Symmetries in Physics
Lecture 3

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Lecture contents

1.2.6. Orthonormality of irrep matrices (REM)

- **Theorem:** Let μ label inequivalent, irreducible representations $D_\mu(g)$ of G. The following *orthonormality condition* holds:

$$\frac{n_\mu}{n_G} \sum_g D^\dagger_\mu(g)^k{}_i D_\nu(g)^j{}_l = \delta_{\mu\nu} \delta^j_i \delta^k_l, \tag{1}$$

where $n_\mu \equiv$ dimension of the μ-representation and $n_G =$ order of G.

I.2.7. Completeness of irrep matrices

- **Cor:** The number of inequivalent irreps of a finite group is restricted by $\sum_\mu n_\mu^2 \leq n_G$.
  Proof: The matrix elements $D_\mu(g)^i{}_j$ can be regarded as vectors of size n_G, with elements corresponding to $g \in G$. The labels (i, j) take n_μ^2 values and $\sum_\mu n_\mu^2$ represents the number of vectors in the set. The inequality follows since the number of mutually orthogonal (also linearly independent) vectors $\leq n_G \equiv$ dimension of the vector space.
  Theorem: (i) n_μ for the inequivalent irreps satisfy $\sum_\mu n_\mu^2 = n_G$; (ii) The representation matrices satisfy the *completeness relation:*

$$\sum_{\mu,l,k} \frac{n_\mu}{n_G} D_\mu(g)^l{}_k D_\mu^\dagger(g')^k{}_l = \delta_{gg'}. \tag{2}$$

  Proof: for (i) will be given later [see Eq. (8)]; accepting (i), the proof of (ii) is automatic, since $D_\mu(g)^l{}_k$ represent n_G mutually orthogonal vectors in a vector space of size n_G, which must therefore be complete.
- **Obs:** Since for abelian groups, $n_\mu = 1$, we must have n_G inequivalent irreps.

I.2.8. Orthonormality and completeness of irreducible χ

- **Lemma:** The sum of $U^\mu(g)$ over any (conjugation) class ζ_i is:

$$A_i^\mu = \sum_{h\in\zeta_i} U_\mu(h) = \frac{n_i}{n_\mu}\chi_i^\mu E. \qquad (3)$$

Proof: It is easy to check that $U_\mu(g)A_i^\mu U_\mu(g)^{-1} = A_i^\mu$, since $ghg^{-1}\in\zeta_i, \forall h\in\zeta_i$ and $g\in G$. By Schur's lemma 1, $A_i = c_i E$. Taking the trace gives $c_i n_\mu = n_i\chi_i^\mu$.

- **Theorem:** The characters of ineq. irreps of G satisfy

Orthonormality: $\qquad$ Completeness:

$$\sum_i \frac{n_i}{n_G}(\chi_i^\mu)^\dagger\chi_i^\nu = \delta_{\mu\nu}, \qquad \frac{n_i}{n_G}\sum_\mu \chi_i^\mu(\chi_j^\mu)^\dagger = \delta_{ij}. \qquad (4)$$

Proof: (i) Multiplying Eq. (1) by $\delta_k^i\delta_j^l$ gives $\frac{n_\mu}{n_G}\sum_g(\chi_g^\mu)^\dagger\chi_g^\nu = n_\mu^2\delta_{\mu\nu}$. Splitting now the sum over g with respect to the conjugation classes ζ_i ($1\le i\le n_c$) and taking into account that $\chi_g^\mu\to\chi_i^\mu, \forall g\in\zeta_i$ proves the Orthonormality relation.
(ii) Summing Eq. (2) over $g\in\zeta_i$ and $g'\in\zeta_j$ and using Eq. (3) gives

$$\sum_{\mu,l,k}\frac{n_\mu}{n_G}\frac{n_i n_j}{n_\mu^2}\chi_i^\mu(\chi_j^\mu)^\dagger\delta_k^l\delta_l^k = \frac{n_i n_j}{n_G}\sum_\mu\chi_i^\mu(\chi_j^\mu)^\dagger.$$

Performing the same summation on the RHS of Eq. (3) gives $n_i\delta_{ij}$.

- χ_i^μ can be viewed as an $n_c \times n_c$ matrix ($\mu \equiv$ line index; $i \equiv$ column index).
- Defining the *normalized characters* via $\tilde{\chi}_i = (n_i/n_G)^{1/2}\chi_i$, Eq. (4) becomes

$$(\tilde{\chi}_i^\mu)^\dagger \tilde{\chi}_i^\nu = \delta_{\mu\nu}, \quad (\tilde{\chi}_i^\mu)^\dagger \tilde{\chi}_j^\mu = \delta_{ij}. \tag{5}$$

- **Cor:** The number of inequivalent irreps for any finite group G is equal to the number of distinct classes n_c of G.
- **Theorem:** In the reduction for a given representation $U(G)$, the number of times a_ν that $U_\nu(G)$ is equal to

$$a_\nu = \sum_i \frac{n_i}{n_G} (\chi_i^\nu)^\dagger \chi_i, \qquad \chi_i = \mathrm{tr}[U(g \in \zeta_i)]. \tag{6}$$

Proof: Since $U(G) = \sum_{\mu\oplus} a_\mu U_\mu(G)$, we have $\chi_i = \sum_\mu a_\mu \chi_i^\mu$. Eq. (6) folllows after using the orthogonality relation (5).

I.2.9. Condition for irreducibility

- **Theorem:** A necessary and sufficient condition for a representation $U(G)$ with characters $\{\chi_i\}$ to be irreducible is that

$$\sum_i \frac{n_i}{n_G} |\chi_i|^2 = \tilde{\chi}^\dagger \cdot \tilde{\chi} = 1 \tag{7}$$

  Proof: Using Eq. (5), $\tilde{\chi}^\dagger \cdot \tilde{\chi} = \sum_{\mu,\nu} (a_\mu \tilde{\chi}^\mu)^\dagger \cdot (a_\nu \tilde{\chi}^\nu) = \sum_\mu |a_\mu|^2$. If there is only one irrep, then clearly $\sum_\mu |a_\mu|^2 = 1$. If $\tilde{\chi}^\dagger \cdot \tilde{\chi} = 1$, then there is one and only one irrep for which $a_\mu = 1$.

I.2.10. The regular representation

- **Def:** The *regular representation* of the group G with multiplication law $g_i g_j = g_k$ is given by the matrix $(\Delta_i)^m{}_j = \delta^m_k$.
- **Theorem** (decomposition of the regular representation): the regular representation contains every inequivalent irrep μ precisely n_μ times, and

$$\sum_\mu n_\mu^2 = n_G. \tag{8}$$

Proof: Let $(\Delta_a)^i{}_j$ be the representation matrix corresponding to element a. For the identity, $(\Delta_e)^i{}_j = \delta^i_j$ and $\chi_e = n_G$. For any other element $b \neq e$, we have $bg_i \neq g_i$ and $\chi_b = (\Delta_b)^i_i = 0$. Eq. (6) implies:

$$a^R_\nu = \sum_i \frac{n_i}{n_G} (\chi^\nu_i)^\dagger \chi^R_i = \frac{n_e}{n_G} (\chi^\nu_e)^\dagger n_G = n_\nu,$$

where we used that $\chi^\nu_e = n_\nu$, $n_e = 1$ and $\chi^R_e = 1$. Since $n_G = \sum_\mu a_\mu n_\mu$, we have $n_G = \sum_\mu n_\mu^2$.

- As a consequence, one can get all irreps by reducing the RR.

I.2.11. Direct product representations

- **Def:** The *direct product vector space* $W = U \otimes V$ of U and V consists of $W = \mathrm{span}\{\hat{\mathbf{w}}_k; k = (i,j); i = 1, \dots n_u; j = 1, \dots n_v\}$, where $n_u = \dim(U)$ and $n_v = \dim(V)$.
- **Def:** The *direct product representation* of the representations $D_\mu(G)$ on U and $D_\nu(G)$ on V is $D_{\mu\times\nu}(G)$, defined on W, and satisfies: $D(g)\,|w_k\rangle = |w_{k'}\rangle\, D(g)^{k'}{}_k$, with $D(g)^{k'}{}_k = (D_\mu)^{i'}{}_i (D_\nu)^{j'}{}_j$.
- It is clear that $\chi_i^{\mu\times\nu} = \chi_i^\mu \chi_i^\nu$.
- $D_{\mu\times\nu} \sim \sum_{\lambda\oplus} D_\lambda$ is usually reducible, even if D_μ and D_ν are irreducible.
- **Def:** The *Clebsch-Gordan coefficients* are the matrix elements $\langle ij(\mu\nu)\alpha\lambda l\rangle$ defined by $|w_{\alpha\lambda l}\rangle = \sum_{i,j} |w_{i,j}\rangle \, \langle ij(\mu,\nu)\alpha\lambda l\rangle$, where $1 \le \alpha \le a_\lambda$ and $1 \le l \le n_\lambda$.
- **Theorem:** The C-B coeffs $\langle \alpha\lambda l(\mu\nu)ij\rangle = \langle ij(\mu\nu)\alpha\lambda l\rangle^*$ satisfy

$$\text{Orthonormality:} \quad \sum_{\alpha\lambda l} \langle i'j'(\mu\nu)\alpha\lambda l\rangle \, \langle \alpha\lambda l(\mu\nu)ij\rangle = \delta_i^{i'}\delta_j^{j'}, \tag{9}$$

$$\text{Completeness:} \quad \sum_{ij} \langle \alpha'\lambda' l'(\mu\nu)ij\rangle \, \langle ij(\mu\nu)\alpha\lambda l\rangle = \delta_\alpha^{\alpha'}\delta_\lambda^{\lambda'}\delta_l^{l'}.$$

Proof: Follows directly from the orthonormality and completeness of the bases $\hat{w}_{i,j}$ and $\hat{w}^\lambda_{\alpha,l}$.

I.2.12. Reduction of product representations

- **Theorem:** The similarity transformation composed of C-B coeffs. decomposes $D^{\mu\times\nu}$ into its irreducible components. The following reciprocal relations hold:

$$D_\mu(g)^{i'}{}_i D_\nu(g)^{j'}{}_j = \langle i'j'(\mu\nu)\alpha\lambda l'\rangle\, D_\lambda(g)^{l'}{}_l\, \langle \alpha\lambda l(\mu\nu)ij\rangle\,,$$
$$\delta^{\alpha'}_{\alpha}\delta^{\lambda'}_{\lambda} D_\lambda(g)^{l'}{}_l = \langle \alpha'\lambda' l'(\mu\nu)i'j'\rangle\, D_\mu(g)^{i'}{}_i D_\nu(g)^{j'}{}_j\, \langle ij(\mu\nu)\alpha\lambda l\rangle\,. \tag{10}$$

Proof: We have $U(g)\,|w_{i,j}\rangle = |w_{i',j'}\rangle\, D_\mu(g)^{i'}{}_i D_\nu(g)^{j'}{}_j$ and $U(g)\,|w_{\alpha\lambda l}\rangle = |w_{\alpha\lambda l'}\rangle\, D_\lambda(g)^{l'}{}_l$. Noting that $|w_{i,j}\rangle = \sum_{\alpha,\lambda,l} |w_{\alpha\lambda l}\rangle\, \langle\alpha\lambda l(\mu\nu)ij\rangle$ proves the first relation. Then writing $|w_{\alpha\lambda l}\rangle = \sum_{i,j} |w_{i,j}\rangle\, \langle ij(\mu\nu)\alpha\lambda l\rangle$ and using Eq. (9) gives the second eq.

I.2.13. Irreducible basis vectors

- **Def:** A set of basis vectors $\{\hat{\mathbf{e}}_{\mu;i}, i = 1, \ldots n_\mu\}$ which transforms under $U(G)$ as $U(g)\,|e_{\mu;i}\rangle = |e_{\mu;j}\rangle\, D_\mu(g)^j{}_i$ forms an *irreducible set* transforming according to the μ-representation of G.
- **Theorem:** Let $\{\hat{\mathbf{u}}_{\mu;i}, i = 1, \ldots n_\mu\}$ and $\{\hat{\mathbf{v}}_{\nu;j}, j = 1, \ldots n_\nu\}$ be two sets of irreducible basis vectors w.r.t. G on V. If the irreps. μ and ν are not equivalent, then the two invariant subspaces are orthogonal.
  Proof: Let us compute the scalar product of two basis vectors:

$$
\begin{aligned}
\langle v_\nu^j | u_{\mu;i}\rangle &= \frac{1}{n_G}\sum_g \langle v_\nu^j | U^\dagger(g) U(g) | u_{\mu;i}\rangle \\
&= \underbrace{\left(\frac{n_\mu}{n_G}\sum_g D_\nu^\dagger(g)^j{}_k D_\mu(g)^l{}_i\right)}_{\text{Orthogonality relation:}\delta_{\mu\nu}\delta_i^j\delta_k^l} \frac{1}{n_\mu}\langle v_\nu^k | u_{\mu;l}\rangle \\
&= \delta_{\mu\nu}\delta_i^j n_\mu^{-1}\langle v_\nu^k | u_{k;\mu}\rangle\,. \qquad (11)
\end{aligned}
$$

I.2.14. Projection operators

- Let $U(G)$ be a rep. of G on V and $D_\mu(G)$ a matrix irrep of G. The operators

$$P^j_{\mu i} = \frac{n_\mu}{n_G} \sum_g D^{-1}_\mu(g)^j{}_i U(g) \tag{12}$$

transform irreducibly according to the μ representation (keeping j fixed).
Proof: Consider $|x\rangle \in V$ and $g \in G$. Then

$$\begin{aligned} U(g)P^j_{\mu i}\,|x\rangle &= \frac{n_\mu}{n_G} \sum_{g'} U(gg')\,|x\rangle\, D^{-1}_\mu(g')^j{}_i \\ &= \frac{n_\mu}{n_G} \sum_{g''} U(g'')\,|x\rangle\, D^{-1}_\mu(g'')^j{}_k D^{-1}_\mu(g^{-1})^k{}_i \\ &= P^j_{\mu k}\,|x\rangle\, D_\mu(g)^k{}_i. \end{aligned} \tag{13}$$

- **Cor:** An irreducible invariant subspace corresponding to the μ irrep can be generated using the basis $\{P^j_{\mu i}\,|x\rangle\,, i = 1, \ldots n_\mu\}$, starting from some (arbitrary) $|x\rangle \in V$.

- **Theorem:** Consider the irreducible basis $\{\hat{\mathbf{e}}_{\nu;k}, k = 1, \ldots n_\nu\}$. Then

$$P^{j}_{\mu i} |e_{\nu;k}\rangle = |e_{\nu;i}\rangle \, \delta_{\mu\nu} \delta^{j}_{k}. \tag{14}$$

  Proof: $P^{j}_{\mu i} |e_{\nu;k}\rangle = |e_{\nu;l}\rangle \frac{n_\mu}{n_G} \sum_g D^{\nu}(g)^{l}{}_{k} D^{\dagger}_{\mu}(g)^{j}{}_{i} = |e\rangle^{\nu}_{i} \, \delta_{\mu\nu} \delta^{j}_{k}$.
- **Cor:** $P^{j}_{\mu i} P^{l}_{\nu k} = \delta_{\mu\nu} \delta^{j}_{k} P^{l}_{\mu i}$.
  Proof: since $P^{l}_{\nu k} |x\rangle$ is an irreducible set $\forall |x\rangle \in V$, Eq. (14) implies the relation in the corollary.
- **Cor:** $U(g) = \sum_{\mu,i,j} P^{j}_{\mu i} D_{\mu}(g)^{i}{}_{j}, \forall g \in G$.
- **Cor:** $U(g) P^{l}_{\nu k} = \sum_i P^{l}_{\nu i} D_{\nu}(g)^{i}{}_{k}$.
- **Def:** $P_{\mu i} \equiv P^{j=i}_{\mu i}$ are projection operators onto the basis $\{\hat{\mathbf{e}}_{\mu;i}\}$.
  Proof: $P_{\mu i} P_{\nu k} = P^{j=i}_{\mu i} P^{l=k}_{\nu k} = \delta_{\mu\nu} \delta^{j=i}_{k} P^{l=k}_{\mu i} = \delta_{\mu\nu} \delta_{ik} P_{\mu k}$.
- **Def:** $P_\mu = \sum_i P_{\mu i} = \frac{n_\mu}{n_G} \sum_g [\chi_\mu(g)]^{-1} U(g)$ are projection operators onto the irreducible invariant space V_μ.
  Proof: $P_\mu P_\nu = P^{i}_{\mu i} P^{j}_{\nu j} = \delta_{\mu\nu} \delta^{i}_{j} P^{j}_{\nu i} = \delta_{\mu\nu} P_\nu$.
- **Theorem:** P_μ and $P_{\mu i}$ are complete in the sense that $\sum_\mu P_\mu = E$.
  Proof: Consider $\{\hat{\mathbf{e}}_{\nu;k}\}$ the basis vectors of any irr. inv. subsp. of V. Then $P_\mu |e_{\nu;k}\rangle = P^{i}_{\mu i} |e_{\nu;k}\rangle = |e_{\nu;i}\rangle \, \delta_{\mu\nu} \delta^{i}_{k} = |e_{\nu;k}\rangle \, \delta_{\mu\nu}$. Summing over all invariant spaces of V gives $\sum_\mu P_\mu |e_{\nu;k}\rangle = |e_{\nu;k}\rangle$.

I.2.15. Wigner-Eckart theorem

- **Def:** A set of operators $\{O_{\mu;i}, i = 1, \ldots n_\mu\}$ on V, transforming under G as

$$U(g)O_{\mu;i}U(g)^{-1} = O_{\mu;j}D_\mu(g)^j{}_i, \tag{15}$$

with $g \in G$ and $D_\mu(G)$ an irr. matrix rep. of G, forms *a set of irreducible operators* corresponding to the μ-representation (a.k.a. *irreducible tensors*).

- Acting with $O_{\mu;i}$ on $|e_{\nu;j}\rangle$ gives a vector transforming under the product representation $D^{\mu\times\nu}$, since

$$\begin{aligned} U(g)O_{\mu;i}\,|e_{\nu;j}\rangle &= U(g)O_{\mu;i}U^{-1}(g)U(g)\,|e_{\nu;j}\rangle \\ &= O_{\mu;k}\,|e_{\nu;l}\rangle\, D_\mu(g)^k{}_i D_\nu(g)^l{}_j. \end{aligned} \tag{16}$$

- **Theorem:** (Wigner-Eckart) Let $\{O_{\mu;i}\}$ be a set of irreducible tensor operators. Then:

$$\langle e^l_{\alpha\lambda}|O_{\mu;i}|e^\nu_j\rangle = \langle \alpha\lambda l(\mu\nu)ij\rangle\,\langle\lambda|O_\mu|\nu\rangle_\alpha\,, \tag{17}$$

where $\langle\lambda|O_\mu|\nu\rangle_\alpha = n_\lambda^{-1}\sum_k \langle e^k_\lambda|\Psi_{\alpha\lambda k}\rangle \equiv$ *reduced matrix element*.

Proof of the Wigner-Eckart theorem: I

- Since $O_{\mu;i}\,|e_{\nu;j}\rangle$ transforms under $U(G)$ as the direct product representation $D_{\mu\times\nu}(g)$, it can be expanded w.r.t. a set of vectors $|\Psi^{\lambda}_{\alpha,l}\rangle$:

$$O_{\mu,i}\,|e_{\nu,j}\rangle = \sum_{\alpha\lambda l} |\Psi_{\alpha\lambda l}\rangle\,\langle \alpha\lambda l(\mu\nu)ij\rangle\,, \tag{18}$$

with $\langle \alpha\lambda l(\mu\nu)ij\rangle$ being the C-G coeffs. connecting representation the copy α of representation λ to the direct product basis $|e^{\mu}_i\rangle \otimes |e^{\nu}_j\rangle$.

- Now we show that $|\Psi_{\alpha\lambda l}\rangle$ forms an irreducible set transforming under the λ representation of G, by applying $U(g)$ on Eq. (18):

$$\sum_{\alpha\lambda l} U(g)\,|\Psi_{\alpha\lambda l}\rangle\,\langle \alpha\lambda l(\mu\nu)ij\rangle = \sum_{\alpha\lambda l;i'j'} |\Psi_{\alpha\lambda l}\rangle\,\langle \alpha\lambda l(\mu\nu)i'j'\rangle\, D_\mu(g)^{i'}{}_i D_\nu(g)^{j'}{}_j.$$

- Multiplying by $\langle ij(\mu\nu)\alpha\lambda l\rangle$, summing over i,j and using the orthogonality relation (9) for the C-G coeffs. gives:

$$U(g)\,|\Psi_{\lambda\alpha l}\rangle = \sum_{\alpha'\lambda' l'} |\Psi_{\alpha'\lambda' l'}\rangle \sum_{i'j';ij} \langle \alpha'\lambda' l'(\mu\nu)i'j'\rangle\, D_\mu(g)^{i'}{}_i D_\nu(g)^{j'}{}_j\,\langle ij(\mu\nu)\alpha\lambda l\rangle\,.$$

- The last sum represents just $\delta_{\alpha\alpha'}\delta_{\lambda\lambda'}D_\lambda(g)^{l'}{}_l = \langle e^{l'}_{\alpha',\lambda'}|U(g)|e_{\alpha\lambda l}\rangle$:

$$\langle e^{l'}_{\alpha'\lambda'}|U(g)|e_{\alpha\lambda l}\rangle = \sum_{ij,i'j'} \langle \alpha'\lambda' l'(\mu\nu)i'j'\rangle\, D_\mu(g)^{i'}{}_i D_\nu(g)^{j'}{}_j\,\langle ij(\mu\nu)\alpha\lambda l\rangle\,.$$

Proof of the Wigner-Eckart theorem: II

- Now we show that $|\Psi_{\alpha\lambda l}\rangle \sim |e_{\alpha\lambda l}\rangle$, where $|e^{\alpha\lambda l}\rangle$ is an orthogonal basis.
- Writing $|\Psi_{\alpha\lambda l}\rangle \equiv \Psi\,|e_{\alpha\lambda l}\rangle = |e_{\alpha\lambda l'}\rangle\,\Psi^{l'}{}_{l}$ and using $U(g)\,|\Psi_{\alpha\lambda l}\rangle = |\Psi_{\alpha\lambda l'}\rangle\,D_\lambda(g)^{l'}{}_{l}$, we find

$$U(g)\,|\Psi_{\alpha\lambda l}\rangle = |e_{\alpha\lambda n}\rangle\,D_\lambda(g)^{n}{}_{m}\Psi^{m}{}_{l} = |e_{\alpha\lambda n}\rangle\,\Psi^{n}{}_{m}D_\lambda(g)^{m}{}_{l}, \tag{19}$$

  which shows that $\Psi D_\lambda(g) = D_\lambda(g)\Psi$.
- According to Schur's lemma 1, Ψ is proportional to the identity operator on the subspace $\{\hat{\mathbf{e}}_{\alpha\lambda l}, l = 1, \ldots n_\lambda\}$, with the proportionality constant being $\langle\lambda|O_\mu|\nu\rangle_\alpha$, the reduced matrix element:

$$O_i^\mu\,|e_{\nu,j}\rangle = \sum_{\alpha\lambda l} |e_{\alpha\lambda l}\rangle\,\langle\alpha\lambda l(\mu\nu)ij\rangle\,\langle\lambda|O_\mu|\nu\rangle_\alpha\,. \tag{20}$$

Exercises

WKT3.5 Find the set of unitary representation matrices $D_\mu(p)$, with $p \in S_3$ for the $2D$ irreps of S_3 ($\mu = 1, 2, 3$) and list the corresponding character table.

WKT3.6 Find the similarity transformation that reduces the following $2D$ representation of the $C_2 = \{e, a\}$ group into diagonal form:

$$e \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad a \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{21}$$

WKT3.8 Let (x_1, y_1) and (x_2, y_2) be coordinates of two 2-vectors which transform independently under D_3 transformations. Consider the function space V spanned by the monomials x_1x_2, x_1y_2, y_1x_2 and y_1y_2. Show that the realization of the group D_3 on this $4D$ space is the direct product representation of that on the $2D$ space with itself.

WKT3.9 Reduce the $4D$ rep. of D_3 from the previous problem into its irreducible comps. Evaluate the Clebsch-Gordan coeffs.

Exercises

WKT3.11 Construct the character table for S_4. [Hint: Make use of the irreps of any factor groups that may exist. Then complete the table by using the orthonormality and completeness relations.]

WKT4.2 Let $G = S_3$ and $V = V_2 \times V_2$ ($V_2 \equiv 2D$ vector space). Starting with the basis vectors $\hat{\mathbf{e}}_x\hat{\mathbf{e}}_x$, $\hat{\mathbf{e}}_x\hat{\mathbf{e}}_y$, $\hat{\mathbf{e}}_y\hat{\mathbf{e}}_x$ and $\hat{\mathbf{e}}_y\hat{\mathbf{e}}_y$, construct four new basis vectors which transform irreducibly under S_3. Use the projection operator technique.