Symmetries in Physics
Lecture 4

Victor E. Ambruș

Universitatea de Vest din Timișoara

Lecture contents

1.4. Representations of the symmetric group

1.4.1. Group algebra

- The regular representation, $g_i g_j = g_m(\Delta_i)^m{}_j$, involves a formal sum over group elements.
- The regular representation provides the natural arena to introduce the *group algebra* $\widetilde{G}$, defined by the elements $r = g_i r^i$, with $r^i \in \mathbb{C}$.
- The group algebra $\widetilde{G}$ represents a *ring* w.r.t.:
  - Addition: $r + q = |g_i\rangle\,(r^i + q^i)$;
  - Multiplication: $rq = g_i g_j r^i q^j = g_m[r^i(\Delta_i)^m{}_j q^j]$.
- **Def:** A *representation* U of $\widetilde{G}$ on V preservers the group algebra structure: $U(\alpha q + \beta r) = \alpha U(q) + \beta U(r)$ and $U(qr) = U(q)U(r)$, $\forall q, r \in \widetilde{G}$.
- **Def:** An *irrep* of $\widetilde{G}$ does not have any non-trivial inv. subsp. in V.
- **Theorem:** (i) A rep. of $\widetilde{G}$ is a rep. of G and *vice-versa*; (ii) An irrep of $\widetilde{G}$ is an irrep of G and *v-v*.
- **Def:** The alternating group A_n consists of all even permutations of n elements.
- The elements of $\widetilde{G}$ can be denoted by $|r\rangle = |g_i\rangle\, r^i$, where $|g_i\rangle$ forms a natural orthonormal basis: $\langle g^j | g_i\rangle = \delta^j_i$.
- Each group element acts as an operator: $r\,|q\rangle = |g_k\rangle\,[r^i(\Delta_i)^k{}_j]q^j$.

I.4.2. Left ideals, projection operators

- **Def:** A subspace L of $\widetilde{G}$ for which $pr \in L, \forall p \in G$ and $r \in L$ is called a *left ideal* of $\widetilde{G}$.
- **Def:** A **minimial** left ideal does not contain smaller left ideals.
- **Obs:** A minimal left ideal corresponds to an irr. inv. subspace.
- Let $L_a^\mu \equiv$ minimal L-id. in $\widetilde{G}$. The projector P_a^μ onto L_a^μ must satisfy:

$$P_a^\mu \left|r\right\rangle \in L_a^\mu, \forall r \in \widetilde{G}; \qquad P_a^\mu \left|q\right\rangle = \left|q\right\rangle, \forall q \in L_a^\mu;$$

$$P_a^\mu r = rP_a^\mu, \forall r \in \widetilde{G}; \qquad P_a^\mu P_b^\nu = \delta^{\mu\nu}\delta_{ab}P_a^\mu.$$

- Consider the decomposition $e = \sum_{\mu,a} e_a^\mu$ of the identity $e \in G$, with $e_a^\mu \in L_a^\mu$ the identity in L_a^μ.
- **Theorem:** P_a^μ can be realized by right multiplication with e_a^μ: $P_a^\mu \left|r\right\rangle \equiv \left|re_a^\mu\right\rangle, \forall r \in \widetilde{G}$.
  Proof: (i) $P_a^\mu \left|\alpha r + \beta q\right\rangle = \alpha P_a^\mu \left|r\right\rangle + \beta P_a^\mu \left|q\right\rangle$ (linearity);
  (ii) Writing $r \in \widetilde{G}$ as $r = \sum_{\mu,a} r_a^\mu$, with $r_a^\mu \in L_a^\mu$, we have $r = re = \sum_{\mu,a} re_a^\mu$, thus $P_a^\mu r = re_a^\mu = r_a^\mu$.
  (iii) For $r, q \in \widetilde{G}$, we have $P_a^\mu q\left|r\right\rangle = \left|qre_a^\mu\right\rangle = qP_a^\mu \left|r\right\rangle$.
  (iv) By construction, $e_a^\mu e_b^\nu = \delta^{\mu\nu}\delta_{ab}e_a^\mu$, such that $P_a^\mu P_b^\nu \left|r\right\rangle = \left|re_b^\nu e_a^\mu\right\rangle = \delta^{\mu\nu}\delta_{ab}P_a^\mu \left|r\right\rangle, \forall r \in \widetilde{G}$.

I.4.3. Idempotents

- **Def:** The elements of the group algebra e_a^μ satisfying $e_a^\mu e_a^\nu = \lambda_a^\mu \delta^{\mu\nu}\delta_{ab} e_a^\mu$ are *essentially idempotents*. If $\lambda_a^\mu = 1$, then e_a^μ are *idempotents*.
- **Def:** A *primitive idempotent* generates a minimal left ideal.
- **Theorem:** e_i is primitive $\Leftrightarrow e_i r e_i = \lambda_r e_i, \forall r \in \widetilde{G}$, with $\lambda_r \in \mathbb{C}$.
  Proof: $\Rightarrow$: Let $R : \widetilde{G} \to \widetilde{G}$, defined by $R\,|q\rangle = |q e_i r e_i\rangle\,, \forall q \in \widetilde{G}$, for some $r \in \widetilde{G}$. Since $Rs = sR, \forall s \in G$, then $R = \lambda_r e_i$ (Schur L1, on L_a^μ).
  $\Leftarrow$: Assume $e_i = e_i' + e_i''$, with e_i' and e_i'' idempotents. By definition, $e_i e_i' = e_i'$ and $e_i e_i' e_i = e_i'$. Also, $e_i e_i' e_i = \lambda' e_i$. Then, $e_i' e_i' = (\lambda')^2 e_i$ and $e_i' e_i' = e_i' = \lambda' e_i$. Hence, $\lambda' = 0$ (then $e_i = e_i''$), or $\lambda' = 1$ and $e_i = e_i'$.
- **Theorem:** Two primitive idempotents e_1 and e_2, corresponding to the minimal left ideals L_1 and L_2, generate equiv. irreps $\Leftrightarrow \exists r \in \widetilde{G}$ s.t. $e_1 r e_2 \neq 0$.
  Proof: $\Rightarrow$: Let $D_{1,2}(\widetilde{G})$ be the irreps corresponding to $e_{1,2}$. Since $e_1 \simeq e_2$, $\exists S : L_1 \to L_2$ s.t. $SD_1(p) = D_2(p)S, \forall p \in G$. Since
  $D_1(p) = D_2(p) = p \in \widetilde{G} \Rightarrow Sp = pS$.
  Let $|s\rangle = S\,|e_1\rangle \in L_2$. Since $S\,|e_1\rangle = Se_1\,|e_1\rangle = e_1\,|s\rangle$, we have $s = e_1 s$.
  Also, $s = se_2$, since $s \in L_2$. Then, $e_1 s = se_2 = s \Rightarrow s = e_1 s e_2$.
  $\Leftarrow$: Let $e_1 r e_2 = s \neq 0$ for some $r \in \widetilde{G}$. Consider $S : L_1 \to L_2$, such that $q_1 \in L_1 \xrightarrow{S} q_2 = q_1 s \in L_2$. Then, $Sp\,|q_1\rangle = pS\,|q_1\rangle$ and Schur-L2 implies $D_1(G) \simeq D_2(G)$.

1.4.4. One-dimensional representations of S_n

- **Def:** An even permutation is one that consists of an even number of simple transpositions.
- **Def:** The alternating group A_n consists of all even permutations of n elements.
- Every symmetric group has the non-trivial subgroup A_n (alternating group). The factor group $S_n/A_n \cong C_2$ induces two $1D$ irreps: $p \to 1$ and $p \to (-1)^p$.
- **Theorem:** The two $1D$ representations correspond to the essentially idempotent and primitive *symmetrizer* $s = \sum_p p$ and *anti-symmetrizer* $a = \sum_p (-1)^p p$.

  Proof: (i) Using the rearrangement lemma, $ps = s$ and $pa = (-1)^p a$. Then, $ss = n!s$ and $aa = n!a$ and thus s and a are essentially idempotent.

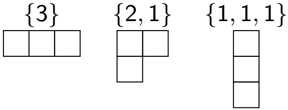
  (ii) Since $sqs = ss = n!s$, while $aqa = (-1)^q n!a$, s and a are primitive idempotents.

  (iii) Since $sqa = sa = \sum_p (-1)^p sp = s \sum_p (-1)^p = 0$, the two representations are inequivalent.

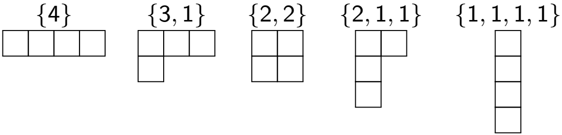
  (iv) The basis vectors are of the form $q|s\rangle = |qs\rangle = |s\rangle$ (hence $q \to 1$ under s); and $q|a\rangle = |qa\rangle = (-1)^q|a\rangle$ (hence $q \to (-1)^q$ under a).

1.4.5. Partitions and Young diagrams

- **Def:** A *partition* $\lambda \equiv \{\lambda_1, \ldots \lambda_r\}$ of the integer n is a sequence of positive integers λ_i, with $\lambda_i \geq \lambda_{i+1}$, such that $\sum_{i=1}^{r} \lambda_i = n$.
- $\lambda = \mu$ if $\lambda_i = \mu_i, \forall i$.
- $\lambda > \mu$ $(\lambda < \mu)$ if the first non-zero number in the sequence $\lambda_i - \mu_i > 0$ (< 0).
- **Def:** λ is represented graphically by a *Young Diagram*, which consists of n squares arranged in r rows, with the ith one containing λ_i squares.
- For $n = 3$, the distinct partitions are:



- For $n = 4$:



1.4.6. Young tableaux

- **Theorem:** The number of Young diagrams for any given n is equal to the number of classes of S_n, and therefore to the number of irreps of S_n.
  Proof: Every class of S_n is characterized by a cycle structure, say ν_1 1-cycles; ν_2 2-cycles; .... Then, $n = \nu_1 + 2\nu_2 + 3\nu_3 + \ldots$.
  Denoting: $\lambda_1 = \nu_1 + \nu_2 + \ldots$, $\lambda_2 = \nu_2 + \nu_3 + \ldots$, etc., we see that λ becomes a partition of n.
- **Def:** A *Young tableau* is obtained by filling a Young diagram with distinct numbers between 1 and n, in any order.
- **Def:** A *normal Young tableau* Θ_λ if one in which the numbers appear in increasing order from left to right and from top to bottom and it is unique for a given partition λ.
- **Def:** A *standard Young tableau* is one in which the numbers appear in increasing order on each row and on each column, but not in strict order.
- **Obs:** Any Young tableau can be obtained from Θ_λ by applying a given permutation p on the numbers $1, \ldots n$, giving Θ_λ^p.

1.4.7. (Anti-)Symmetrizers of Young tableaux

- **Def:** A *horizontal permutation* h_λ^p leaves invariant the sets of numbers appearing in each row of Θ_λ^p.
- **Def:** A *vertical permutation* h_λ^p leaves invariant the sets of numbers appearing in each column of Θ_λ^p.
- **Def:** The *symmetrizer* s_λ^p, *anti-symmetrizer* a_λ^p and *irreducible (Young) symmetrizer* e_λ^p associated with the Young tableau Θ_λ^p are

$$s_\lambda^p = \sum_h h_\lambda^p, \quad a_\lambda^p = \sum_v (-1)^{v_\lambda} v_\lambda^p, \quad e_\lambda^p = \sum_{h,v} (-1)^{v_\lambda} h_\lambda^p v_\lambda^p. \qquad (1)$$

- Example for S_3:

$\Theta_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}$: $\{h_\lambda\} = S_3 \Rightarrow s_1 = s$; $\{v_\lambda\} = \{e\} \Rightarrow a_1 = e$; $e_1 = s$.

$\Theta_2 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$: $\{h_\lambda\} = \{e, (12)\} \Rightarrow s_2 = e + (12)$; $\{v_\lambda\} = \{e, (13)\} \Rightarrow a_1 = e - (13)$; $e_2 = e + (12) - (13) - (132)$.

$\Theta_3 = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$: $\{h_\lambda\} = \{e\} \Rightarrow s_3 = e$; $\{v_\lambda\} = S_3 \Rightarrow a_3 = a$; $e_3 = a$.

$\Theta_2^{(23)} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$: $\{h_\lambda^p\} = \{e, (13)\} \Rightarrow s_2^p = e + (13)$; $\{v_\lambda^p\} = \{e, (12)\} \Rightarrow a_1^p = e - (12)$; $e_2^p = e + (13) - (12) - (123)$.

- For each Θ_λ^p, $\{h_\lambda^p\}$ and $\{v_\lambda^p\}$ each form a subgroup of S_n;
- $s_\lambda^p h_\lambda^p = h_\lambda^p s_\lambda^p = s_\lambda^p$ and $a_\lambda^p v_\lambda^p = v_\lambda^p a_\lambda^p = (-1)^{v_\lambda^p} a_\lambda^p$, as well as $s_\lambda^p s_\lambda^p = n_\lambda s_\lambda^p$ $(n_\lambda = \lambda_1!\lambda_2!\cdots\lambda_r!)$ and $a_\lambda^p a_\lambda^p = \tilde{n}_\lambda a_\lambda^p$ $(\tilde{n}_\lambda = (1!)^{\nu_1}(2!)^{\nu_2}\cdots(r!)^{\nu_r}) \Rightarrow s_\lambda^p$ and a_λ^p are essentially idempotent (but not primitive).
- e_λ^p are primitive idempotents.
- e_λ generate all inequivalent irreducible representations; e_λ^p generate irreducible representations that are equivalent to e_λ.
- The left ideals generated by e_λ^p corresponding to the standard Young tableaux Θ_λ^p are non-overlapping, spanning (through direct sum) $\widetilde{S}_n$.
- For S_3, e_1 and e_3 generate the $1D$ irreps; while e_2 and $e_2^{(12)}$ generate:

$$ee_2 = (12)e_2 = e_2, \quad (23)e_2 = (132)e_2 = r_2, \quad (13)e_2 = (123)e_2 = -e_2 - r_2,$$
$$ee_2^p = (13)e_2^p = e_2^p, \quad (23)e_2^p = (123)e_2^p = r_2^p, \quad (12)e_2^p = (132)e_2^p = -e_2^p - r_2^p,$$

where $r_2 = (23) + (132) - (123) - (12)$ and $r_2^p = (23) + (123) - (132) - (13)$.
- It can be checked that $\{e_1, e_2, r_2, e_2^p, r_2^p, e_3\}$ are mutually orthogonal and together, they span $\widetilde{S}_3$:

$$e = \frac{1}{6}e_1 + \frac{1}{3}e_2 + \frac{1}{3}e_2^p + \frac{1}{6}e_3. \tag{2}$$

I.4.8. Useful lemmas

- **Lemma:** Consider a Young tableau θ_λ and $\theta_\lambda^p = p\theta_\lambda$. Then,

$$\{h_\lambda^p, v_\lambda^p, s_\lambda^p, a_\lambda^p, e_\lambda^p\} = p\{h_\lambda, v_\lambda, s_\lambda, a_\lambda, e_\lambda\}p^{-1}.$$

  Proof: Obvious, since for each f_λ on θ_λ, there is an associated $f_\lambda^p = pf_\lambda p^{-1}$ acting on θ_λ^p as f_λ acts on θ_λ, i.e. $f_\lambda^p \theta_\lambda^p = pp^{-1} f_\lambda^p p\theta_\lambda = pf_\lambda \theta_\lambda$.

- **Lemma:** Given θ_λ and $p \in S_n$ defining θ_λ^p, a necessary and sufficient condition that $p \neq h_\lambda v_\lambda$ is that there are at least two numbers in one row of θ_λ which appear in the same column of θ_λ^p.

  Proof: $\Leftarrow$: Consider $p = h_\lambda v_\lambda$. Notice $v_\lambda^{h_\lambda} = h_\lambda v_\lambda h_\lambda^{-1}$ is a vertical permutation for $\theta_\lambda^{h_\lambda} = h_\lambda \theta_\lambda$. Neither h_λ nor $v_\lambda^{h_\lambda}$ bring two numbers from the same column onto the same row, thus $p \neq h_\lambda v_\lambda$.

  $\Rightarrow$: Assuming θ_λ^p does not have on any column two elements from the same row in θ_λ^p, we construct $\theta_\lambda^{h_\lambda} = h_\lambda \theta_\lambda$ such that each column in $\theta_\lambda^{h_\lambda}$ has the same elements as in θ_λ^p. Then, $\theta_\lambda^p = v_\lambda^{h_\lambda} \theta_\lambda^{h_\lambda}$ is obtained by applying a vertical permutation, arranging the elements in each column. Conversely, if $p \neq h_\lambda v_\lambda$, then there must be a pair of numbers appearing in one row of θ_λ and one column of θ_λ^p.

- **Lemma:** If $p \neq h_\lambda v_\lambda$, $\exists$ the transpositions $\tilde{h}_\lambda$ and $\tilde{v}_\lambda$ s.t. $p = \tilde{h}_\lambda p \tilde{v}_\lambda$.

  Proof: Since $p \neq h_\lambda v_\lambda$, there is a pair of numbers on some row of θ_λ that appears on the same column in θ_λ^p. Conversely, there will be a pair of numbers on some column in θ_λ that appear on the same row in θ_λ^p. Let $\tilde{v}_\lambda$ and $\tilde{h}_\lambda^p$ be the vertical and horizontal transpositions swapping these numbers in θ_λ and θ_λ^p, such that $\tilde{h}_\lambda^p p = p\tilde{v}_\lambda$. Then, $\tilde{h}_\lambda^p p \tilde{v}_\lambda = p$.

  Example: Consider $n = 8$ and $p = (13472)$:

$$\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 5&6\\\cline{1-2} 7\\\cline{1-1} 8\\\cline{1-1}\end{array} \quad \xrightarrow{\;p\;} \quad \begin{array}{|c|c|c|c|}\hline 2&7&1&3\\\hline 5&6\\\cline{1-2} 4\\\cline{1-1} 8\\\cline{1-1}\end{array}. \tag{3}$$

  Then, $\tilde{v}_\lambda = (17)$ and $\tilde{h}_\lambda = (23)$, such that

$$\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 5&6\\\cline{1-2} 7\\\cline{1-1} 8\\\cline{1-1}\end{array} \xrightarrow{\tilde{v}_\lambda} \begin{array}{|c|c|c|c|}\hline 7&2&3&4\\\hline 5&6\\\cline{1-2} 1\\\cline{1-1} 8\\\cline{1-1}\end{array} \xrightarrow{p} \begin{array}{|c|c|c|c|}\hline 2&1&7&3\\\hline 5&6\\\cline{1-2} 4\\\cline{1-1} 8\\\cline{1-1}\end{array} \xrightarrow{\tilde{h}_\lambda^p} \begin{array}{|c|c|c|c|}\hline 2&7&1&3\\\hline 5&6\\\cline{1-2} 4\\\cline{1-1} 8\\\cline{1-1}\end{array}. \tag{4}$$

- **Lemma:** If $h_\lambda r v_\lambda = (-1)^{v_\lambda} r, \forall h_\lambda, v_\lambda$ defined on θ_λ, then $r = \xi_r e_\lambda$ for some number ξ_r.
  Proof: Consider the decomposition $r = \sum_p \alpha_p p$. Then:
  $$h_\lambda r v_\lambda = \sum_q \alpha_q (h_\lambda q v_\lambda) = \sum_p \alpha_{h_\lambda^{-1} p v_\lambda^{-1}} p.$$
  Equating with $(-1)^{v_\lambda} p$, we find $\alpha_{h_\lambda^{-1} p v_\lambda^{-1}} = (-1)^{v_\lambda} \alpha_p$. If $p \neq h_\lambda v_\lambda$, then we can find $\tilde{h}_\lambda$ and $\tilde{v}_\lambda$ s.t. $p = \tilde{h}_\lambda p \tilde{v}_\lambda$ and $\alpha_p = (-1)^{\tilde{v}_\lambda} \alpha_p = -\alpha_p$, by which $\alpha_p = 0$. If $p = h_\lambda v_\lambda$, then $\alpha_p = (-1)^{v_\lambda} \xi$, with $\xi = \alpha_e$. Then, $r = \xi \sum_{h_\lambda, v_\lambda} (-1)^{v_\lambda} h_\lambda v_\lambda = \xi e_\lambda$.

I.4.9. Irreps of S_n

- **Theorem:** Within Θ_λ^p, $s_\lambda^p r a_\lambda^p = \xi_r e_\lambda^p, \forall r \in \widetilde{S}_n$ and with ξ_r an r-dependent number. Moreover, $e_\lambda^p e_\lambda^p = \eta e_\lambda^p$, with $\eta \neq 0$. Hence, e_λ is an idempotent.
  Proof: (i) Omitting p, let $q = s_\lambda r a_\lambda$. Since $h_\lambda s_\lambda = s_\lambda$ and $a_\lambda v_\lambda = (-1)^{v_\lambda} a_\lambda$, $h_\lambda q v_\lambda = (-1)^{v_\lambda} q, \forall h_\lambda, v_\lambda$. Hence, by **Lemma**, $q = s_\lambda r a_\lambda = \xi e_\lambda$.
  (ii) Writing $e_\lambda e_\lambda = (s_\lambda a_\lambda)(s_\lambda a_\lambda)$, we can apply (i) with $r = a_\lambda s_\lambda$ to conclude $e_\lambda e_\lambda = \eta e_\lambda$.
  (iii) As a matter of principle, e_λ contains e, whose coefficient is nonvanishing. Therefore, $e_\lambda e_\lambda$ also contains e with a nonvanishing coefficient $\Rightarrow \eta \neq 0$.
- **Theorem:** e_λ associated with Θ_λ is a primitive idempotent, generating an irrep of S_n on $\widetilde{S}_n$.
  Proof: We already know that e_λ is idempotent. It is primitive because $e_\lambda r e_\lambda = s_\lambda(a_\lambda r s_\lambda) a_\lambda = \xi e_\lambda, \forall r \in \widetilde{S}_n$.
- **Theorem:** The irreps generated by e_λ and $e_\lambda^p, \forall p \in S_n$, are equivalent.
  Proof: Since $e_\lambda^p = p e_\lambda p^{-1}$, we have $e_\lambda^p p e_\lambda = p e_\lambda e_\lambda = \eta p e_\lambda \neq 0$. Hence, e_λ^p and e_λ generate equivalent irreps.

- **Lemma:** For two distinct Young diagrams with $\lambda > \mu$, we have $a^q_\mu s^p_\lambda = s^p_\lambda a^q_\mu = 0$ and $e^q_\mu e^p_\lambda = 0$.
  Proof: There is at least one pair of numbers that appears simultaneously in one row of θ^p_λ and in one column of θ^q_μ. Let $\tilde{h}^p_\lambda$ and $\tilde{v}^q_\mu$ be the transpositions associated with these numbers. By Lemma, $\tilde{h}^p_\lambda s^p_\lambda = s^p_\lambda \tilde{h}^p_\lambda = s^p_\lambda$ and $\tilde{v}^q_\mu a^q_\mu = a^q_\mu \tilde{v}^q_\mu = -a^q_\mu$. Then,
- **Theorem:** e_λ and e_μ generate inequivalent irreps if the corresponding Young diagrams are different (i.e., if $\lambda > \mu$).
  Proof: For $p \in S_n$, we have $e_\mu p e_\lambda = e_\mu (p e_\lambda p^{-1}) p = e_\mu e^p_\lambda p = 0$, s.t. $e_\mu r e_\lambda = 0, \forall r \in \widetilde{S}_n$. Then Theorem guarantees e_μ and e_λ generate inequivalent representations.
- **Corr:** If $\lambda \neq \mu$, $e^p_\lambda e^q_\mu = 0, \forall p, q \in S_n$.
- **Theorem:** The irreducible symmetrizers $\{e_\lambda\}$ associated with the normal Young tableaux $\{\Theta_\lambda\}$ generate all inequivalent irreps of S_n.
  Proof: (i) The number of ineq. irreps of S_n = no. of Young diagrams.
  (ii) There is one e_λ associated with each Young diagram.
  (iii) Every e_λ generates an ineq. irrep.
- **Theorem:** (i) The left ideals generated by the idempotents associated with distinct standard Young tableaux are linearly independent; (ii) the direct sum of the left ideals generated by all standard tableaux spans the whole $\widetilde{S}_n$.

Exercises

1. Consider the group algebra $\widetilde{C}_3$ of the cyclic group, $\{C_3 : e, a, a^{-1}\}$. Compute the reduction of the regular representation of C_3, following these steps:
   a) Show that $e_1 = \frac{1}{3}(e + a + a^{-1})$ is a primitive idempotent.
   b) Construct $e_2 = xe + ya + za^{-1}$ and find x, y and z such that $e_1 e_2 = 0$ and $e_2 e_2 = e_2$.
   c) Check that e_2 is a primitive idempotent, i.e. that $e_2 r e_2 = \lambda_r e_2, \forall r \in C_3$.
2. Find the coefficients α, β, γ, δ, β' and γ' such that $e = \alpha e_1 + \beta e_2 + \gamma e_2^p + \delta e_3 + \beta' r_2 + \gamma' r_2^p$, i.e. derive Eq. (2).
3. Derive the representation matrices $D_2(p)$ and $D_2'(p)$ for S_3, corresponding to the normal Young tableau Θ_2 and $\Theta_2^{(23)}$. Find the similarity transformation S such that $D_2'(p) = S D_2(p) S^{-1}, \forall p \in S_3$.

Exercises

WKT5.1 Display all the standard Young tableaux of the group S_4. Enumerate the inequivalent irreducible representations of S_4 and specify their dimensions. Check the validity of the hook formula (see below).

WKT5.2 Repeat the above for S_5.

The hook formula: Consider a Young diagram Θ_λ corresponding to S_n. For each box (i, j) in the diagram, the hook length $h_\lambda(i, j)$ is defined as the number of boxes to the right, plus the number of boxes below, plus 1, for example:

8	6	5	3	2
7	5	4	2	1
4	2	1		
1				

Then the dimension of the irreducible representation associated with this Young diagram is:

$$\dim(V_\lambda) = n! \prod_{(i,j)\in\Theta_\lambda} \frac{1}{h_\lambda(i,j)}. \tag{5}$$