Symmetries in Physics
Lecture 2

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Lecture contents

Chapter 1. Discrete symmetry groups

- I.1. Basic notions of abstract group theory
- I.2. **Group representations**
- I.3. Representations of the symmetric group; Young diagrams

I.1.8. Cosets

- **Def:** Let H be a subgroup of G and let $p \in G$ but $p \notin H$. Then, the set pH is called a *left coset* of H. Similarly, Hp is a *right coset* of H.
- **Obs:** The cosets of H are not subgroups, because they do not contain e.
- **Obs:** Each coset has exactly the same number of elements as H.
- **Lemma:** Two left (right) cosets of H either coincide completely, or else they have no elements in common at all.
  Proof: Let pH and qH be the two cosets. Assume $ph_i = qh_j$ for some $h_i, h_j \in H$. Then
  $q^{-1}p = h_j h_i^{-1} \in H \Rightarrow q^{-1}pH = H \Rightarrow pH = qH$. $\square$
- **Theorem (Lagrange):** The order of a finite group must be an integer multiple of the order of any of its subgroups.

I.1.9. Factor (quotient) groups

- **Obs:** If H is an invariant subgroup, then $pHp^{-1} = H$ and its left cosets are also right cosets: $pH = Hp$.
- **Obs:** The cosets of an invariant subgroup form a group:
  - $(pH)(qH) = (pq)H$, since $ph_i qh_j = pq(q^{-1}h_i q)h_j = pqh_k$, with $h_k \in H$;
  - $H = eH$ is the identity element;
  - $p^{-1}H$ is the inverse of pH;
  - $pH(qH \cdot rH) = (pH \cdot qH) \cdot rH) = (pqr)H$.
- **Theorem:** If H is an invariant subgroup of G, the set of cosets endowed with the law of multiplication $pH \cdot qH = (pq)H$ form a group, called the *factor* (or *quotient*) *group* of G. The factor group G/H is of order n_G/n_H.
- **Ex:** $H = \{e, a^2\}$ is an invariant group of C_4. Together with its coset $M = \{a, a^3\}$, they form the factor group C_4/H. Since $HM = M = MH$, $HH = H$ and $MM = H$, both H and C_4/H are of order 2 and are $\simeq C_2$.

I.1.10. Homomorphisms

- **Def:** A *homomorphism* from a group G to another group G' is a mapping (not necessarily one-to-one) which preserves group multiplication. In other words, if $g_i \in G \to g_i' \in G'$ and $g_1 g_2 = g_3$, then $g_1' g_2' = g_3'$.
- **Obs:** The isomorphism is a particular case of homomorphism.
- **Theorem:** Let f be a homomorphism from G to G'. Denote $K = \{a \in G; a \xrightarrow{f} e' \in G'\}$. Then K forms an invariant subgroup of G. Moreover, the factor group G/K is isomorphic to G'.

  Proof: (i) Since $e \xrightarrow{f} e'$, $e \in K$. For $a, b \in K$, $ab \xrightarrow{f} e' \cdot e' = e'$, hence $ab \in K$. If $a \in K$, then $a^{-1} \xrightarrow{f} (e')^{-1} = e'$ and $a^{-1} \in K$. Therefore, K is a subgroup.
  (ii) Let $a \in K$ and $g \in G$. Then $gag^{-1} \xrightarrow{f} g'e'(g')^{-1} = e'$ and $gag^{-1} \in K$. Hence, K is an invariant subgroup.
  (iii) The elements of G/K are the cosets pK. Let $pK \xrightarrow{\rho} p' \in G'$, where $p \xrightarrow{f} p'$. If $\rho(pK) = \rho(qK)$, then $\rho(q^{-1}pK) = (q^{-1}p)' = \rho^{-1}(qK)\rho(pK) = e'$, hence $q^{-1}pK = K$ and $pK = qK$. Thus, the mapping $G/K \xrightarrow{\rho} G'$ is one-to-one. Since $\rho(pK)\rho(qK) = \rho(pqK)$, multiplication is preserved by ρ and ρ is an isomorphism.
- **Obs:** K is called the *kernel* or *center* of the homomorphism f.

I.1.11. Direct products

- **Def:** Let H_1 and H_2 be subgroups of G with the properties: (i) $\forall h_1 \in H_1$ and $h_2 \in H_2$, $h_1 h_2 = h_2 h_1$ (the elements commute); and (ii) $\forall g \in G$, $\exists h_1 \in H_1$ and $h_2 \in H_2$ s.t. $g = h_1 h_2$. In this case, G is said to be the *direct product group* of H_1 and H_2; symbolically, $G = H_1 \otimes H_2$.
- **Obs:** $G = H_1 \otimes H_2 \Rightarrow H_1$ and H_2 must be invariant subgroups of G.
- **Obs:** $G = H_1 \otimes H_2 \Rightarrow G/H_2 \simeq H_1$ and $G/H_1 \simeq H_2$.
- **Ex:** Consider $C_6 = \{e = a^6, a, a^2, a^3, a^4, a^5\}$ with subgroups $H_1 = \{e, a^3\}$ and $H_2 = \{e, a^2, a^4\}$. Since G is abelian, $h_1 h_2 = h_2 h_1$. Moreover, $a = a^3 a^4$ and $a^5 = a^3 a^2$, hence $C_6 = H_1 \otimes H_2$. Since $H_1 \simeq C_2$ and $H_2 \simeq C_3$, we have $C_6 = C_2 \otimes C_3$.

I.2. Group representations

I.2.2. Linear vector spaces. Bra-ket notation

- Vectors in general linear vector spaces are denoted using Dirac's $|\rangle$ (*ket*) or $\langle|$ (*bra*) symbols.
- Multiplication by scalars is $|\alpha x\rangle = \alpha \cdot |x\rangle = |x\rangle \, \alpha$.
- A basis of the vector space is denoted by $|e_i\rangle$.
- Its dual basis $\langle e^i|$ is defined such that $\langle e^i|e_j\rangle = \delta^i_j$.
- A vector $\mathbf{x}$ has components $|x\rangle = |e_i\rangle \, x^i$.
- Its dual is defined by Hermitian conjugation, $\langle x| = (|x\rangle)^\dagger \equiv x^\dagger_i \, \langle e^i|$, s.t. $x^\dagger_i = (x^i)^*$.
- The bra-ket between two vectors defines the scalar product on the vector space, $\langle x|y\rangle = x^\dagger_i y^i$.
- An operator A is a linear functional $A : V \to V$ s.t. $A\,|x\rangle = |Ax\rangle \in V$.
- ${A^i}_j = \langle e^i|A|e_j\rangle$ represent the components of A.
- The product of two operators is defined by $AB\,|x\rangle = A\,|Bx\rangle$ and ${(AB)^i}_j = {A^i}_k {B^k}_j$.

I.2.2. Representations

- In physics, we are interested in the effect of symmetry transformations on the solutions of partial differential or integral eqs.
- These solutions usually form a *linear vector space* (e.g., the Hilbert space in QM).
- Group theory describes the *realization* of group transformations as *linear transformations on vector spaces*.
- Linear transformations (or operators) on linear vector spaces form a (generally non-abelian) group.
- **Def:** If there is a homomorphism from a group G to a group of operators $U(G)$ on a linear vector space V, we say $U(G)$ forms a *representation of the group G*.
- The dimension of $U(G)$ is the dimension of V.
- $U(G)$ is *faithful* if the homomorphism is also an isomorphism.
- A *degenerate representation* is one which is not faithful.
- $g \in G \to U(g)$, s.t. $U(g_1)U(g_2) = U(g_1 g_2)$.

Finite-dimensional representations

- Consider a basis $\{\hat{\mathbf{e}}_i, i = 1, 2, \ldots n\}$ in V_n.
- The operators $U(g)$ are realized as $n \times n$ matrices $D(g)$, defined via $U(g)\,|e_i\rangle = |e_j\rangle\, D(g)^j{}_i$.
- Because $U(g_1)U(g_2) = U(g_1 g_2)$, we have $D(g_1)D(g_2) = D(g_1 g_2) \Rightarrow$ the matrices $D(G)$ form the *matrix representation* of G.
- $|x\rangle$ transforms as $U(g)\,|x\rangle = |x'\rangle = |e_i\rangle\, x'^i$, with $x'^i = D(g)^i{}_j x^j$.
- **Ex:** $\{D_2 : e, h(\text{refl. about } y), v(\text{refl. about } x), r(\text{rot. by } \pi)\}$ and $V = \mathrm{span}(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2)$. Then $D(e) = \mathrm{diag}(1, 1)$ and

$$D(h) = \mathrm{diag}(-1, 1); \quad D(v) = \mathrm{diag}(1, -1); \quad D(r) = \mathrm{diag}(-1, -1).$$

- **Ex:** $G = \{R(\phi), 0 \le \phi < 2\pi\}$ is the group of $2D$ rotations and:

$$\mathbf{x}' = U(\phi)\mathbf{x} = \hat{\mathbf{e}}_i x'^i, \quad x'^i = D(\phi)^i{}_j x^j, \quad D(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}.$$

- Consider a function $f : \mathbb{R}^2 \to \mathbb{C}$. Under a rotation,

$$f \xrightarrow{g \in G} f'(\mathbf{x}) = f(\mathbf{x}'), \quad \mathbf{x}' = U(g^{-1})\mathbf{x}.$$

- **Theorem:** (i) If the group G has a non-trivial invariant subgroup H, then any representation of the factor group $K = G/H$ is also a (degenerate) representation of G; (ii) Conversely, if $U(G)$ is a degenerate representation of G, then G has at least one invariant subgroup H such that $U(G)$ defines a faithful representation of the factor group G/H.
  Proof: Let $\{K : gH, g \in G\}$ be the set of cosets of H. Then $g \in G \to k = gH \in K \to U(k)$ on V is a homomorphism from G to $U(K)$, forming a representation. Since H is a non-trivial invariant subgroup, $g \to k = gH$ is a many-to-one mapping $\Rightarrow$ the representation is not faithful. (ii) Proof already given.

I.2.3. Irreducible, inequivalent representations

- **Def:** Two representations $U(G)$ and $U'(G)$ are *equivalent* if they are related by a similarity transformation S, i.e. $U'(G) = SU(G)S^{-1}$.
- **Def:** The *character* $\chi(g)$ of $g \in G$ in a representation $U(G)$ is defined as $\chi(g) = \mathrm{Tr}[U(g)]$.
- **Obs:** All elements in a given class of G have the same characters, because $\mathrm{Tr}\, U(p)U(g)U(p^{-1}) = \mathrm{Tr}\, U(g)$.
- Direct sum representations: If for some choice of basis on V_n, $D(g \in G) = \mathrm{diag}(D_1(g), D_2(g))$ is block diagonal, then $D = D_1 \oplus D_2$.
- **Def:** V_1 is an *invariant subspace* of V w.r.t. $U(G)$ if $U(g)\,|x\rangle \in V_1, \forall \mathbf{x} \in V_1$ and $g \in G$. An invariant subspace is *minimal* (*proper*) if it does not contain any non-trivial invariant subspace w.r.t. $U(G)$.
- **Def:** $U(G)$ on V is *irreducible* if there is no non-trivial invariant subspace in V w.r.t. $U(G)$. Otherwise, the representation is *reducible* and, if the orthogonal complement of the invariant subspace is also invariant w.r.t. $U(G)$, then the representation is *fully reducible* or *decomposable*.
- **Ex:** R_2 on E_2: $\hat{\mathbf{e}}_\pm = \frac{1}{\sqrt{2}}(\mp\hat{\mathbf{e}}_1 - i\hat{\mathbf{e}}_2)$ span 1-dimensional invariant subspaces, since $U(\phi)\hat{\mathbf{e}}_\pm = \hat{\mathbf{e}}_\pm e^{\mp i\phi}$.

Reducible representations

- Consider an invariant subspace V_1 of V w.r.t. $U(G)$, having $\dim(V_1) = n_1 < n = \dim(V)$. Arranging the basis vectors such that $\hat{\mathbf{e}}_i \in V_1$ for $1 \leq i \leq n_1$, we have

$$U(g)\,|e_i\rangle = |e_j\rangle\, D(g)^j{}_i \in V_1 \quad \Rightarrow \quad D(g) = \begin{pmatrix} D_1(g) & D'(g) \\ 0 & D_2(g) \end{pmatrix}, \tag{1}$$

  where $D_i(g)$ are $n_i \times n_i$ matrices $(n_2 = n - n_1)$.
- If $V_2 = \mathrm{span}(\hat{\mathbf{e}}_i, i = n_1 + 1, \ldots n)$ is also invariant, then $D'(g) = 0$ and $D(g)$ becomes block-diagonal.
- If V^μ is an invariant subspace of V, restricting $U(G)$ to V^μ gives a lower-dimensional representation $U^\mu(G)$ of G.
- If V^μ cannot be further reduced, $U^\mu(G)$ is an irreducible representation and V^μ is a *proper* (*irreducible invariant*) *subspace* w.r.t. G.

I.2.4. Unitary representations

- **Def:** If V is an inner product space and if $U(g)^\dagger = U(g)^{-1}$ are unitary $\forall g \in G$, then $U(G)$ is a *unitary representation*. Equivalently,

$$\langle U(g)x|U(g)y\rangle = \langle x|y\rangle\,. \tag{2}$$

- **Def:** For completeness: an operator is Hermitian if $A^\dagger = A$, i.e. $\langle Ax|y\rangle = \langle x|Ay\rangle$.
- **Theorem:** If a unitary representation is reducible, then it is also decomposable (i.e., fully reducible).
  Proof: Let $V_1 = \mathrm{span}(\hat{\mathbf{e}}_i, i = 1, 2 \ldots n_1)$ be an invariant subspace and $V_2 = \mathrm{span}(\hat{\mathbf{e}}_i, i = n_1 + 1, \ldots n)$ be its complement. Since V_1 is invariant, $|e_i(g)\rangle = U(g)\,|e_i\rangle \in V_1$ for $1 \leq i \leq n_1$. Since $U(G)$ is unitary, $\langle e^j(g)|e_i(g)\rangle = \langle e^j|U^\dagger(g)U(g)|e_i\rangle = \delta^j{}_i$ vanishes for all $n_1 < j \leq n$ and $1 \leq i \leq n_1$. Since $|e_1(g)\rangle \in V_1$, this means $|e_2(g)\rangle \in V_2 \Rightarrow U(g)\,|x\rangle \in V_2 \forall \mathbf{x} \in V_2$ and V_2 is an invariant subspace w.r.t. $U(g)$.

Unitary representations of finite groups

- **Maschke's theorem:** Every representation $D(G)$ of a finite group on an inner product space is equivalent to a unitary representation.
  Proof: We need the similarity transformation S s.t. $SD(g)S^{-1} = U(g)$ is unitary $\forall g \in G$. Since unitarity is established based on a scalar product, we introduce a new scalar product $(x, y) \equiv \langle Sx|Sy\rangle = \sum_g \langle D(g)x|D(g)y\rangle$. Then, S can be regarded as a transformation from the basis orthogonal w.r.t. $\langle|\rangle$ to one orthogonal w.r.t. $(,)$. We now show $U(g)$ is unitary:

$$\begin{aligned}\langle U(g)x|U(g)y\rangle &= \langle SD(g)S^{-1}x|SD(g)S^{-1}y\rangle \\ &= \sum_{g'} \langle D(g')D(g)S^{-1}x|D(g')D(g)S-1y\rangle \\ &= \sum_{g''} \langle D(g'')S^{-1}x|D(g'')S^{-1}y\rangle \\ &= (S^{-1}x, S^{-1}y) = \langle x|y\rangle \,.\end{aligned}$$

  Key elements in the above proof include: (i) the summation over all group elements (non-trivial for continuous groups); and (ii) the validity of the rearrangement lema.
- **Corr:** All reducible reps. of finite groups are fully reducible.

1.2.5. Schur's lemmas

- **Def:** Let V_1 and V_2 be complementary subspaces w.r.t. $U(G)$, and $U_1(G)$, $U_2(G)$ denote operators which coincide with $U(G)$ on these subspaces. Then clearly $V = V_1 \oplus V_2$ and $U(g) = U_1(g) + U_2(g)$ is the *direct sum representation* of $U_1(G)$.
- If either V_1 and V_2 are reducible w.r.t. G, the representation can be further decomposed, until $U(G)$ is fully reduced: $U(G) = \sum_{\mu\oplus} a_\mu U_\mu(G)$, where $\mu = 1, 2, \dots$ labels the inequivalent irreducible representations $U_\mu(G)$ and $a_\mu \equiv$ their multiplicity.
- **Schur's Lemma 1:** Let $U(G)$ be an irreducible representation of G on V, and A an arbitrary operator on V. If $AU(g) = U(g)A, \forall g \in G$, then $\exists \lambda \in \mathbb{C}$ s.t. $A = \lambda E$.
  Proof: (i) Without loss of generality, we take $U(G)$ unitary and A hermitian.
  (ii) We take a basis $\{\hat{\mathbf{u}}_{\alpha,i}\}$ of V consisting of the eigenvectors of A: $A\,|u_{\alpha,i}\rangle = |u_{\alpha,i}\rangle\,\lambda_i$, with λ_i being the distinct eigenvalues of A and α labels different vectors with the same λ_i.
  (iii) Consider $V_i = \mathrm{span}(\hat{\mathbf{u}}_{\alpha,i}, \alpha = 1, 2, \dots)$. Then $AU(g)\,|u_{\alpha,i}\rangle$ $= U(g)\,|u_{\alpha,i}\rangle\,\lambda_i \in V_i$, s.t. V_i is an invariant subspace of V.
  (iv) Since $U(G)$ is irreducible on V, V has no non-trivial invariant subspaces $\Rightarrow V_i = V$ and $A = \lambda E$ has a single eigenvalue λ.

- **Schur's lemma 2:** Let $U(G)$ and $U'(G)$ be two irreps of G on V and V', resp, and let $A : V' \to V$ be a linear transf. satisfying $AU'(g) = U(g)A, \forall g \in G$. It follows that either (i) $A = 0$, or (ii) $V \simeq V'$ and $U(G)$ is equivalent to $U'(G)$.
  Proof: (i) Consider the range
  $R = AV' = \{\mathbf{x} \in V; \mathbf{x} = A\mathbf{x}' \text{ for some } \mathbf{x}' \in V'\}$. Then,
  $U(g)\,|x\rangle = U(g)A\,|x'\rangle = AU'(g)\,|x'\rangle = A\,|U'(g)x'\rangle \in R, \forall g \in G$,
  hence R is an invariant subspace of V. Since $U(G)$ is irreducible, $R = 0$ (hence $A = 0$) or $R = V$.
  (ii) Let $N' = \{\mathbf{x}' \in V' \text{ s.t. } A\mathbf{x}' = 0\}$ be the null space of A. Then
  $AU'(g)\,|x'\rangle = U(g)A\,|x'\rangle = U(g)\,|0\rangle = 0$, s.t.
  $U'(g)\,|x'\rangle \in N', \forall g \in G$, implying N' is an invariant subspace of V'.
  Since $U'(G)$ is irreducible $\Rightarrow N' = V'$ (hence $A = 0$) or $N' = 0$.
  (iii) if $R = V$ and $N' = 0$, then $A\,|x'\rangle = A\,|y'\rangle$ implies $|x'\rangle = |y'\rangle$ and A is an isomorphism, while $U(G) = AU'(G)A^{-1}$ is equivalent to $U'(G)$.

1.2.6. Orthonormality of irrep matrices

- **Theorem:** Let μ label inequivalent, irreducible representations $D_\mu(g)$ of G. The following *orthonormality condition* holds:

$$\frac{n_\mu}{n_G}\sum_g D^\dagger_\mu(g)^k{}_i D_\nu(g)^j{}_l = \delta_{\mu\nu}\delta^j_i\delta^k_l, \qquad (3)$$

where $n_\mu \equiv$ dimension of the μ-representation and $n_G =$ order of G.
Proof: (i) Without loss of generality, we consider $D^\dagger_\mu(g) = D^{-1}_\mu(g), \forall g \in G$. In order to apply Schur's 2^{nd} lemma, we construct $M_x = \sum_g D^\dagger_\mu(g) X D^\nu(g)$, with X some $n_\mu \times n_\nu$ matrix. Since $D^{-1}_\mu(p) M_x D^\nu(p) = M_x, \forall p \in G$, Schur's second lemma implies either $\mu \neq \nu$ and $M_x = 0$, or $\mu = \nu$ and $M_x = c_x E$.
(ii) We take $X \to (X^k_l)^i{}_j = \delta^k_j \delta^i_l$. Then:

$$(M^k_l)^m{}_n = \delta_{\mu\nu}\sum_g D^\dagger_\mu(g)^m{}_i (X^k_l)^i{}_j D_\mu(g)^j{}_n = \delta_{\mu\nu}\sum_g D^\dagger_\mu(g)^m{}_l D_\mu(g)^k{}_n.$$

By (i), $M^k_l = c^k_l E$ and c^k_l can be found by tracing both sides:

$$n_\mu c^k_l = \sum_g [D_\mu(g) D^\dagger_\mu(g)]^k{}_l = n_G \delta^k_l,$$

by which $c^k_l = (n_G/n_\mu)\delta^k_l$.

Examples of irreps

- For abelian groups, where all irreps are 1D, the *orthonormality of irreducible representation matrices theorem* implies $n_G^{-1} \sum_g d_\mu^\dagger(g) d_\nu(g) = \delta_{\mu\nu}$.
- For $\{C_2 : e, a\}$, we have the identity representation $(e, a) \xrightarrow{d_1} (1, 1)$. A second inequivalent representation d_2 must be orthogonal to d_1: $(e, a) \xrightarrow{d_2} (1, -1)$. There is no other irreducible representation of C_2.
- For $\{D_2 : e, a, b, c\}$, with $a^2 = b^2 = c^2 = e$ and $ab = c$, we have:

$\mu\backslash g$	e	a	b	c
1	1	1	1	1
2	1	1	−1	−1
3	1	−1	1	−1
4	1	−1	−1	1

d_1: The identity representation: $(e, a, b, c) \xrightarrow{d_1} (1, 1, 1, 1)$;

d_2: The invariant subgroup $\{e, a\}$ induces the factor group $\{(e, a), (b, c)\} \sim C_2$. C_2 has 2 inequivalent irreps: identity (equivalent to d_1) and $(1, -1)$, such that $(e, a, b, c) \xrightarrow{d_2} (1, 1, -1, -1)$.

d_3: Same for $\{e, b\}$: $(e, a, b, c) \xrightarrow{d_3} (1, -1, 1, -1)$.

d_4: Same for $\{e, c\}$: $(e, a, b, c) \xrightarrow{d_3} (1, -1, -1, 1)$.

Exercises

WKT2.7 Prove that $G = H_1 \otimes H_2$ implies $G/H_1 \simeq H_2$ and $G/H_2 \simeq H_1$.

WKT3.1 Consider the six transformations associated with the dihedral group D_3. Let $V = \mathrm{span}\{\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y\}$ be the $2D$ Euclidean space. Write down the matrix rep. $D(g)$ on V for all $g \in D_3$.

WKT3.2 Let $R_2 = \{R(\phi), 0 \leq \phi < 2\pi\}$ be the group of continuous rotations in a plane around the origin and $V = E_2$ the $2D$ Euclidean plane.
a) Write down the 2D representation of these rotations with respect to the basis $\{\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y\}$.
b) Show that the representation at a) can be decomposed into two $1D$ representations.

WKT3.4 Prove that if $D(G)$ is any representation of a finite group G on an inner product space V, and $\mathbf{x}, \mathbf{y} \in V$, then $(x, y) = \sum_{g \in G} \langle D(g)x | D(g)y \rangle$ defines a new scalar product on V.

Appendix 1: Linear vector spaces

- **Def:** A *linear vector space* V is a set $\{|x\rangle, |y\rangle, \dots\}$ on which two operations, $+$ (*addition*) and $\cdot$ (*multiplication by a number*) are defined, s.t. the following basic axioms hold:

(i) If $|x\rangle \in V$ and $|y\rangle \in V$, then $|x\rangle + |y\rangle \equiv |z\rangle \in V$;

(ii) If $|x\rangle \in V$ and α is a (real or complex) number, then $|\alpha x\rangle = |x\rangle \alpha \in V$;

(iii) There exists a *null vector* $|0\rangle$ s.t. $|x\rangle + |0\rangle = |x\rangle, \forall |x\rangle \in V$.

(iv) $\forall |x\rangle \in V$, $\exists |-x\rangle \in V$ s.t. $|x\rangle + |-x\rangle = |0\rangle$.

(v) The operation $+$ is *commutative* and *associative*:

$$|x\rangle + |y\rangle = |y\rangle + |x\rangle, \qquad (|x\rangle + |y\rangle) + |z\rangle = |x\rangle + (|y\rangle + |z\rangle).$$

(vi) $1 \cdot |x\rangle = |x\rangle$;

(vii) Multiplication by a number is associative: $\alpha \cdot |\beta x\rangle = (\alpha\beta) \cdot |x\rangle = |x\rangle (\alpha\beta)$.

(viii) The two operations are *distributive*:

$$(\alpha + \beta) \cdot |x\rangle = |x\rangle \alpha + |x\rangle \beta, \qquad \alpha \cdot (|x\rangle + |y\rangle) = |x\rangle \alpha + |y\rangle \alpha.$$

- **Def:** A set of vectors $\{\mathbf{x}_i \in V, i = 1, \ldots m\}$ are *linearly independent* if $|x_i\rangle \alpha^i = 0$ implies $\alpha^i = 0 \forall i$. Conversely, the vectors $\mathbf{x}_i$ are linearly dependent if $\exists \{\alpha^i\}$ not all zero, s.t. $|x_i\rangle \alpha^i = 0$.
- **Def:** A set of vectors $\{\hat{\mathbf{e}}_i, i = 1, \ldots n\}$ forms a *basis* of V if (i) they are linearly independent; and (ii) $\forall \mathbf{x} \in V, \exists \{x^i\}$ s.t. $|x\rangle = |e_i\rangle x^i$.
- The numbers x^i are the *components* of $\mathbf{x}$ w.r.t. the basis $\{\hat{\mathbf{e}}_i\}$.
- Vector spaces which have a basis with a finite number of elements are *finite dimensional*.
- **Theorem:** All bases of a finite dimensional vector space have the same number of elements.
- **Def:** The number of elements n in a basis of a finite dimensional vector space V is called the *dimension* of V.
- **Def:** Two vector spaces V and V' are said to be isomorphic to each other if there exists a $1:1$ mapping $\mathbf{x} \in V \to \mathbf{x}' \in V'$ s.t. $(|x\rangle + |y\rangle \alpha)' = |x'\rangle + |y'\rangle \alpha, \forall \mathbf{x}, \mathbf{y}$ and α.
- **Theorem:** Every n-dimensional linear vector space V_n is isomorphic to the space of n ordered complex numbers $\mathbb{C}^n$; hence all n-dimensional linear vector spaces are isomorphic to each other.

- **Def:** A subset V_n of V forming a linear vector space w.r.t. the same $+$ and $\cdot$ as in V is called a *subspace* of V.
- **Theorem:** Given V_n and a subspace V_m ($m < n$), one can always choose a basis $\{\hat{\mathbf{e}}_i, i = 1, \ldots n\}$ for V_n s.t. the first m basis vectors lie in V_m.
- **Def:** Let V_1 and V_2 be subspaces of V. We say $V = V_1 \oplus V_2$ is the *direct sum* of V_1 and V_2, provided (i) $V_1 \cap V_2 = 0$; and (ii) $\forall \mathbf{x} \in V$, $\exists \mathbf{x}_1 \in V_1$ and $\mathbf{x}_2 \in V_2$ s.t. $|x\rangle = |x_1\rangle + |x_2\rangle$.
- If m_1 and m_2 are the dimensions of V_1 and V_2, then V has dimension $n = m_1 + m_2$.
- Let $V = V_1 \oplus V_2$. The elements of V can be denoted $(\mathbf{x}_1, \mathbf{x}_2)$, where $\mathbf{x}_1 \in V_1$ and $\mathbf{x}_2 \in V_2$. Then,

$$(\mathbf{x}_1, \mathbf{x}_2) + (\mathbf{y}_1, \mathbf{y}_2) = (\mathbf{x}_1 + \mathbf{y}_1, \mathbf{x}_2 + \mathbf{y}_2), \qquad \alpha(\mathbf{x}_1, \mathbf{x}_2) = (\alpha\mathbf{x}_1, \alpha\mathbf{x}_2).$$

Linear functionals and dual space

- **Def:** *Linear functionals* $f : V \to \mathbb{C}$ satisfy $f(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha f(\mathbf{x}) + \beta f(\mathbf{y})$.
- The set of linear functionals $f : V \to \mathbb{C}$ forms the vector space $\widetilde{V}$ dual to V, such that f becomes $\langle f|$ and $f(\mathbf{x}) = \langle f|x\rangle$, satisfying

$$\langle f|\alpha x + \beta y\rangle = \alpha \langle f|x\rangle + \beta \langle f|y\rangle , \qquad \langle \alpha f + \beta g|x\rangle = \alpha^* \langle f|x\rangle + \beta^* \langle g|x\rangle .$$

- The dual basis $\hat{\mathbf{e}}^j \in \widetilde{V}$ is defined by $\langle \tilde{e}^j|e_i\rangle = \delta^j{}_i$.
- **Theorem:** (i) the linear functionals $\tilde{\mathbf{e}}^j$ are linearly independent and (ii) $\exists f_i^\dagger$ s.t. $f \to \langle f| = f_i^\dagger \langle \tilde{e}^j|$.
- It follows that (i) $\widetilde{V}$ has dimension n and (ii) it is isomorphic to V.
- **Def:** An *inner* (*scalar*) *product* on V, $\langle|\rangle : V \times V \to \mathbb{C}$ satisfies: (i) $\langle x|y\rangle = \langle y|x\rangle^*$; (ii) $\langle x|\alpha_1 y_1 + \alpha_2 y_2\rangle = \alpha_1 \langle x|y_1\rangle + \alpha_2 \langle x|y_2\rangle$; (iii) $\langle x|x\rangle \geq 0$; and (iv) $\langle x|x\rangle = 0 \Leftrightarrow \mathbf{x} = 0$.
- **Def:** The *length* (*norm*) of a vector $\mathbf{x} \in V$ is $|x| = \langle x|x\rangle^{1/2}$.
- **Def** The angle θ between two vectors $\mathbf{x}, \mathbf{y}$ of a *real* vector space satisfies $\cos\theta = \langle x|y\rangle / |x||y|$.
- $V + \langle|\rangle \equiv$ *inner product space*.

- **Def:** $\langle x|y\rangle = 0 \Leftrightarrow \mathbf{x}$ and $\mathbf{y} = 0$ are orthogonal.
- **Theorem:** Any set of n orhtonormal vectors $\{\hat{\mathbf{u}}_i\}$ in an n-dimensional vector space V_n forms an orthonormal basis, s.t. (i) $|x\rangle = |u_i\rangle\, x^i$ with $x^i = \langle u^i|x\rangle$; (ii) $\langle x|y\rangle = x_i^\dagger y^i = \langle x|u_i\rangle\,\langle u^i|y\rangle$; (iii) $|x|^2 = x_i^\dagger x^i$, $\forall \mathbf{x}, \mathbf{y} \in V_n$.
- **Theorem:** Let $E_i = |e_i\rangle\,\langle e^i|$ (no summation) be the mapping $|x\rangle \to E_i\,|x\rangle = |e_i\rangle\, x^i$ (no summation). Then: (i) $E_i = 1, 2, \ldots n$ are linear operators on V; (ii) E_i are *projection operators* (idempotents); (iii) $\sum_{i=1}^{n} E_i = |e_i\rangle\,\langle e^i| = E$ is the identity operator (*the completeness relation*).
- **Theorem:** Let $\{\hat{\mathbf{e}}_i\}$ and $\{\hat{\mathbf{u}}_i\}$ be two orthonormal bases on V_n. If $|u_k\rangle = |e_i\rangle\, S^i{}_k$, with $(S^\dagger)^k{}_j = (S^j{}_k)^*$, then: (i) $(S^\dagger),_i S^i{}_k = \delta^l{}_k$; (ii) $S^i{}_k (S^\dagger)^k{}_j = \delta^i{}_j$; and (iii) $|e_i\rangle = |u_k\rangle\,(S^\dagger)^k{}_i$. Hence, $S^\dagger = S^{-1}$ and S is a unitary matrix.

Linear transformations (operators) on vector spaces

- **Def:** *Linear functionals* $F : V \to \mathbb{C}$ satisfy $F(\alpha \mathbf{x} + \beta \mathbf{y}) = \alpha F(\mathbf{x}) + \beta F(\mathbf{y})$.
- Denoting $f_i^\dagger = F(\hat{\mathbf{e}}_i)$ the components of F w.r.t. the basis $\hat{\mathbf{e}}_i$, we have $F(\mathbf{x}) = f_i^\dagger x^i = \langle f | x \rangle$, where $\langle f | = f_i^\dagger \langle e^i |$.
- **Def:** A *linear transformation* (operator) $A : V \to V'$ satisfies: (i) $|x\rangle \underset{A}{\longrightarrow} |Ax\rangle \equiv A\,|x\rangle \in V'$; and (ii) if $|y\rangle = |x_1\rangle\,\alpha_1 + |x_2\rangle\,\alpha_2 \in V$ then $|Ay\rangle = |Ax_1\rangle\,\alpha_1 + |Ax_2\rangle\,\alpha_2 \in V'$.