Rigidly rotating Maxwell-Jüttner states on spherically symmetric space-times using the tetrad formalism

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Outline

Motivation

- Rigidly-rotating thermal states represent toy-models for more complex space-times where thermal equilibrium implies a rigid rotation (as is the case inside the ergosphere of a Kerr black hole).
- Rigidly-rotating states can studied analytically and important conclusions can be drawn which carry through to more complex scenarios.
- The analysis of such states performed using kinetic theory can serve as a good starting point for quantum field theory computations.

Relativistic Boltzmann equation

- The Boltzmann equation in conservative form can be written as:[1]

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}p^{\hat{\alpha}}e_{\hat{\alpha}}^{\mu}f\right) - \Gamma^{\hat{i}}{}_{\hat{\alpha}\hat{\beta}}p^{\hat{\alpha}}p^{\hat{\beta}}\frac{\partial f}{\partial p^{\hat{i}}} = J[f],$$

where $\{e_{\hat{\alpha}}\}$ is an orthonormal tetrad and $p^{\hat{\alpha}} \equiv \omega^{\hat{\alpha}}_{\mu}p^{\mu}$ $(\omega^{\hat{\alpha}}_{\mu}e^{\mu}_{\hat{\beta}} = \delta^{\hat{\alpha}}{}_{\hat{\beta}})$ is the momentum of the microscopic constituents, obeying the mass-shell condition:

$$g_{\mu\nu}p^\mu p^\nu = -m^2.$$

- For non-degenerate fluids, $J[f]$ drives the relaxation of f towards the Maxwell-Jüttner equilibrium distribution function:

$$f^{(\mathrm{eq})}_{\mathrm{M-J}} = Z\exp(\beta\mu_E + \beta p^\lambda u_\lambda),$$

where Z is the number of degrees of freedom, β is the local temperature, μ_E is the chemical potential and u^λ is the fluid four-velocity.

- The fluid is in global thermodynamic equilibrium if:

$$\nabla_\lambda \beta\mu_E = 0, \qquad (\beta u_\lambda)_{;\nu} + (\beta u_\nu)_{;\lambda} = 0.$$

[1] C. Y. Cardall, E. Endeve, A. Mezzacappa, Phys. Rev. D **88** (2013) 023011.

Spherically-symmetric space-times

- The line element for a spherically-symmetric ST can be written as [2]:

$$ds^2 = w^2 \left[-dt^2 + \frac{dr^2}{u^2} + \frac{r^2}{v^2}(d\theta^2 + \sin^2\theta d\varphi^2) \right],$$

  where w, u and v only depend on r.
- A possible choice for the tetrad is given below:

$$\begin{aligned} e_{\tilde{0}} &= \frac{1}{w}\partial_t, & e_{\tilde{r}} &= \frac{u}{w}\partial_r, & e_{\tilde{\theta}} &= \frac{v}{rw}\partial_\theta, & e_{\tilde{\varphi}} &= \frac{v}{\rho w}\partial_\varphi, \\ \omega^{\tilde{0}} &= w dt, & \omega^{\tilde{r}} &= \frac{w}{u} dr, & \omega^{\tilde{\theta}} &= \frac{rw}{v} d\theta, & \omega^{\tilde{\varphi}} &= \frac{\rho w}{v} d\varphi. \end{aligned}$$

- The Killing vector representing rotations with respect to the z axis is:

$$k^{\tilde{\alpha}} = \beta u^{\tilde{\alpha}} = \beta_0 (1, 0, 0, \Omega)^T.$$

- β and $u^{\tilde{\alpha}}$ can be found as the norm of, and unit vector along $k^{\tilde{\alpha}}$:

$$\beta = w\gamma^{-1}\beta_0, \qquad u^{\tilde{\alpha}} = w^{-1}\gamma \left(1, 0, 0, \frac{\rho\Omega}{v}\right)^T, \qquad \gamma = \frac{1}{\sqrt{1 - \left(\frac{\rho\Omega}{v}\right)^2}}.$$

[2] I. I. Cotaescu, J. Phys. A **33**, 9177 (2000).

Co-moving frame for rigidly-rotating flows

- The co-moving tetrad $\{e_{\hat{\alpha}}\}$ can be obtained from $\{e_{\tilde{\alpha}}\}$ by applying the following Lorentz boost:

$$L^{\tilde{a}}{}_{\hat{\alpha}} = \begin{pmatrix} u^{\tilde{0}} & u_{\tilde{\jmath}} \\ u^{\tilde{\imath}} & \delta^{\tilde{\imath}}{}_{\tilde{\jmath}} + \dfrac{u^{\tilde{\imath}} u_{\tilde{\jmath}}}{u^{\tilde{0}} + 1} \end{pmatrix}.$$

giving rise to:

$$\begin{aligned}
e_{\hat{t}} &= \frac{\gamma}{w}(\partial_t + \Omega \partial_\varphi), & \omega^{\hat{t}} &= \gamma w \left(dt - \frac{\rho^2 \Omega}{v^2} d\varphi \right), \\
e_{\hat{r}} &= \frac{u}{w} \partial_r, & \omega^{\hat{r}} &= \frac{w}{u} dr, \\
e_{\hat{\theta}} &= \frac{v}{rw} \partial_\theta, & \omega^{\hat{\theta}} &= \frac{rw}{v} d\theta, \\
e_{\hat{\varphi}} &= \frac{\gamma}{w} \left(\frac{\rho \Omega}{v} \partial_t + \frac{v}{\rho} \partial_\varphi \right), & \omega^{\hat{\varphi}} &= \frac{\rho \gamma w}{v}(-\Omega dt + d\varphi).
\end{aligned}$$

- With respect to this tetrad, the fluid velocity is:

$$u^{\hat{\alpha}} = (1, 0, 0, 0)^T.$$

Equilibrium states in the co-moving frame

- When the co-moving tetrad is employed, the Maxwell-Jüttner distribution function has the form ($\mu_E = 0$):

$$f^{(\mathrm{eq})} = Z e^{-p^{\hat{t}}}.$$

- The resulting particle flux $N^{\hat{\alpha}}$ and stress-energy tensor $T^{\hat{\alpha}\hat{\beta}}$ are:

$$N^{\hat{\alpha}} = \int \frac{d^3p}{p^{\hat{t}}} f^{(\mathrm{eq})} p^{\hat{\alpha}} = (n, 0, 0, 0)^T,$$

$$T^{\hat{\alpha}\hat{\beta}} = \int \frac{d^3p}{p^{\hat{t}}} f^{(\mathrm{eq})} p^{\hat{\alpha}} p^{\hat{\beta}} = \mathrm{diag}(E, P, P, P).$$

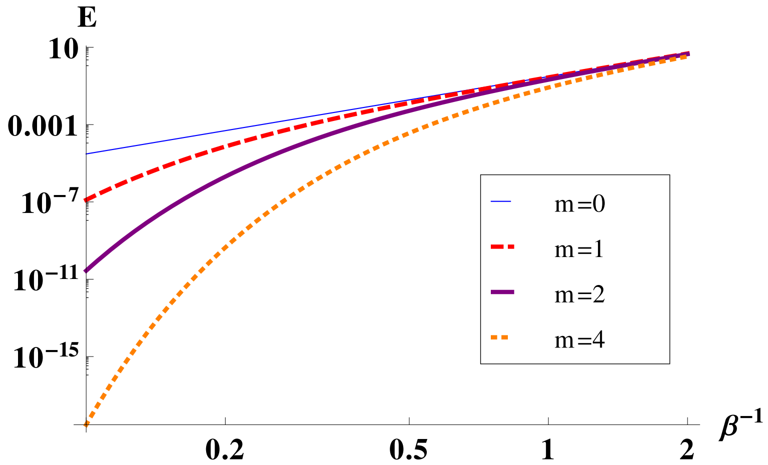
- The hydrostatic pressure P and energy density E are given by ($Z = 1$):[3]

$$P = \frac{m^2}{2\pi^2\beta^3} K_2(m\beta), \qquad E = \frac{3m^2}{2\pi^2\beta^2}\left[K_2(m\beta) + \frac{m\beta}{3} K_1(m\beta)\right],$$

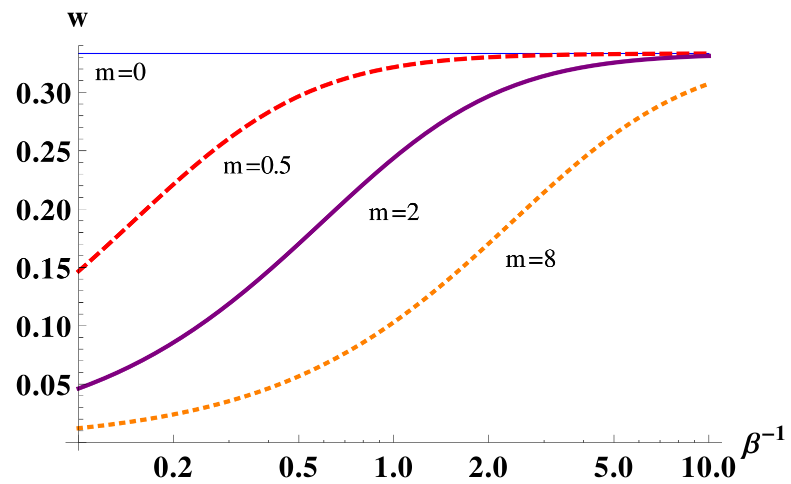
while $n = \beta P$.

[3] V. E. Ambruș, R. Blaga, Annals of West University of Timisoara - Physics **58** (2015) 89.

Maxwell-Jüttner equilibrium states



Energy density E



Equation of state $w = P/E$

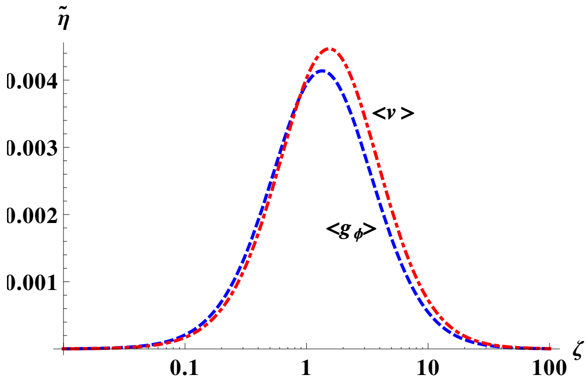
- In the massless case, n, P and E reduce to $(Z = 1)$:

$$n = \frac{1}{\pi^2 \beta^3}, \qquad P = \frac{1}{\pi^2 \beta^4}, \qquad E = \frac{31}{\pi^2 \beta^4}.$$

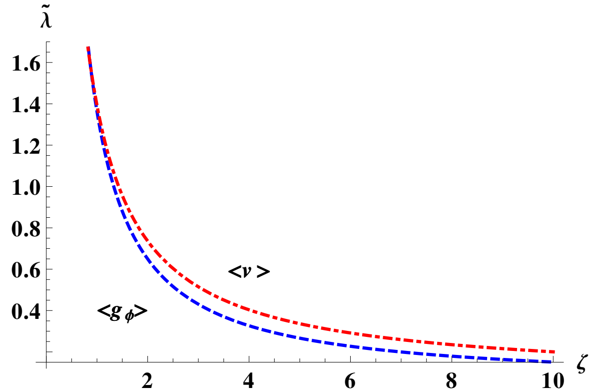
- n, P and E are monotonic functions with respect to β.

Transport coefficients

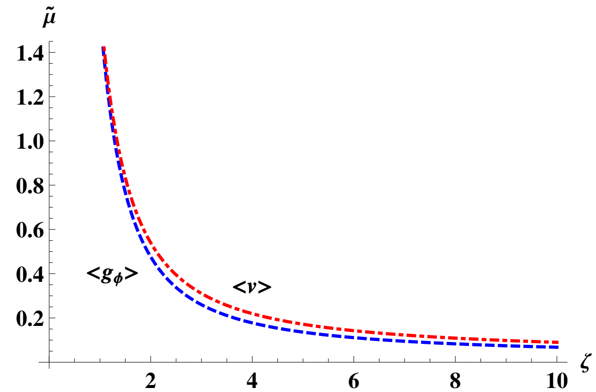
Marle model: $J[f] = -\frac{m}{\tau}(f - f^{(\mathrm{eq})})$, $\qquad \tau = \frac{1}{n\pi a^2 \mathcal{V}}$



Bulk viscosity η



Thermal conductivity λ



Shear viscosty μ

- The transport coefficients have the same expression on curved spaces as on Minkowski space.[4]
- They characterise the flows in the hydrodynamic regime close to the equilibrium state.
- While λ and μ are monotonic in $\zeta = m\beta$, the bulk viscosity η attains a maximum value at $\zeta_{\max} \simeq 1.535$.

[4] C. Cercignani, G. M. Kremer, *The relativistic Boltzmann equation: theory and applications*, Birkhäuser Verlag, Basel, Switzerland (2002).

Speed of light surface and horizon structure

- In co-rotating coordinates $t = t_{\text{static}}$ and $\varphi = \varphi_{\text{static}} - \Omega t_{\text{static}}$, the line element becomes:

$$ds^2 = w^2 \left[-\gamma^{-2} dt^2 + \frac{2\rho^2 \Omega}{v^2} dt d\varphi + \frac{dr^2}{u^2} + \frac{r^2}{v^2} d\Omega^2 \right].$$

- The surfaces where $g_{00} = 0$ represent Killing horizons (βu^μ becomes null) and the temperature β^{-1} diverges:[5]

$$-g_{00} = w^2 \left(1 - \frac{\rho^2 \Omega^2}{v^2} \right) = \frac{\beta^2}{\beta_0^2} = 0.$$

- The horizons are:
  - Rotation horizons (speed of light surfaces): where $1 - \rho^2\Omega^2/v^2 \to 0$;
  - Cosmological or event horizons: due to the properties of w and v.
- The horizon structure is complicated, revealing an interplay between the properties of the space-time and the rigid rotation.

[5] R. C. Tolman, Phys. Rev. **35** (1930) 904;
R. C. Tolman, P. Ehrenfest, Phys. Rev. 36 (1930) 1791.

Maximally-symmetric space-times

- The line element on maximally-symmetric space-times is:

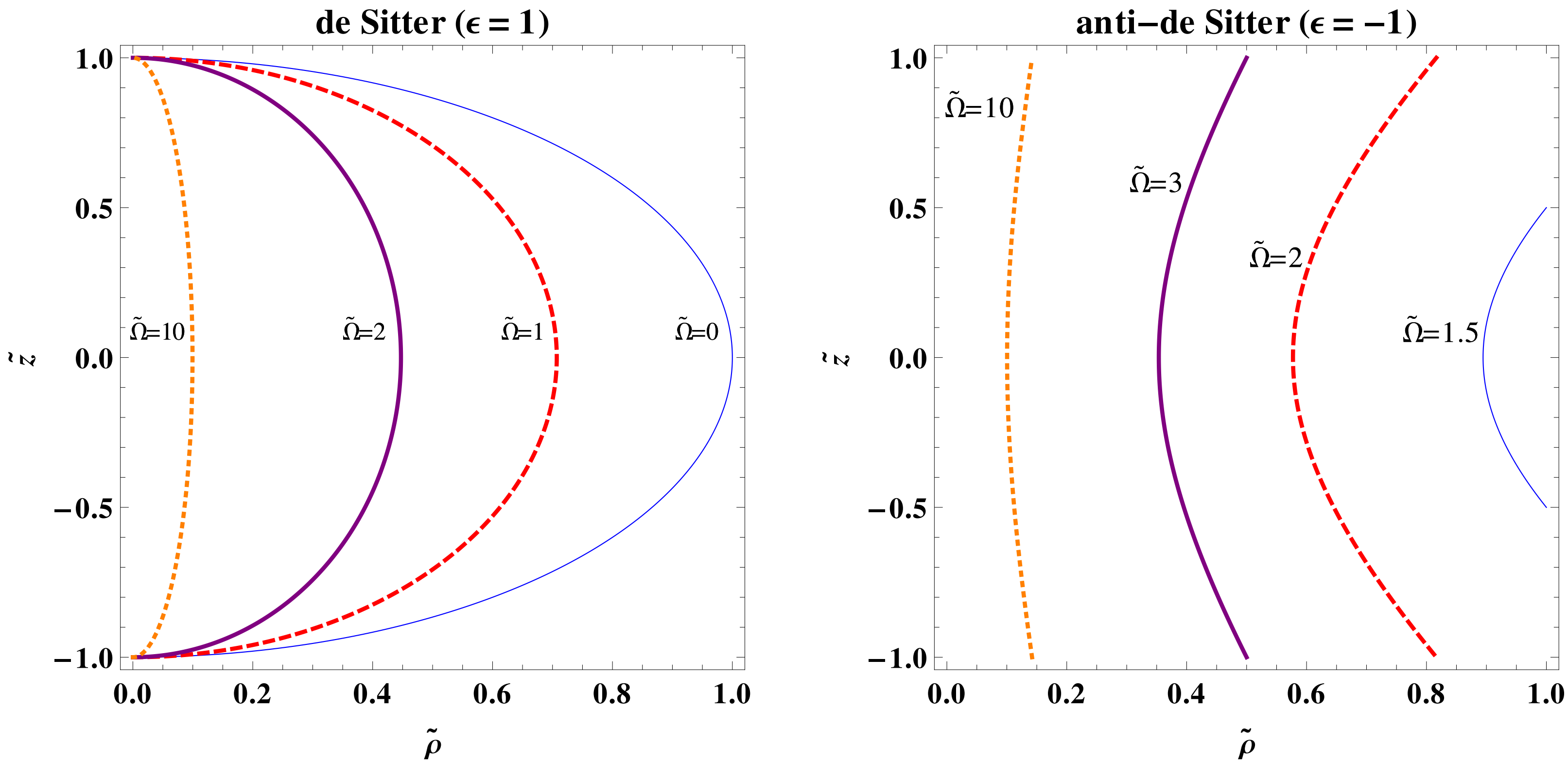
$$ds^2 = -(1 - \epsilon\omega^2 r^2)dt^2 + \frac{dr^2}{1 - \epsilon\omega^2 r^2} + r^2 d\Omega^2,$$

  where $\epsilon = 0$ (Minkowski), 1 (dS) and -1 (adS).
- The inverse temperature takes the form:

$$\beta = \beta_0 \sqrt{1 - (\epsilon\omega^2 + \Omega^2 \sin^2\theta) r^2},$$

  where $\beta_0 = \beta(r = 0)$.
- At $\Omega = 0$:
  - β^{-1} remains constant (Minkowski);
  - β^{-1} decreases to 0 as $r \to \infty$ (adS);
  - β^{-1} increases to ∞ as the cosmological horizon $r = \omega^{-1}$ is approached (dS).

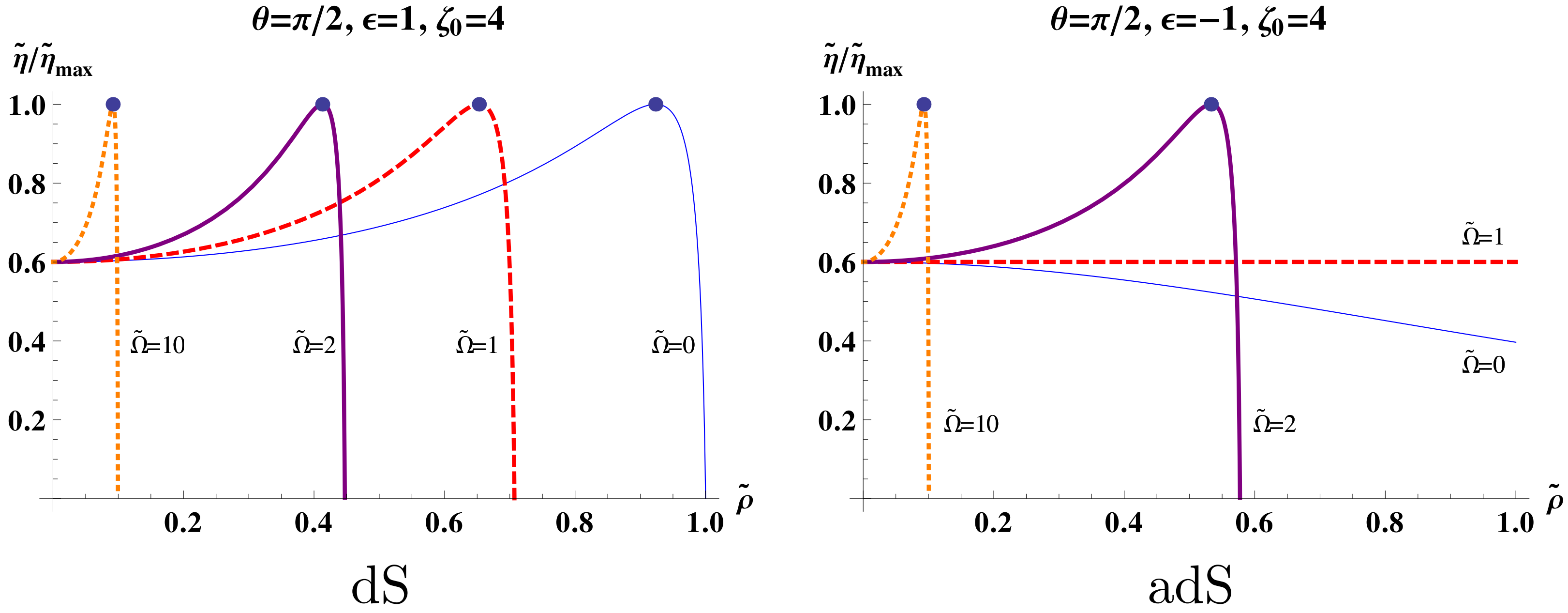
Killing horizons of rigidly-rotating observers



dS adS

- The rotation induces an SOL where $\beta = 0$, i.e.:
  - $\rho = \Omega^{-1}$ (Minkowski)
  - $\omega r_{\text{SOL}} = (1 + \Omega^2 \sin^2 \theta / \omega^2)^{-1/2}$ (dS)
  - $\omega r_{\text{SOL}} = (-1 + \Omega^2 \sin^2 \theta / \omega^2)^{-1/2}$ (adS)
- Decreasing Ω decreases the distance to the SOL.

Bulk viscosity in rigidly-rotating thermal states



- No SOL forms on adS for $\Omega < \omega$, while for $\Omega = \omega$, β is constant in the equatorial plane.
- As ζ decreases from $\zeta_0 = 4$ at the origin to 0 on the SOL, η attains a maximum when $\zeta = \zeta_{\max} \simeq 1.535$.

Reissner-Nordström black-holes

- The line element of the Reissner-Nordström metric is given by:

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r} + \frac{Q^2}{r^2}} + r^2 d\Omega^2,$$

- The inverse temperature β seen by rigidly-rotating observers is:

$$\beta = \beta_0 \sqrt{1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \rho^2 \Omega^2}, \tag{1}$$

where $\beta_0 = \beta(\rho = 0, z \to \pm\infty)$.

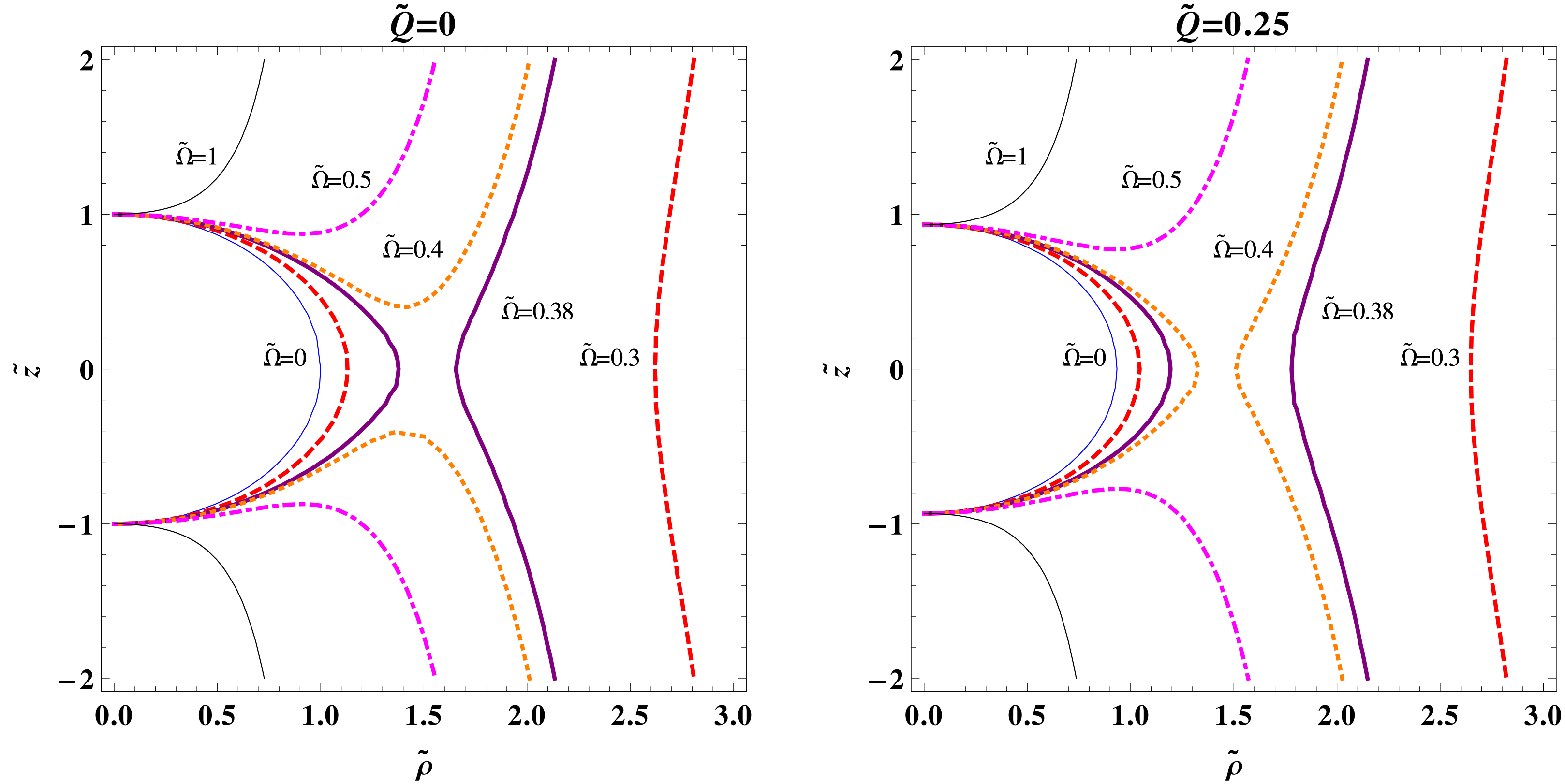
- Killing horizons at:

$$1 - \frac{1}{\widetilde{r}} + \frac{\widetilde{Q}^2}{\widetilde{r}^2} - \widetilde{\rho}^2 \widetilde{\Omega}^2 = 0, \tag{2}$$

where $\widetilde{r} = \frac{r}{2M}$, $\widetilde{Q} = \frac{Q}{2M}$ and $\widetilde{\Omega} = 2M\Omega$.

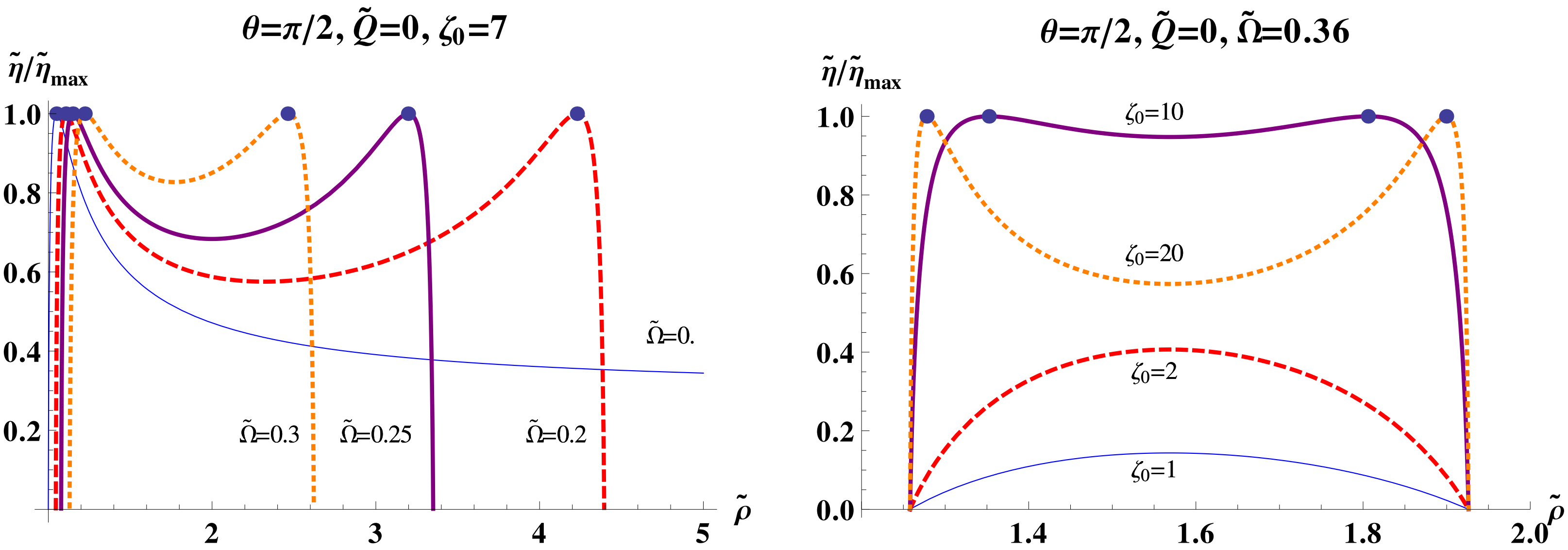
- At $\Omega = 0$, β decreases from β_0 at ∞ to 0 on the the black hole outer horizon at $r \to M + \sqrt{M^2 - Q^2}$.

Killing horizons of rigidly-rotating observers



- Increasing $\widetilde{\Omega} = 2M\Omega$ brings the SOL closer to, and pushes the event horizon away form, the rotation axis.
- Increasing $\widetilde{Q} = Q/2M$ has an inverse effect.
- At large enough $\widetilde{\Omega}$, the SOL and event horizon join, such that β becomes imaginary throughout the equatorial plane.

Bulk viscosity in rigidly-rotating thermal states



- At $\widetilde{\Omega} \neq 0$, $\zeta = m\beta$ increases from 0 on the event horizon to ζ_M, from where it decreases to 0 on the SOL;
- If $\zeta_M > \zeta_{\rm max} \simeq 1.535$, η peaks twice between the two horizons, presenting a minimum between the two peaks.

Conclusion

- On maximally-symmetric spaces, the SOL forms closer (farther) to the rotation axis on dS (adS) compared to Minkowski.
- No SOL forms on adS if $\Omega < \omega$. At $\Omega > \omega$, β (and n, η, $\dots$) are constant on cones defined by $\Omega \sin\theta = \omega$.
- On Reissner-Nordström, the rotation enhances the EH, which joins the SOL at large Ω.
- At large enough β_0, η can exhibit two points of maxima between the EH and the SOL.
- The analysis serves as a good starting point for the interpretation of rigidly-rotating quantum thermal states.
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