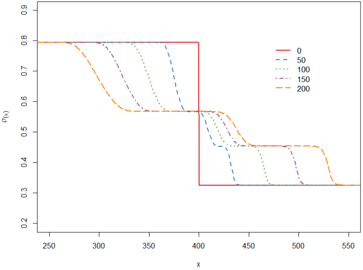
# Quadrature-based Lattice Boltzmann model for simulating relativistic flows

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# Outline

1. **Relativistic Boltzmann equation**
   - the equation
   - the moments
   - the collision operator

2. **Lattice Boltzmann models**
   - collision-streaming method
   - finite difference method

3. **Application**
   - The Riemann problem

# Boltzmann equation

- Tool for investigating fluids at a mesoscopic level
- Different levels
  - Macroscopic: continuous fields (hydrodynamics)
  - Mesoscopic: distribution function (*Boltzmann equation*)
  - Microscopic: individual molecules (molecular dynamics)
- Provides a good description of fluids from the hydrodynamic to the ballistic regime

# Boltzmann equation

- The relativistic Boltzmann equation[1]

$$p^\mu \frac{\partial f}{\partial x^\mu} - \Gamma^i_{\mu\nu} p^\mu p^\nu \frac{\partial f}{\partial p^i} = C[f] \tag{1}$$

$$+ \text{ mass-shell condition: } p_\mu p^\mu = -m^2 c^2$$

- In conservative form, using a local orthonormal (tetrad) frame[2]

$$\frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} p^{\hat\alpha} e^\mu_{\hat\alpha} f \right) - \Gamma^{\hat i}{}_{\hat\alpha\hat\beta} p^{\hat\alpha} p^{\hat\beta} \frac{\partial f}{\partial p^{\hat i}} = C[f] \tag{2}$$

[1] C. Cercignani, G. M. Kremer, *The relativistic Boltzmann equation: theory and application*, Birkhäuser Verlag, Basel, Switzerland (2002).

[2] C. Y. Cardall, E. Endeve, A. Mezzacappa, Phys. Rev. D **88** 023011 (2013).

# Moments

- Macroscopic description given by the moments of the distribution function

$$
\begin{aligned}
N^{\hat{\alpha}} &= c \int p^{\hat{\alpha}} f \sqrt{g} \frac{d^3p}{p^{\hat{0}}} \qquad \text{(particle four-flow)} && (3) \\
T^{\hat{\alpha}\hat{\beta}} &= c \int p^{\hat{\alpha}} p^{\hat{\beta}} f \sqrt{g} \frac{d^3p}{p^{\hat{0}}} \quad \text{(energy-momentum)} && (4) \\
&\dots
\end{aligned}
$$

- The corresponding conservation equations

$$
{N^{\hat{\alpha}}}_{;\hat{\alpha}} = 0, \qquad {T^{\hat{\alpha}\hat{\beta}}}_{;\hat{\alpha}} = 0, \qquad \dots \tag{5}
$$

- The particle number density and the macroscopic velocity of the fluid are

$$
n = \sqrt{-N_{\hat{\alpha}} N^{\hat{\alpha}}}, \qquad u^{\hat{\alpha}} = N^{\hat{\alpha}}/n \tag{6}
$$

# The collision operator - $C[f]$

- The full collision operator is complicated to work with. Close to equilibrium we can use a simplified model (BGK) . The two main choices are:
  - Marle: $C[f] = -\frac{m}{\tau}(f - f^{eq})$
  - Anderson-Witting: $C[f] = \frac{p_{\hat{\alpha}} u_L^{\hat{\alpha}}}{\tau}(f - f^{eq})$
- The Maxwell-Juttner equilibrium distribution function:

$$f^{eq} = \frac{n}{8\pi T^3} \exp\left(\frac{p^{\hat{\alpha}} u_{\hat{\alpha}}}{T}\right), \qquad (\mu_E = 0) \tag{7}$$

- Landau-velocity $u_L^{\mu}$: the only future-oriented timelike eigenvector of $T^{\mu\nu}$

$$T^{\hat{\alpha}}{}_{\hat{\beta}} u_L^{\hat{\beta}} = -\epsilon\, u_L^{\hat{\alpha}} \tag{8}$$

# Lattice Boltzmann models

1. Decide up to which moment (N) we want to recover the physics
2. Discretize space
3. Take from the continuous, infinite momentum-space a discrete, finite set of momenta/velocities.

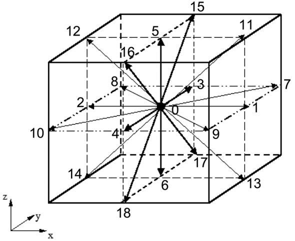
$$p^{\alpha} \to p_k^{\hat{\alpha}}, \qquad \int \frac{d^3 p}{p^{\hat{0}}} f A(p^{\hat{\mu}}) \to \sum_k w_k f_k A(p_k^{\hat{\mu}}) \tag{9}$$

4. Expand the equilibrium distribution function and truncate the series at order N
5. "Synchronize" the truncations such that the momenta, up to the desired order, are recovered exactly

# Collision-streaming

- The essence of the method is to chose a simple set of velocities, such that the transport is always done along the straight lines of the spatial lattice.[3]

- Pros:
  - easy to implement
  - time-efficient
- Cons:
  - hard to extend
  - works only for on-lattice velocities




[3] Image taken from K. N. Premnath, M. J. Pattison, S. Banerjee, J. Fluids Eng. **135**, 051401 (2013).

# Quadrature method + finite difference

- We project the distribution functions onto a set of orthogonal polynomials, and truncate the series

$$f^{eq}(x,\vec{p},t) = e^{-p} \sum_{k=0}^{Q_p-1} \sum_{n=0}^{Q_\xi-1} a^{i_1\dots i_n}_{(nk)}(x,t) P^{(n)}_{i1..i_n}\left(\frac{\vec{p}}{p}\right) L^{(1)}_j(p) \qquad (10)$$

- Use Gauss-quadrature method to write integrals as sums [4,5]

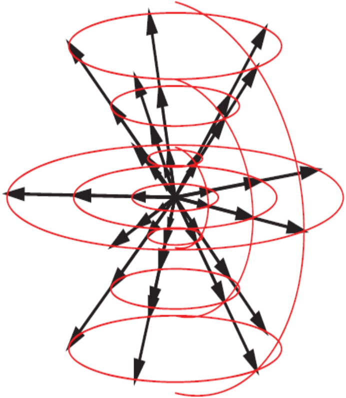
$$\begin{aligned}
\mathcal{Q}(p,\theta) &= \int_0^{2\pi} d\phi f^{eq} A_n(p,\xi,\phi) = \sum\nolimits_{i=0}^{Q_\phi-1} \frac{2\pi}{Q_\phi} f^{eq}_i A_n(p,\theta,\phi_i) \\
\mathcal{E}(p) &= \int_{-1}^{1} d(\cos\theta)\,\mathcal{Q}(p,\theta) = \sum\nolimits_{j=0}^{Q_\xi-1} w^\xi_j \mathcal{Q}(p,\theta_i) \\
\mathcal{M} &= \int_0^\infty dp\, p\, e^{-p}\mathcal{E}(p) = \sum\nolimits_{k=0}^{Q_p-1} w^p_k\, \mathcal{E}(p_k) \qquad (11)
\end{aligned}$$

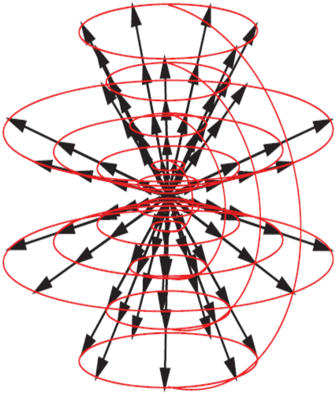
[4] V.E. Ambruş, V. Sofonea, Phys. Rev. E **86**, 016708 (2012)

[5] P. Romatschke, M. Mendoza, and S. Succi, Phys. Rev. C **84**, 034903 (2012)

# Quadrature method + finite difference

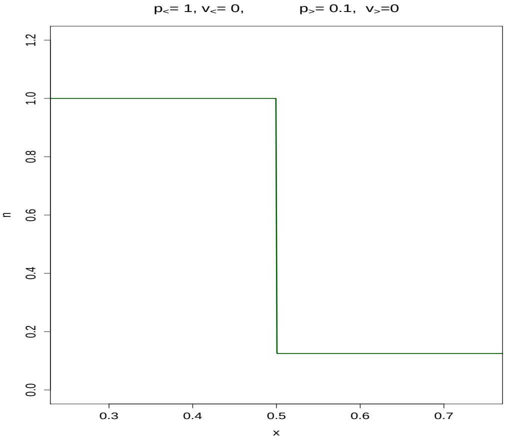
- Examples for sets of momenta:[6]





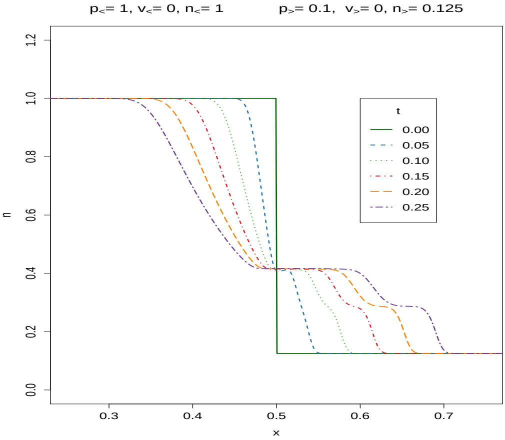
[6] V.E. Ambruş, V. Sofonea, Phys. Rev. E **86**, 016708 (2012)

# The Riemann problem



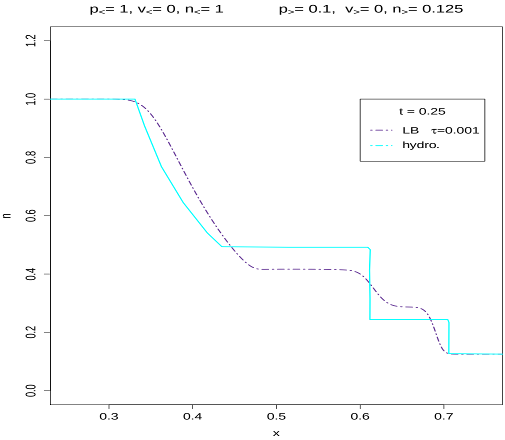

- Canonical problem with discontinuous initial conditions
- Simple setup, but hard test for the numerical model

# The Riemann problem




- Time evolution : formation of three features
- Compare with the result from hydrodynamics

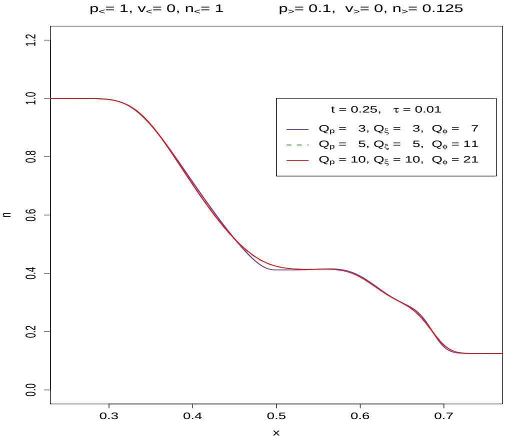
# The Riemann problem




- Rarefaction wave - Contact discontinuity - Shock-wave[7]
- Difference may be due to numerical viscosity

[7] L. Rezzolla, O. Zanotti, *Relativistic hydrodynamics*, Oxford University Press, New York, U.S.A.(2013).

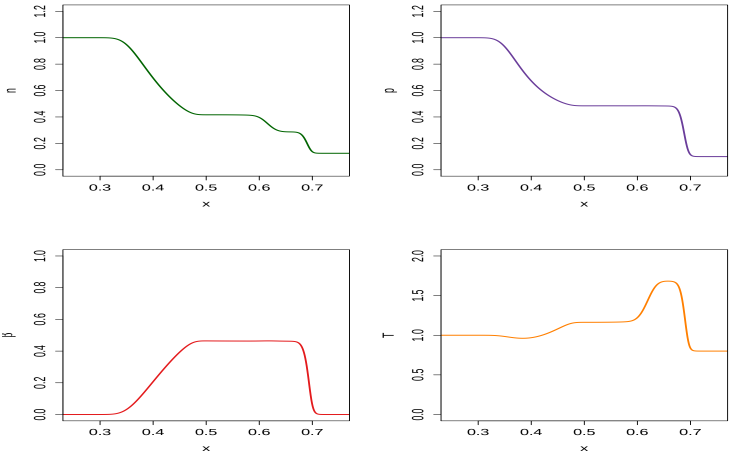
# The Riemann problem




- The curves for $N_p = N_\xi = 5, 10$ are indistinguishable
- The procedure converges for increasing order of the quadrature

# The Riemann problem

$p_< = 1,\ v_< = 0,\ n_< = 1$ $\qquad$ $p_> = 0.1,\ v_> = 0,\ n_> = 0.125$




$\tau = 0.001,\ t = 0.25$

# Conclusion

1. We have developed a quadrature-based Lattice Boltzmann model appropriate for simulating relativistic flows
2. The procedure is systematic and allows expansion to arbitrarily high orders
3. We have tested our model for a setup with discontinuous initial conditions (the Riemann problem)
4. Our procedure is convergent with respect to increasing order of the quadrature