Symmetries in Physics
Lecture 1

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Lecture contents

I.1.1. Introduction
From mathematics to physics

- Group theory provides the natural mathematical language for describing symmetries of the physical world.
- The connection between the mathematical theory and its physical consequences was clearly formulated by Wigner and Weyl (and others), before 1930.
- The connection is of paramount importance in quantum mechanics, but it is very useful also in classical physics.
- Since the 1950's, group theory has become increasingly important. Nowadays, it permeates every branch of physics or of physical and life sciences.
- Examples: Internal symmetries in particle physics (isospin, colour); External symmetries (translations, rotations, Lorentz boosts).

Examples of continuous symmetries

- Spatial translations, $\mathbf{x} \to \mathbf{x} + \mathbf{a}$ with constant $\mathbf{a}$:
  - Based on the assumption of homogeneity of space;
  - Applicable to isolated systems;
  - Leads to conservation of linear momentum $\mathbf{p}$.
- Time translations, $t \to t + a_0$ with constant a_0:
  - Based on the assumption of homogeneity of time;
  - Applicable to isolated systems;
  - Leads to conservation of energy E.
- Rotations in 3D space: $x^i \to x^{i'} = R^{i'}{}_j x^j$
  - Based on the assumption of isotropy of space;
  - Applicable to central potentials;
  - Leads to conservation of angular momentum $\mathbf{L}$.
- Lorentz transformations: $x^\mu \to x^{\mu'} = \Lambda^{\mu'}{}_\nu x^\nu$:
  - Comprise rotations and boosts;
  - Boosts generalize Galilei transformations to special relativity;
  - Severely constrain the equations of relativistic quantum mechanics.

Examples of discrete symmetries

- Space inversion (parity transformation): $\mathbf{x} \to -\mathbf{x}$:
  - Equivalent to reflection in a plane + rotation through angle π;
  - Obeyed by most interactions, except the weak interaction.
- Time reversal, $t \to -t$:
  - A system with T symmetry would evolve backwards in time if all velocities simultaneously flip sign;
  - Obeyed by most interactions, except isolated instances (e.g., neutral K meson decay);
  - Not an exact symmetry in real systems, due to second law of thermodynamics.
- Discrete translations on a lattice: $\mathbf{x} \to \mathbf{x} + n\mathbf{a}$:
  - Appears in systems with periodic potentials (e.g., crystals);
  - Leads to quantization of momentum and infrared cutoff;
  - Leads to Bloch theorem.
- Discrete rotational symmetry of a lattice (point groups):
  - Subsets of the 3D rotation group and parity that leave a given lattice structure invariant;
  - There are 32 crystallographic point groups;
  - Together with discrete translations $\Rightarrow$ space groups of solid state physics.

Other important symmetries

- Permutation symmetry:
  - Permutations form the *symmetry group*;
  - Systems with identical particles are invariant under the interchange of these particles;
  - When the particles have several degrees of freedom, group theory is essential to extract symmetry properties of permissible physical states.
- Gauge invariance and charge conservation:
  - Both classical and quantum electrodynamics are invariant under *gauge transformations*;
  - Gauge symmetry is intimately related to charge conservation.
- Internal symmetries (nuclear and elementary particle physics):
  - These are symmetries of the internal degrees of freedom (spin, flavour, colour, etc);
  - Usually they are the unitary $U(N)$ or special unitary $SU(N)$ groups.
  - Isospin is the $SU(2)$ symmetry of invariance under $u \leftrightarrow d$ interchange of nuclear interaction;
  - $SU(3)_c$ represents the colour symmetry of QCD;
  - Gauging an internal symmetry leads to the introduction bosonic fields mediating the corresponding interaction.

I.1.2. What is a group?

Def: A set $\{G : a, b, c, \ldots\}$ is said to form a *group* if there is a binary operation $\cdot : G^2 \to G$, called *group multiplication*, which associates $a, b \to a \cdot b \in G$, such that:

- $\cdot$ is associative: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$, $\forall a, b, c \in G$;
- $\exists e \in G$ called identity, such that $a \cdot e = a$, $\forall a \in G$;
- $\forall a \in G$, $\exists a^{-1} \in G$ called the *inverse* of a, such that $a \cdot a^{-1} = e$.

The above imply $e^{-1} = e$; $a^{-1} \cdot a = e$; and $e \cdot a = a$, $\forall a \in G$.
Def: An *abelian group* G is one for which $ab = ba$, $\forall a, b \in G$.
Def: The *order* of a group is the number of elements of the group (if it is finite).

I.1.3. The cyclic groups C_n: multiplication tables

Group multiplication tables

	e	a
e	e	a
a	a	e

C_2

e	a	b
a	b	e
b	e	a

C_3

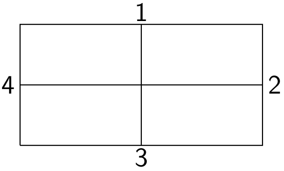
- The cyclic group C_n contains the $n \in \mathbb{N}^*$ elements given as $\{e, a, a^2, \ldots a^{n-1}\}$, with $a^n = e$ or $a^{n-1} = a^{-1}$.
- **Ex:** C_1 is the group that consists of a single element, e. Ex: $\{1\}$ and usual multiplication.
- **Ex:** C_2 consists of $\{e, a\}$, with $a \cdot a = e$. Ex: $\{1, -1\}$ and usual multiplication; parity inversion.
- **Ex:** C_3, with $\{e, a, a^{-1}\}$. Ex: $\{1, e^{2i\pi/3}, e^{4i\pi/3}\}$ under multiplication; symmetry operations of the equilateral triangle (rotations by $2\pi/3$ and $4\pi/3$).

I.1.4. Noncyclic groups: dihedral group

Group multiplication table of D_2

e	a	b	c
a	e	c	b
b	c	e	a
c	b	a	e

Configuration with D_2 symmetry



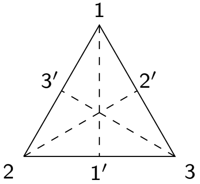
- C_1, C_2 and C_3 are necessarily cyclic.
- The simplest non-cyclic group has order R and is called the *dihedral group*, $\{D_2 : e, a, b, c\}$.
- Since the multiplication table is symmetric, D_2 is abelian.
- One example of D_2 is the symmetry operations on the rectangle above:
  - $e =$figure stays the same;
  - $a =$ reflection about $(1, 3)$;
  - $b =$ reflection about $(2, 4)$;
  - $c =$ rotation around the center by π.
- D_n represents the symmetry group of rotations of a regular polygon with n undirected sides.

I.1.5. Nonabelian groups: D_3

Group multiplication table of D_3

e	(12)	(23)	(13)	(123)	(132)
(12)	e	(123)	(132)	(23)	(13)
(23)	(132)	e	(123)	(13)	(12)
(13)	(123)	(132)	e	(12)	(23)
(123)	(13)	(12)	(23)	(132)	e
(132)	(23)	(13)	(12)	e	(123)

Config. w. D_3 symmetry



- The smallest *non-abelian group* is the *dihedral group* D_3, consisting of 6 elements:
  - identity transformation e;
  - reflection about the $(3, 3')$ axis, flipping 1 and 2: (12);
  - reflection about the $(1, 1')$ axis, flipping 2 and 3: (23);
  - reflection about the $(2, 2')$ axis, flipping 3 and 1: (31);
  - rotation by $2\pi/3$, permutting $[1, 2, 3] \to [3, 1, 2]$, denoted (123);
  - rotation by $4\pi/3$, permutting $[1, 2, 3] \to [2, 3, 1]$, denoted (132).
- The parentheses notation denotes permutations applied on (123):

$$(123) : 1 \to 2\&2 \to 3\&3 \to 1 \Rightarrow (123) \to (231).$$

- $(123) \cdot (12) = (13) \neq (12) \cdot (123) = (23) \Rightarrow$ nonabelian group.

I.1.6. Subgroups

Def: A subset $H \in G$ which forms a group under the same multiplication law as G is said to form a *subgroup* of G.

- D_2 has 3 subgroups identical to C_2, consisting of $\{e, a\}$; $\{e, b\}$; and $\{e, c\}$, since $a^2 = b^2 = c^2 = e$.
- D_3 has 4 subgroups: $\{e, (12)\}$; $\{e, (23)\}$; $\{e, (31)\}$; and $\{e, (123), (321)\}$. The first 3 are identical to C_2 and the fourth has the structure of C_3.

Matrix groups

- An important class of groups are those involving $n \times n$ matrices with standard matrix multiplications. Examples include:
- The *general linear group* $GL(n)$ consists of all invertible matrices;
- The *unitary group* $U(n)$ consists of all unitary matrices, satisfying $UU^\dagger = 1$;
- The *special unitary group* $SU(n)$ consists of unitary matrices with *unit determinant*;
- The *orthogonal group* $O(n)$ consists of real orthogonal matrices satisfying $OO^T = 1$.

Rearrangement lemma and Symmetric (permutation) group

Rearrangement Lema: If $p, b, c \in G$ and $pb = pc$ then $b = c$.

- Consider $\{G : g_1, g_2, \dots g_n\}$. Multiplication to the left by h gives

$$\{hg_1, hg_2, \dots hg_n\} = \{g_{h_1}, g_{h_2}, \dots g_{h_n}\}, \tag{1}$$

  where $(h_1, h_2, \dots h_n)$ is the permutation of $(1, 2, \dots n)$ determined by h.

- A permutation p can be represented as

$$p = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ p_1 & p_2 & p_3 & \cdots & p_n \end{pmatrix}, \qquad p^{-1} = \begin{pmatrix} p_1 & p_2 & p_3 & \cdots & p_n \\ 1 & 2 & 3 & \cdots & n \end{pmatrix}. \tag{2}$$

- A more convenient representation is based on the cyclical structure:

$$p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 4 & 1 & 2 & 6 \end{pmatrix} = (143)(25)(6) \equiv (143)(25). \tag{3}$$

- The inverse permutation can be obtained by listing each sequence in inverse order, $p^{-1} = (431)(52)(6) \equiv (143)(25)$.
- The set of $n!$ permutations of n objects forms the *permutation (symmetric) group* S_n.

Composition of cycles

- Let us consider the composition of two cycles: (123)(12).
- With the explicit representation of the permutations, we have

$$(132)(12) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = (23), \qquad (4)$$

  where the rules in the second permutation (12) are applied on the bottom row of the first permutation.
- By successive application on a set of objects (abc), we get

$$(132)(12)[abc] = (132)[bac] = [acb] = (23)[abc], \qquad (5)$$

  where after the first equal sign: position 1 is moved to pos. 2 and vice-versa; after the second equal sign, pos. 1 is moved to pos. 3 (b); pos. 3 moves to pos. 2 (c); pos. 2 moves to pos. 1 (a).
- Using cycle multiplication gives:

$$(132)(12) = (23). \qquad (6)$$

  The permutations are applied right-to-left. Within a permutation, the rules are read left-to-right: Pos. 1 goes to pos. 2 by (12); and pos. 2 goes to pos. 1 by (132) $\Rightarrow$ (1); pos. 2 goes to pos. 1 by (12); and pos. 1 goes to pos. 3 by (132) $\Rightarrow$ (23).

Isomorphism and Cayley's theorem

Def: Two groups G and G' are said to be *isomorphic* ($G \sim G'$) if there exists a one-to-one correspondence between their elements which preserves the law of group multiplication, e.g. if $g_i \in G \leftrightarrow g_i' \in G'$ and $g_1 g_2 = g_3$ in G, then $g_1' g_2' = g_3'$ in G' and vice-versa.

- The groups $\{e, a\}$ are isomorphic to C_2;
- The group $\{\pm 1, \pm i\}$ with usual multiplication is isomorphic to C_4.
- D_3 is isomorphic to S_3.

Cayley's theorem: Every group G of order n is isomorphic to a subgroup of S_n.
Proof: The rearrangement lema guarantees that $ag_i = g_{a_i}$ for $a, g_i \in G$, defining the permutation

$$a \in G \to p_a = \begin{pmatrix} 1 & 2 & \cdots & n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix} \in S_n. \tag{7}$$

Consider now $ab = c$. Then, $abg_i = ag_{b_i} = g_{a_{b_i}}$, while $cg_i = g_{c_i}$. Then, $p_c = p_a p_b$ and group multiplication is preserved. Thus, the group formed by the permutations p_a corresponding to $a \in G$ is a subgroup of S_n which is isomorphic to G. $\square$

- **Obs:** The subgroup of S_n isomorphic to the group G cannot contain elements with one-cycles (except e).
  Proof: A one-cycle corresponds to an element which is unchanged: $ag_i = g_i$, which can hold true only when $a = e$.
- **Obs:** The cycles which occur in any permutation associated with a given group element must all be of the same length.
  Proof: If p_a has multiple cycles and l is the length of the shortest cycle, then $p_{a^l} = p_a^l$ will contain l 1-cycles, which is forbidden unless all cycles in p_a are of the same length l and $p_a^l = e$.
- **Theorem:** If the order n of a group G is a prime number, it must be isomorphic to the cyclic group C_n.

Examples

C_3 vs. S_3:

- Consider $\{C_3 : e, a, b = a^2 = a^{-1}\}$ and let $(g_1, g_2, g_3) = (e, a, b)$.
- Since $eg_i = g_i$, we have $e \in C_3 \to e = (1)(2)(3) = () \in S_3$.
- Applying $g_2 = a$ gives $(a, b, e) = (g_2, g_3, g_1)$, hence $p_a = (132)$.
- Applying $g_3 = b = a^2$ gives $(b, e, a) = (g_3, g_1, g_2)$, hence $p_b = (123)$.

D_2 vs. S_4:

- $a(e, a, b, c) = (a, e, c, b) \Rightarrow p_a = (12)(34)$;
- $b(e, a, b, c) = (b, c, e, a) \Rightarrow p_b = (13)(24)$;
- $c(e, a, b, c) = (c, b, a, e) \Rightarrow p_c = (14)(23)$;
- D_2 is isomorphic to $\{e, (12)(34), (13)(24), (14)(23)\} \in S_4$.

$\{C_4 : e, a, b, c\} = \{C_4 : e, a, a^2, a^3\}$ vs. S_4:

- $a(e, a, a^2, a^3) = (a, a^2, a^3, e) \Rightarrow p_a = (1432)$;
- $a^2(e, a, a^2, a^3) = (a^2, a^3, e, a) \Rightarrow p_b = (13)(24)$;
- $a^3(e, a, a^2, a^3) = (a^3, e, a, a^2) \Rightarrow p_c = (1234)$;
- C_4 is isomorphic to $\{e, (1234), (13)(24), (1432)\} \in S_4$.

I.1.7. Conjugate elements and classes

- **Def:** An element $b \in G$ is said to be *conjugate* to $a \in G$ if $\exists p \in G$ s.t. $b = pap^{-1}$. Then, $b \sim a$ or b is *conjugate* to a.
  Ex: In S_3, $(12) \sim (13)$ because $(23)(12)(23)^{-1} = (13)$; also $(123) \sim (321)$ since $(12)(123)(12)^{-1} = (132)$.
- **Obs:** Conjugation is an *equivalence* relation, since it is:
  - *reflexive:* $a \sim a$;
  - *symmetric:* $a \sim b \Leftrightarrow b \sim a$;
  - *transitive:* $a \sim b$ and $b \sim c \Rightarrow a \sim c$.
- **Def:** Elements of a group which are conjugate to each other are said to form a (conjugate) *class*.
- **Obs:** Each element of a group belongs to one and only one class and e forms a class by itself.
- **Ex:** S_3 contains 3 classes: $\zeta_1 = \{e\}$; $\zeta_2 = \{(12), (23), (31)\}$; and $\zeta_3 = \{(123), (321)\}$. Generally, permutations with the same cycle structure belong to the same class.
- **Ex:** For 3D rotations, let $R_{\mathbf{n}}(\psi)$ denote the rotation of angle ψ about $\mathbf{n}$. Then $\{R_{\mathbf{n}}(\psi); \text{all } \mathbf{n}\}$ belong to the same class.

Conjugate and invariant subgroups

- **Def:** If H is a subgroup of G and $a \in G$, then $H' = \{aha^{-1}; h \in H\}$ also forms a subgroup of G and H' is said to be a *conjugate subgroup* to H.
- **Obs:** It can be shown that either H and H' are isomorphic, or they have only e in common.
- **Def:** An *invariant subgroup* H of G is one which is identical to all its conjugate subgroups.
- **Obs:** H is invariant $\Leftrightarrow$ it contains elements of G in complete subclasses.
- **Obs:** All subgroups of an abelian group are invariant subgroups.
- **Ex:** $H = \{e, a^2\}$ is an invariant subgroup of $C_4 = \{e = a^4, a, a^2, a^3\}$.
- **Ex:** $\{e, (123), (132)\}$ forms an invariant subgroup of S_3; but $\{e, (12)\}$ does not.
- **Def:** A group is *simple* if it does not contain any non-trivial invariant subgroup. A group is *semi-simple* if it does not contain any abelian invariant subgroup.

Exercises

WKT2.2 Show that there is only one group of order three, by explicitly constructing the multiplication table.

HFJ1.2 Write the following permutations in cyclic notation:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 1 & 4 & 8 & 5 & 7 & 2 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 5 & 4 & 1 & 8 & 9 & 6 & 7 & 2 \end{pmatrix}$$

WKT2.3 Construct the multiplication table of the permutation group S_3 using the cycle structure notation.

WKT2.5 Consider the permutation group S_4.

- a Enumerate the subgroups and classes of the group S_4.
- b Which of the subgroups are invariant ones?
- c Find the factor groups of the invariant subgroups.

Probleme

WKT2.8 Consider the dihedral group D_4, representing the symmetry group of the square, consisting of rotations around the center and reflections about the vertical, horizontal and diagonal axes.

- a Enumerate the group elements.
- b Enumerate the classes.
- c Enumerate the subgroups.
- d Identify the invariant subgroups.
- e Identify the factor groups.
- f Is D_4 the direct product of some of its subgroups?

HFJ1.6 The *centre* Z of a group G is defined as the set of elements z which commute with all elements of the group, i.e. $Z = \{z \in G | zg = gz, \forall g \in G\}$. Show that Z is an Abelian subgroup of G.

HFJ1.7 Using the multiplication table for D_3, write down the isomorphism of Cayley's theorem between D_3 and the relevant subgroup of S_6.