.
- By Schur's Lemma 1, $\mathbf{J}^2 \sim E$ and all vectors in an irrep are eigenvectors of $\mathbf{J}^2$ with the same eigenvalue.
- The basis vectors defining the irrep are taken as eigenvectors of J_3 and $\mathbf{J}^2$; while $J_\pm = J_1 \pm iJ_2 = J_\mp^\dagger$ represent the *raising* and *lowering operators*, satisfying:

$$[J_3, J_\pm] = \pm J_\pm, \ [J_+, J_-] = 2J_3, \ \mathbf{J}^2 = J_3^2 - J_3 + J_+J_- = J_3^2 + J_3 + J_-J_+.$$

- Consider $|m\rangle$ s.t. $J_3 |m\rangle = |m\rangle m$. Then $J_3 J_+ |m\rangle = J_+ |m\rangle (m+1)$.
- Applying J_+^k on $|m\rangle$ will generate a vector proportional to $|m+k\rangle$. Imposing $J_+ |j\rangle = 0$ implies

$$\mathbf{J}^2 |j\rangle = |j\rangle j(j+1). \tag{20}$$

- Similarly, imposing $J_- |j'\rangle = 0 \Rightarrow j(j+1) = j'(j'-1)$, hence $j' = -j$.
- $|-j\rangle$ is obtained by applying J_- on $|j\rangle$ an integer number of times $\Rightarrow 2j = n = 0, 1, 2, \cdots \Rightarrow j = 0, \frac{1}{2}, 1, \ldots$.

Exercises

1. Compute $\mathbf{J}^2$ in the standard ($3D$) representation (18) and thus find the associated value of j.
2. Show that $J_\pm |jm\rangle = |jm \pm 1\rangle\, [j(j+1) - m(m\pm 1)]^{1/2} e^{i\theta}$, where θ is a real number. The case $\theta = 0$ corresponds to the Condon-Shortley convention.

WKT7.1 Derive the general expression for the 3×3 matrix $R(\alpha, \beta, \gamma)$.

WKT7.2 Derive the relation between the Euler angle variables (α, β, γ) and the angle-axis parameters (ψ, θ, ϕ) for a general rotation:

$$\phi = \frac{\pi + \alpha - \gamma}{2}, \quad \tan\theta = \frac{\tan(\beta/2)}{\sin[(\gamma+\alpha)/2]},$$
$$\cos\psi = 2\cos^2\left(\frac{\beta}{2}\right)\cos^2\left(\frac{\alpha+\gamma}{2}\right) - 1. \qquad (21)$$

WKT7.3 From geometrical considerations, derive the following result which describes the effect of the rotation $R_{\mathbf{n}}(\psi)$ on an arbitrary vector $\hat{\mathbf{r}}$:

$$R_{\mathbf{n}}(\psi)\hat{\mathbf{r}} = \hat{\mathbf{r}}\cos\psi + \hat{\mathbf{n}}(1-\cos\psi)(\hat{\mathbf{r}}\cdot\hat{\mathbf{n}}) + (\hat{\mathbf{n}}\times\hat{\mathbf{r}})\sin\psi. \qquad (22)$$

Exercises

WKT7.4 An alternative way of writing the Lie algebra for $SO(3)$ can be obtained by defining $J^{kl} = \varepsilon^{klm} J_m$ (i.e., $J^{12} = J_3$, etc) as the generator for rotations in the $k - l$ plane. Show that

$$[J^{kl}, J^{mn}] = i(\delta^{km} J^{ln} - \delta^{kn} J^{lm} - \delta^{lm} J^{kn} + \delta^{ln} J^{km}). \tag{23}$$

WKT7.6 Find the similarity transformation relating the Cartesian generators J_k in Eq. (18) with those in the canonical basis $\{|m\rangle, -1 \leq m \leq 1\}$ shown below:

$$J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad J_+ = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}, \quad J_- = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}. \tag{24}$$