Bouncing Dirac particles: compatibility between MIT boundary conditions and Thomas precession

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The problem:

- **Dirac particle** which bounces off from a perfectly reflecting plane

- Determine the **rotation of the spin**

- Repeat the problem for a **classical particle**

- Compare the quantum and the classical rotation angles

Background facts:

- Long studied problem in GR: the dynamics of a spinning body

- The basic result: **Mathisson–Papapetrou–Dixon equations**

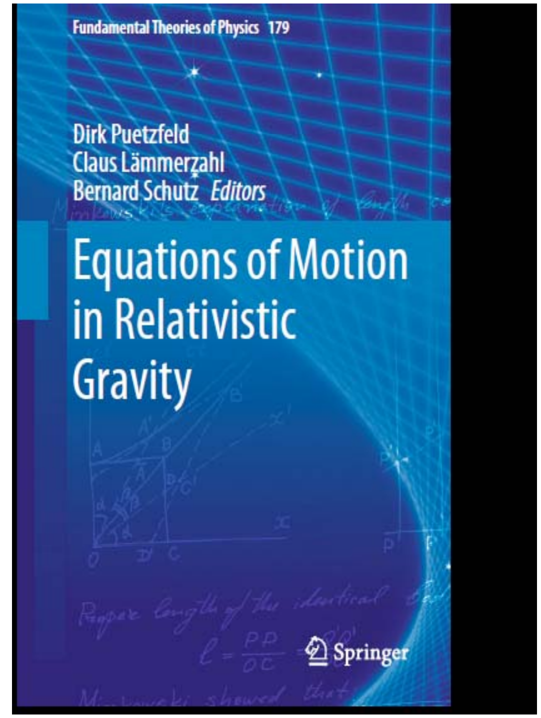
$$\begin{cases} \dfrac{D}{Ds}P^{\mu} = \dfrac{1}{2}R_{\rho\sigma\nu}{}^{\mu}S^{\rho\sigma}U^{\nu}, \\ \dfrac{D}{Ds}S^{\mu\nu} = P^{\mu}U^{\nu} - P^{\nu}U^{\mu}, \\ P^{\mu} = mU^{\mu} - \dfrac{DS^{\mu\nu}}{Ds}U_{\nu}, \end{cases}$$

- Supplementary conditions:

$$S^{\mu 0} = 0, \qquad S^{\mu\nu}U_{\nu} = 0, \qquad S^{\mu\nu}P_{\nu} = 0$$

Resource literature:

- Proceedings of the 2013 Bad Honnef Conference



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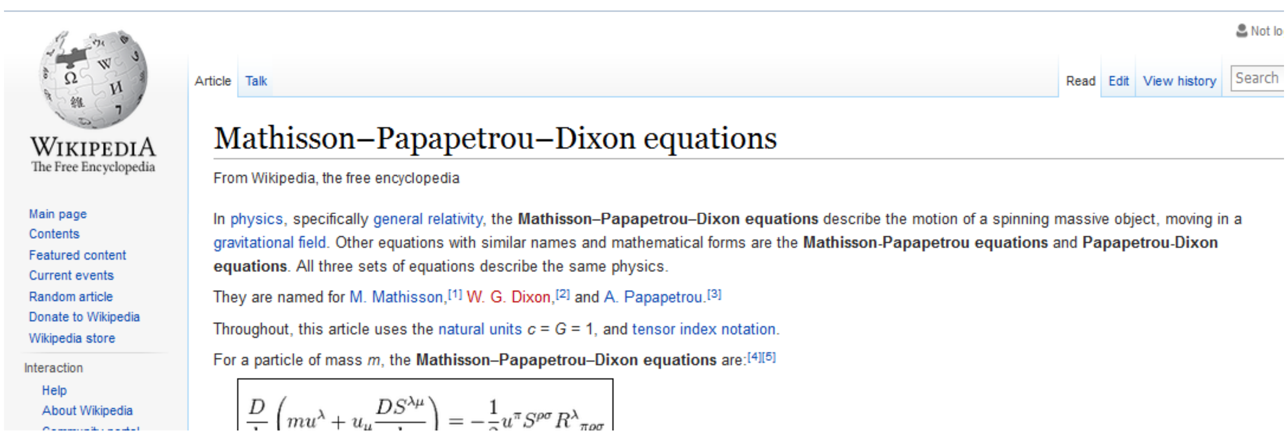
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Mathisson–Papapetrou–Dixon equations

From Wikipedia, the free encyclopedia

In physics, specifically general relativity, the **Mathisson–Papapetrou–Dixon equations** describe the motion of a spinning massive object, moving in a gravitational field. Other equations with similar names and mathematical forms are the **Mathisson-Papapetrou equations** and **Papapetrou-Dixon equations**. All three sets of equations describe the same physics.

They are named for M. Mathisson,[1] W. G. Dixon,[2] and A. Papapetrou.[3]

Throughout, this article uses the natural units $c = G = 1$, and tensor index notation.

For a particle of mass m, the **Mathisson–Papapetrou–Dixon equations** are:[4][5]

$$\frac{D}{ds}\left(mu^\lambda + u_\mu \frac{DS^{\lambda\mu}}{ds}\right) = -\frac{1}{2}u^\pi S^{\rho\sigma} R^\lambda{}_{\pi\rho\sigma}$$

• The **Dirac** particle = a **quantum** spinning particle

• The **MPD equations** can be indeed recovered from the **Dirac equation** in the classical limit

• Numerous calculations (various definitions of the spin operator / procedures to extract the classical limit)

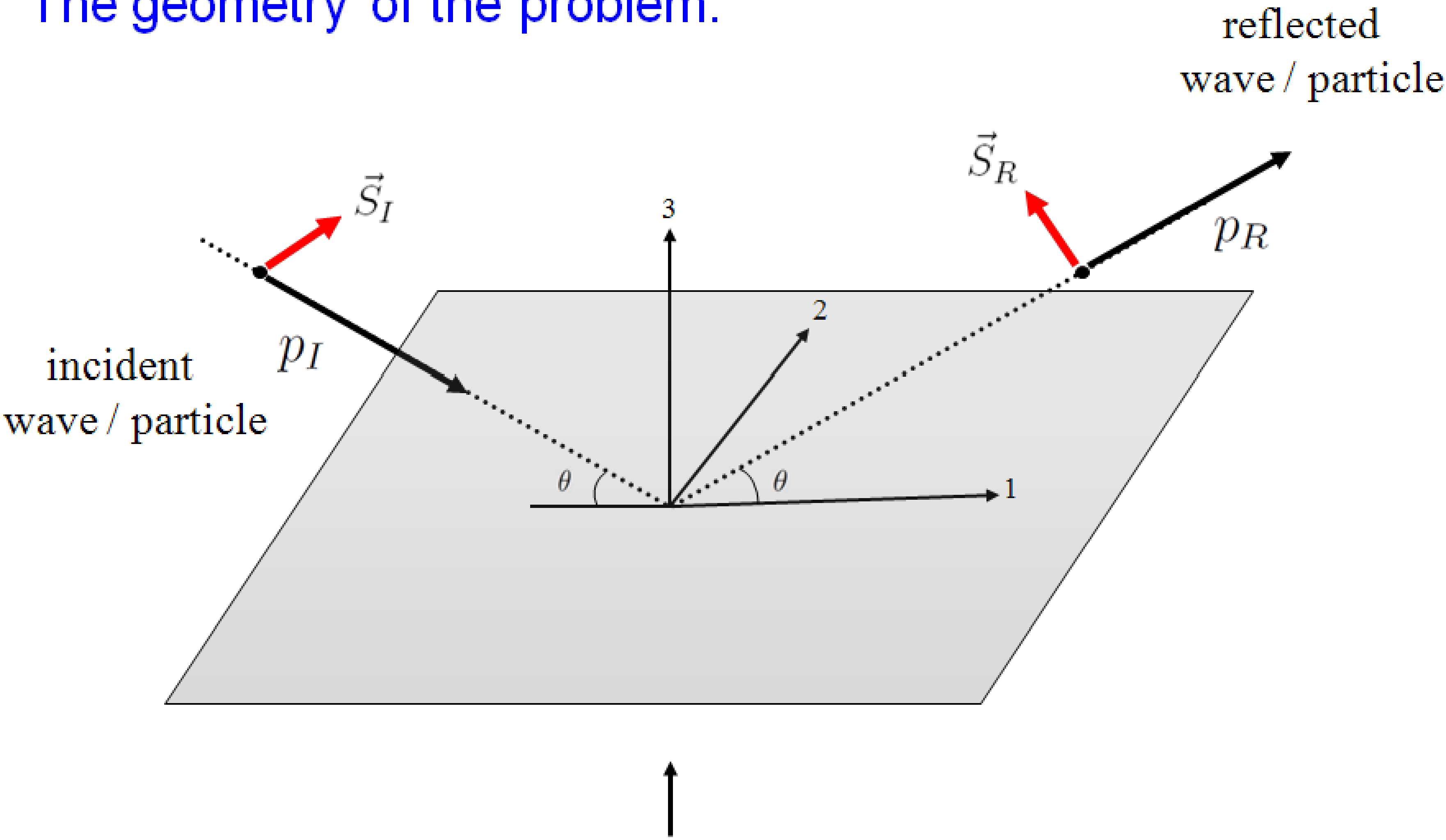
R. Rudiger, *``The Dirac equation and spinninng particles in general relativity,”* J. Proc. R. Soc. Lond. A **377,** 417 (1981)

J. Audretsch, *``Trajectories and spin motion of massive spin ½ particles in gravitational fields,”* J. Phys. A: Math. Gen. **14**, 411 (1981)

F. Cianfrani and G. Montani, *``Dirac equation in curved space-time vs. Papapetrou spinning particles”*, EPL **84**, 30008 (2008)

Y. Obukhov, A. Silenko and O. Teryaev, *``Spin in an arbitrary gravitational field”*, Phys. Rev. D **88**, 084014 (2013)

The geometry of the problem:



perfectly reflecting plane

The quantum solution:

- Boundary condition on the plane – MIT with chiral angle

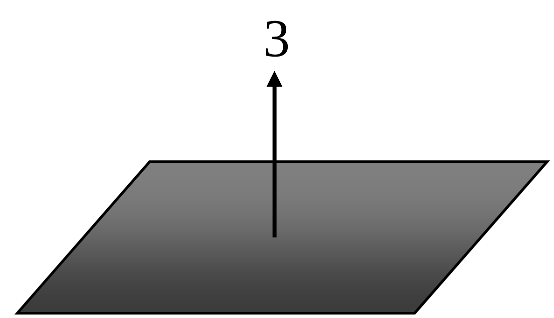
Chodos, Jaffe, Johnson and Thorn, ``*Baryon Structure in the Bag Theory,''* Phys. Rev. D 1974.

$$N^\mu \gamma_\mu \psi = i e^{i\eta\gamma_5} \psi$$

η - the chiral angle

ensures that the Dirac current through the plane **is zero**

$$N^\mu = (0,\ \vec{e}_3)$$



- MIT in the Dirac representation:

$$\gamma^0 = \left| \begin{matrix} I & 0 \\ 0 & -I \end{matrix} \right| \qquad \gamma^i = \left| \begin{matrix} o & \sigma_i \\ -\sigma_i & 0 \end{matrix} \right| \qquad \gamma^5 = \left| \begin{matrix} 0 & I \\ I & 0 \end{matrix} \right|$$

$$N^\mu = (0, \vec{e}_3) \qquad N^\mu \gamma_\mu \psi = i e^{i\eta\gamma_5} \psi \qquad \psi = \left| \begin{matrix} \Phi \\ \chi \end{matrix} \right|$$

$$\downarrow$$

$$\boxed{\begin{aligned} (\sigma_3 + \sin\eta)\,\Phi - i\cos\eta\,\chi = 0 \\ (\sigma_3 - \sin\eta)\,\chi + i\cos\eta\,\Phi = 0 \end{aligned}}$$

• **Plane wave** solutions of the Dirac equation:

$$u_{p,\xi}(x) = u(p,\xi)\, e^{-ip\cdot x} \qquad (1)$$

$$u(\hat{p},\xi) = \begin{vmatrix} \xi \\ 0 \end{vmatrix}, \quad \xi^{+}\xi = 1 \qquad (2)$$

ξ - defines the **spin** in the **proper frame** of the particle

$$u(p,\xi) = S(\Lambda_p)\, u(\hat{p},\xi)$$

pure boost

$$S(\Lambda_p) = e^{-i\alpha\vec{n}\cdot\vec{K}}, \quad \alpha = \operatorname{arctanh}\frac{|\vec{p}\,|}{(m^2+\vec{p}^{\,2})^{1/2}}, \quad \vec{n} = \frac{\vec{p}}{|\vec{p}|}$$

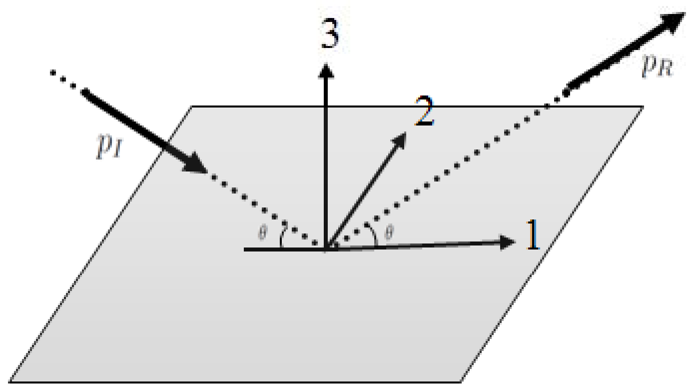
- General form of the Dirac spinors:

$$u(p,\,\xi) = \begin{vmatrix} \cosh\frac{\alpha}{2}\,\xi \\ \sinh\frac{\alpha}{2}\,\sigma_{\vec{n}}\,\xi \end{vmatrix}, \qquad \sigma_{\vec{n}} = \vec{n}\cdot\vec{\sigma}.$$

- The incident and reflected momenta:

$$p^{\mu}_{I/R} = m\,(\cosh\alpha,\ \sinh\alpha\,\vec{n}_{I/R})$$

$$\vec{n}_I = (\cos\theta,\ 0,\ -\sin\theta), \qquad \vec{n}_R = (\cos\theta,\ 0,\ \sin\theta).$$



• The total wave function:

$$\psi(x) = u(p_I,\, \xi_I)\, e^{-ip_I \cdot x} + u(p_R,\, \xi_R)\, e^{-ip_R \cdot x} \qquad (1)$$

$$u(p_I,\, \xi_I) = \begin{vmatrix} \cosh \frac{\alpha}{2}\, \xi_I \\ \sinh \frac{\alpha}{2}\, \sigma_I\, \xi_I \end{vmatrix}, \quad \sigma_I = \vec{n}_I \cdot \vec{\sigma} \qquad (2)$$

$$u(p_R,\, \xi_R) = \begin{vmatrix} \cosh \frac{\alpha}{2}\, \xi_R \\ \sinh \frac{\alpha}{2}\, \sigma_R\, \xi_R \end{vmatrix}, \quad \sigma_R = \vec{n}_R \cdot \vec{\sigma} \qquad (3)$$

ξ_I **-** defines the **initial (**incident) spin

ξ_R **-** defines **the final** (reflected) spin

• MIT selects the following solution:

(1) introduce:

$$A = 1 + i\cos\eta\sin\theta\tanh\frac{\alpha}{2}, \qquad (1)$$

$$\vec{B} = \cos\eta\cos\theta\tanh\frac{\alpha}{2}\,\vec{e}_2 + \sin\eta\,\vec{e}_3.$$

(2) construct the operator:

$$Q = AI + \vec{B}\cdot\vec{\sigma} \qquad (2)$$

(3) the initial and final two-spinors are related by:

$$\xi_R = \mathcal{U}\,\xi_I \qquad \mathcal{U} = (Q^+)^{-1}Q \qquad (3)$$

unitary matrix by definition *(2)*

• Extract the physical content from:

$$\mathcal{U} = e^{i\chi}\left(\cos\frac{\Delta\phi}{2}\,I - i\sin\frac{\Delta\phi}{2}\vec{\mathcal{N}}\cdot\vec{\sigma}\right)$$

irrelevant phase factor

SU(2) matrix = **rotation matrix** in the spin space

- Rotation **axis**: $$\vec{\mathcal{N}} = \frac{\vec{B}}{|\vec{B}|}$$

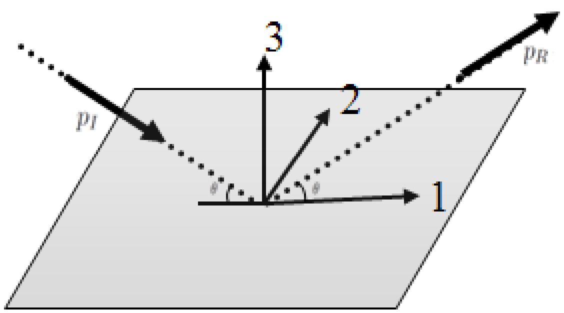
- Rotation **angle**:

$$\cos\frac{\Delta\phi}{2} = \frac{|A|^2 - \vec{B}^{\,2}}{|A^2 - \vec{B}^{\,2}|} \qquad \sin\frac{\Delta\phi}{2} = \frac{2\,\mathrm{Im}A\,|\vec{B}|}{|A^2 - \vec{B}^{\,2}|}.$$

- Axis of rotation:

$$\vec{\mathcal{N}} = \frac{\vec{B}}{|\vec{B}|}$$

$$\vec{B} = \underline{\cos\eta}\cos\theta\tanh\frac{\alpha}{2}\,\vec{e}_2 + \underline{\sin\eta}\,\vec{e}_3$$



$3^{\rm rd}$ component absent for $\eta = 0$ or π

the same as in the **classical** problem (!)

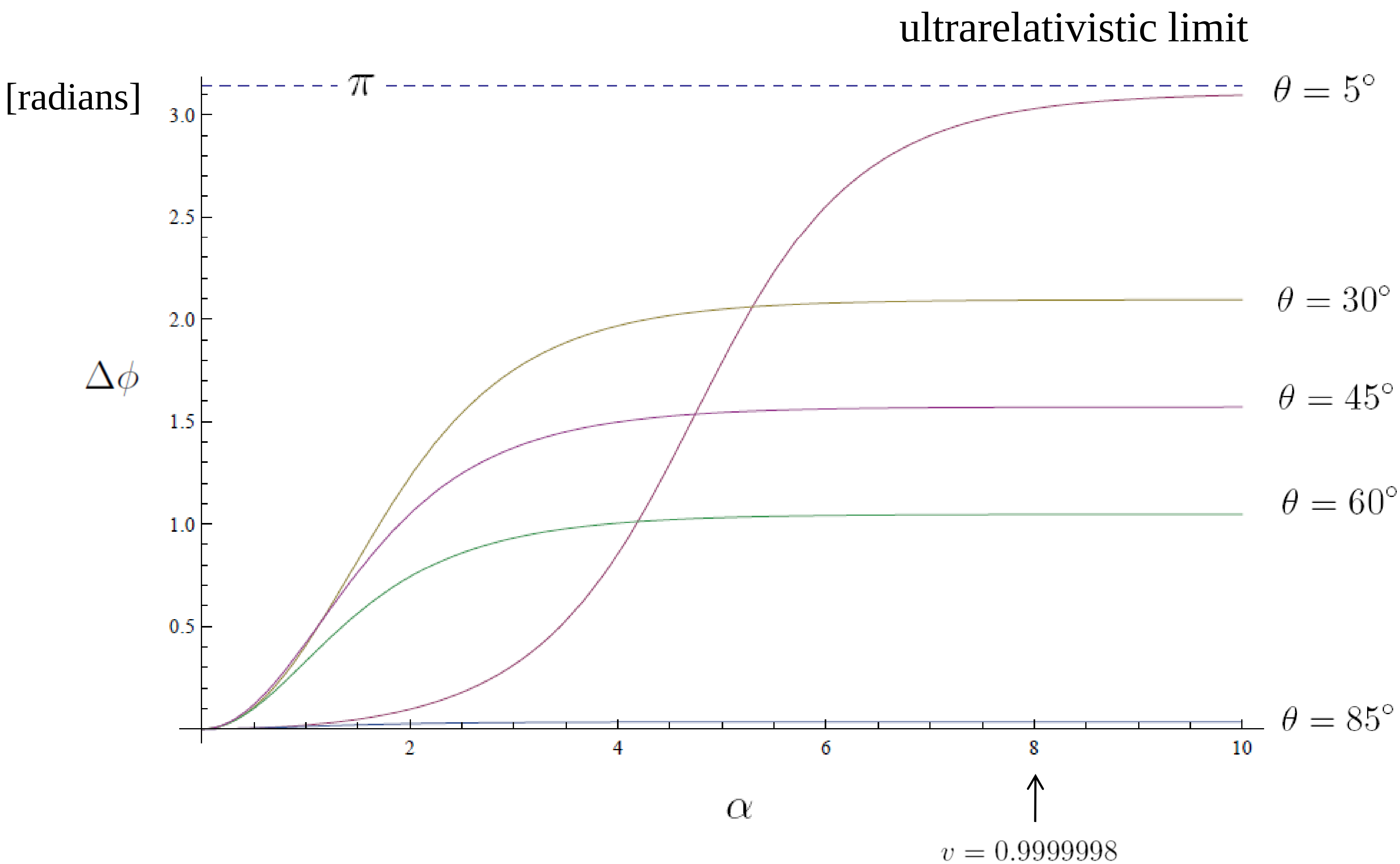
- The case $\eta = 0$:

$$\boxed{\vec{\mathcal{N}} = \vec{e}_2}$$

$$\boxed{\tan\frac{\Delta\phi}{2} = \frac{\sin 2\theta\ \tanh^2(\alpha/2)}{1-\cos 2\theta\ \tanh^2(\alpha/2)}.} \qquad (1)$$

- Nonrelativistic limit:

$$\alpha \ll 1\,, \qquad \alpha \simeq v\,, \qquad \Delta\phi^{NR} \simeq \frac{1}{2}\,v^2 \sin 2\theta. \qquad (2)$$



The rotation angle $\Delta\varphi$ as a function of the velocity parameter α for different incident angles θ

• Ultrarelativistic limit:

$$\boxed{\tan\frac{\Delta\phi}{2} = \frac{\sin 2\theta\ \tanh^2(\alpha/2)}{1-\cos 2\theta\ \tanh^2(\alpha/2)}.} \qquad (1)$$

$\alpha \longrightarrow \infty$

$$\tan\frac{\Delta\phi^{UR}}{2} \simeq \frac{\sin 2\theta}{1-\cos 2\theta} = \tan\left(\frac{\pi}{2}-\theta\right) \qquad (2)$$

$$\Delta\phi^{UR} \simeq \pi - 2\theta \qquad (3)$$

• The geometrical picture in the UR limit:

$$\Delta\phi^{UR} \simeq \pi + (-2\theta)$$

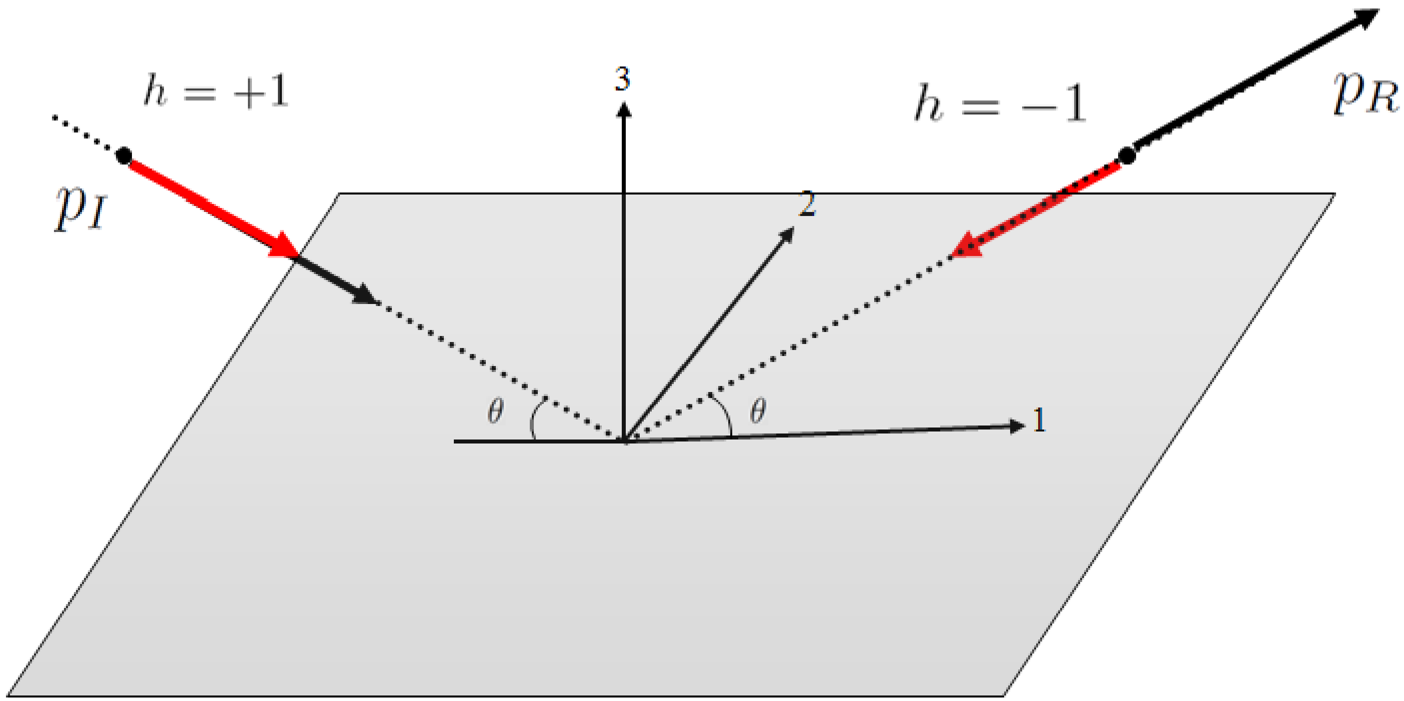
rotation angle of the 3-momentum around axis 2

The rotation angle of the spin **relative to the 3-momentum**:

$$\Delta\phi^{UR}_{rel} \simeq \pi$$

the helicity is **reversed** after reflection

• MIT + zero chiral angle = helicity flip in the UR limit



Puzzle:

- UR particles tend to behave like massless particles
- for massless chiral fermions the helicity is **fixed**

Answer:

MIT **cannot** be applied to **chiral** particles

For chiral particles: $\gamma_5\psi = \pm\psi$ (1)

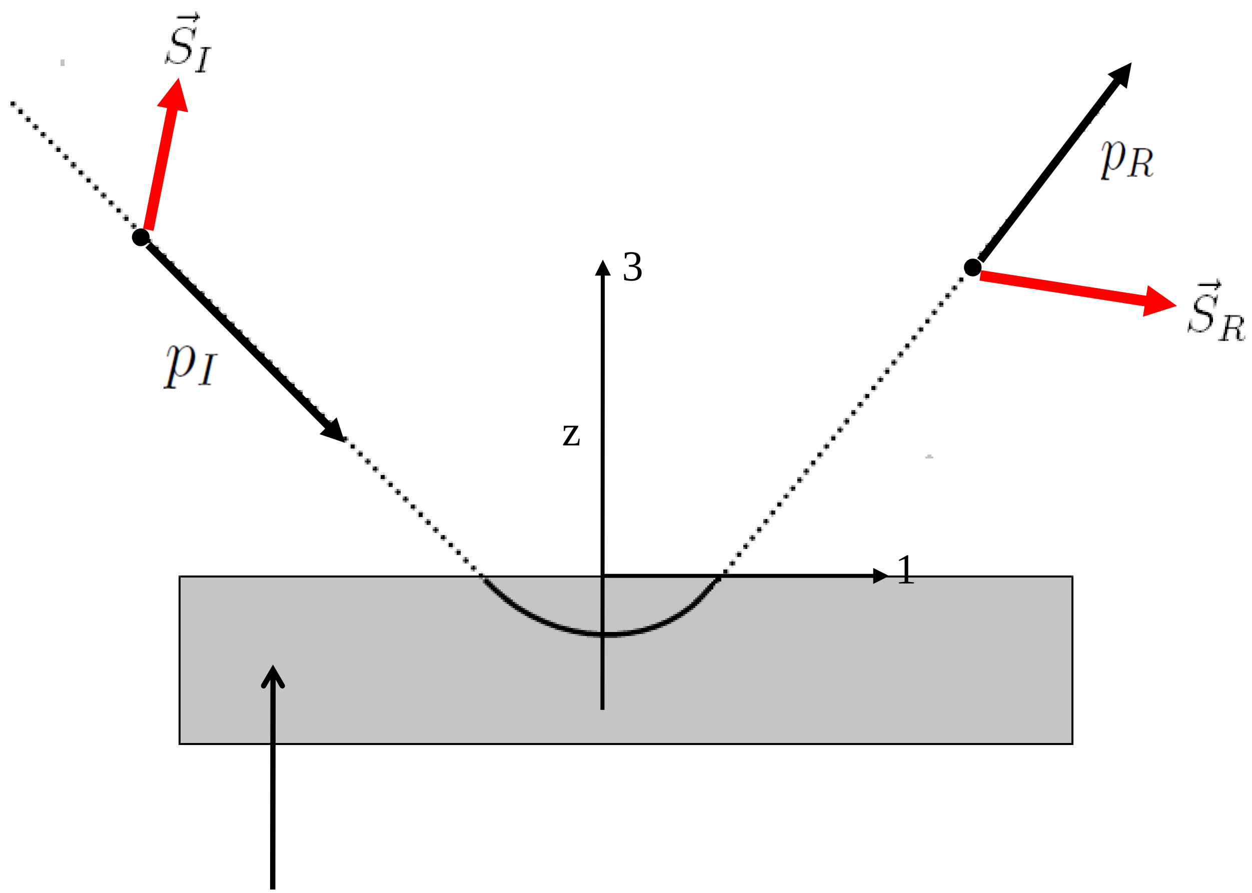
Incompatibility with MIT: $N^\mu\gamma_\mu\psi = ie^{i\eta\gamma_5}\psi$ (2)

γ_μ: anticommutes with γ_5

$e^{i\eta\gamma_5}$: commutes with γ_5

The classical solution:

First step - **smooth** the trajectory at the impact:



repulsive scalar potential V(z) $\longrightarrow$ assures that the trajectory is contained in the plane **13**

- The evolution of the spin defined the **Thomas precession**:

$$\vec{\omega}_T = \frac{\gamma^2}{\gamma+1}\,\vec{a}\times\vec{v}, \qquad (1)$$

$$\vec{v} = \frac{d\vec{x}}{dt}, \quad \vec{a} = \frac{d\vec{v}}{dt},$$

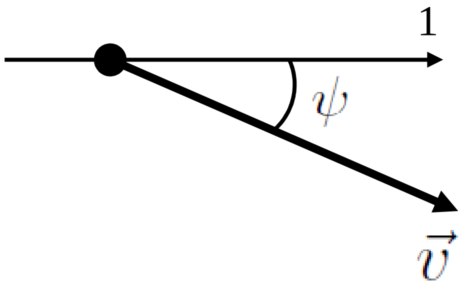
- The trajectory of the particle defined by:

$$\frac{d\vec{p}}{dt} = -\nabla V \qquad (2)$$

• The calculation:

Trick: consider the velocity of the particle as a function of ψ

$$\vec{v}(\psi) = v(\psi)\,\vec{n}(\psi)$$



$$\vec{n}(\psi) = (\cos\psi,\ 0,\ \sin\psi), \qquad \psi \in [-\theta,\ \theta]$$

The Thomas precession vector: $\boxed{\vec{\omega}_T = \frac{d\phi_{c\ell}}{dt}\,\vec{e}_2}$

the **same rotation axis** as in the quantum problem with $\psi = 0$

The angular velocity:

$$\frac{d\phi_{c\ell}}{dt} = \frac{\gamma^2 v^2}{\gamma + 1} \frac{d\psi}{dt} \qquad (1)$$

$$\gamma v \cos\psi = \gamma_I v_I \cos\psi_I \qquad (2)$$

conservation of the horizontal component of the momentum

allows to find the **total rotation angle** by integrating w.r.t. $\psi \in [-\theta,\ \theta]$

- The total classical rotation angle:

$$\boxed{\Delta\phi_{c\ell} = (\gamma v \cos\theta)^2 \times \int_{-\theta}^{\theta} \frac{d\psi}{\cos^2\psi} \left(\sqrt{1 + \frac{(\gamma v \cos\theta)^2}{\cos^2\psi}} + 1 \right)^{-1}}$$

↑

independent of the form of the surface potential V(z)

$$\boxed{\Delta\phi_{c\ell} = (\gamma v \cos\theta)^2 \times \int_{-\theta}^{\theta} \frac{d\psi}{\cos^2\psi} \left(\sqrt{1 + \frac{(\gamma v \cos\theta)^2}{\cos^2\psi}} + 1 \right)^{-1}} \tag{1}$$

- Nonrelativistic limit: $v \ll 1, \quad \gamma \simeq 1$

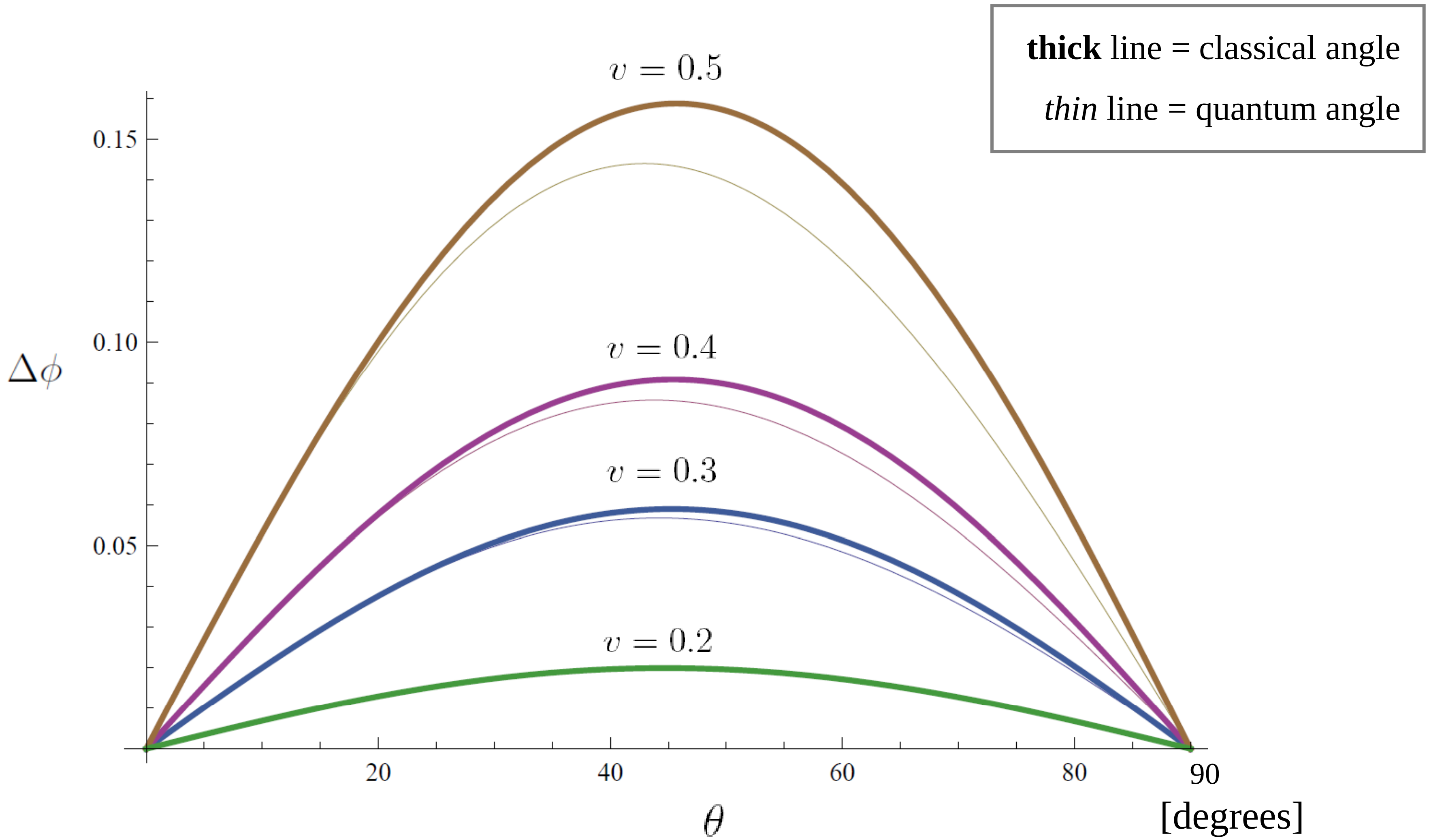
$$\Delta\phi_{c\ell}^{NR} \simeq (v\cos\theta)^2 \times \int_{-\theta}^{\theta} \frac{d\psi}{2\cos^2\psi} = v^2 \sin\theta\cos\theta \tag{2}$$

the **same** as in the quantum problem !

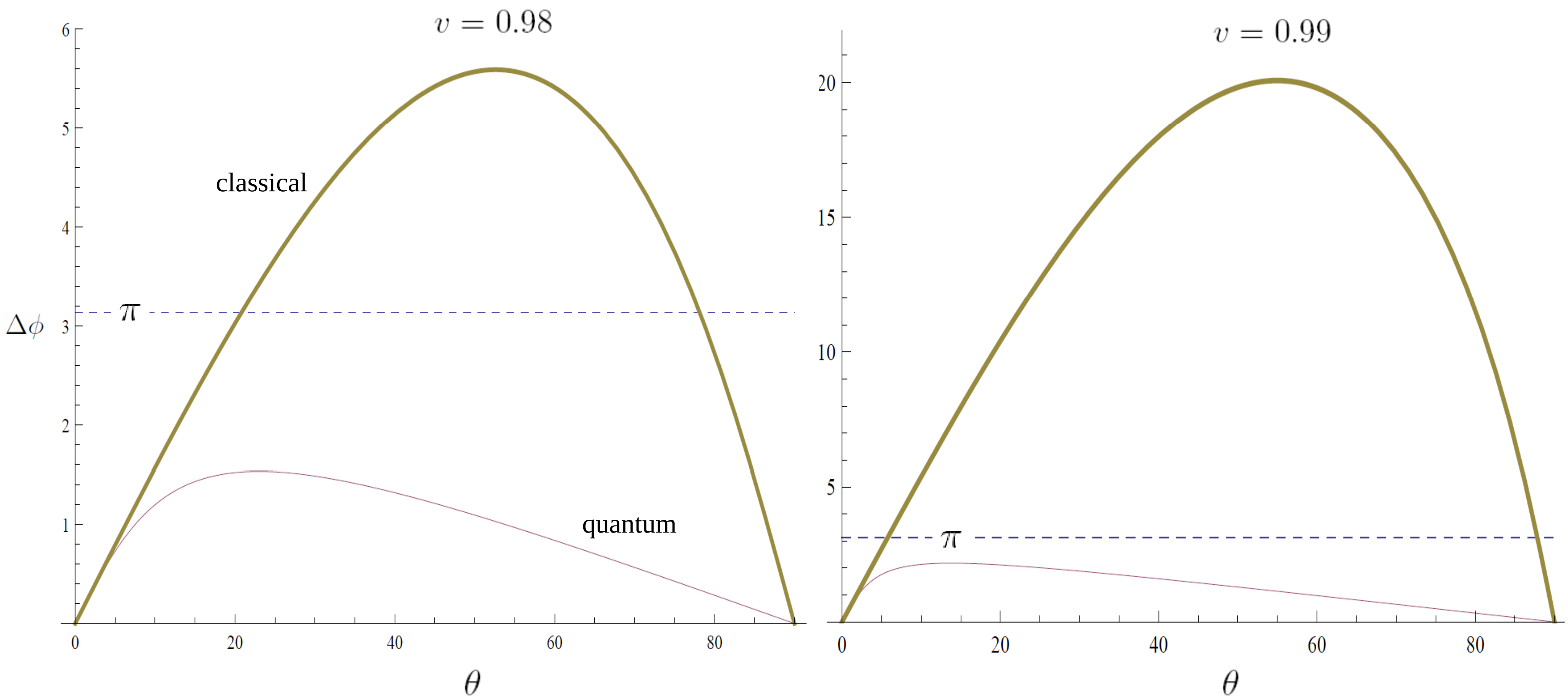
- Ultrarelativistic limit: $v \to 1, \quad \gamma \to \infty$

$$\Delta\phi_{c\ell}^{UR} \sim \gamma \to \infty$$

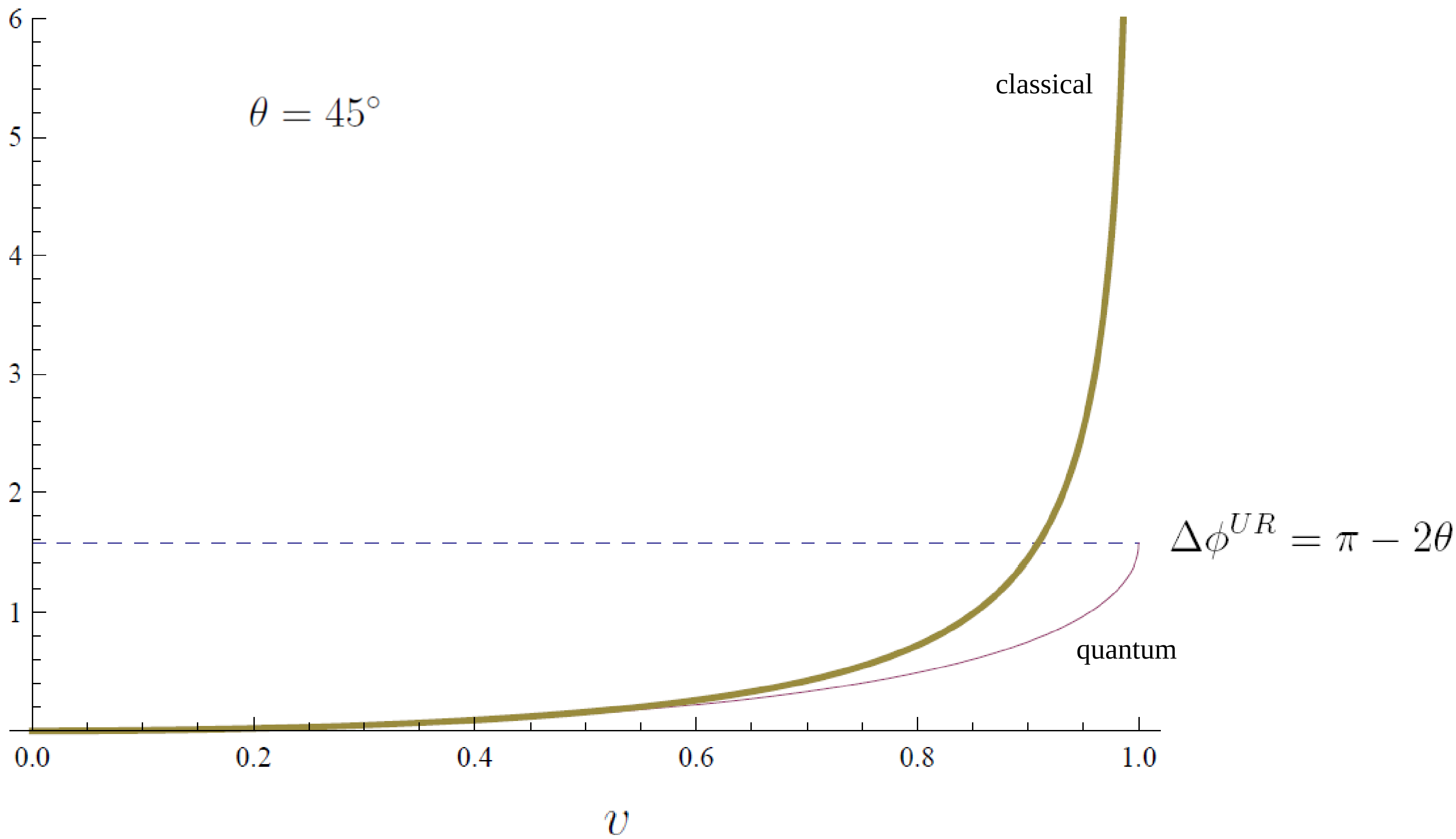
the rotation angle **diverges** !



The quantum and the classical rotation angles as functions of the angle θ for different velocities.



The quantum and the classical rotation angles as functions of the angle θ for different velocities.



The quantum and the classical rotation angles as functions of the velocity v for a fixed θ.

Discrepancy in the UR limit $\gamma \to \infty$

❖ The quantum result - stays **finite**

$$\Delta\phi^{UR} \simeq \pi - 2\theta$$

❖ The classical result - **diverges**

$$\Delta\phi_{c\ell}^{UR} \sim \gamma$$

• Two options regarding the picture in the UR limit:

(I) A classical spinning body **is not** a quantum particle with spin

↓

$\boxed{\textbf{no}\text{ problem}}$

(II) The divergent classical rotation angle is **unphysical**

↓

$\boxed{\text{something }\textbf{goes wrong}\text{ with the Thomas precession formula}}$

↑

theoretical models often fail due to **unrealistic idealizations**

- **Idealizations** in the Thomas precession formula:

- A well-defined precession vector **=** **infinitely rigid** body

- The spinning body **=** **point-like** particle

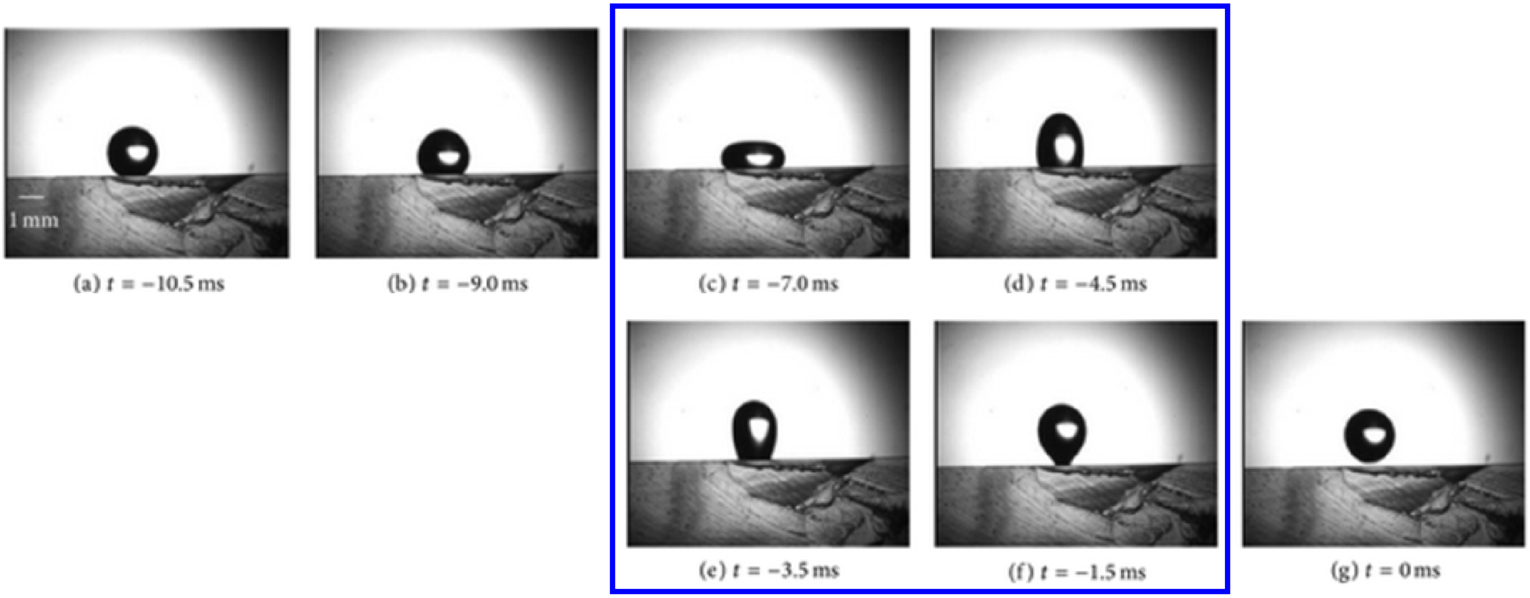
no such bodies exist in special relativity

- **Solution**: consider a **finite-size elastic** body

Conjecture: $\Delta\phi$ will stay **finite** in the UR limit for **elastic bodies**

Analogy: the dynamics of point-like / **extended charges** in classical ED

A **high velocity impact** should look like this:



Thomas precession formula here clearly **inapplicable**

Practical implications:

Mathisson–Papapetrou–Dixon equations + supp. condition $S^{\mu\nu}U_\nu = 0$

$$\downarrow$$

the spin evolves according to the **Thomas precession** formula

$$\downarrow$$

problems with the MPD equations in the UR limit ?

Increased interest in MPD eqs. in the UR limit in recent years:

R. M. Plyatsko and M. T. Fenyk,
``Highly relativistic spinning particle in the Schwarzschild field: Circular and other orbits," Phys. Rev. **D** 85 (2012)

R. Plyatsko and M. Fenyk,
``Highly relativistic circular orbits of spinning particle in the Kerr field," Phys. Rev. D **87** (2013)

R. Plyatsko, M. Fenyk and O. Stefanyshyn, \`
\`Solutions of Mathisson Equations for Highly Relativistic Spinning Particles," Fund. Theor. Phys. **179** (2015)

R. Plyatsko and M. Fenyk,
`` Highly relativistic spin-gravity coupling for fermions," Phys. Rev. D **91** (2015)

S. A. Hojman and F. A. Asenjo, \`
\`Comment on "Highly relativistic spin-gravity coupling for fermions," Phys. Rev. D **93** (2016)

R. Plyatsko and M. Fenyk,
``Reply to "Comment on 'Highly relativistic spin-gravity coupling for fermions," Phys. Rev. D 93 (2016)

Acknowledgments: This work was partially supported by a grant of Romanian Authority for Scientific Research and Innovation, CNCS-UEFISCDI, project number PN-II-RU-TE-2014-4-2910.