Quantum-induced non-equilibrium effects in thermal states undergoing rigid rotation

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Outline

Motivation

- Kinetic theory requires that, in global thermodynamic equilibrium, the stress-energy tensor (SET) has the ideal form $T^{\mu\nu} = (E + P)u^\mu u^\nu + P g^{\mu\nu}$.[1]
- When the quantum nature of the gas is taken into account, the resulting SET departs from the equilibrium form and non-equilibrium terms appear.[2]
- Using the Eckart decomposition, the hydrodynamic content of the quantum-induced corrections can be highlighted.

[1] L. Rezzolla, O. Zanotti, *Relativistic hydrodynamics*, Oxford University Press, Oxford, UK (2013).

[2] F. Becattini, E. Grossi, Phys. Rev. D **92**, 045037 (2015).

Relativistic Boltzmann equation

- The relativistic Boltzmann equation can be written as[3]:

$$p^\mu \frac{\partial f}{\partial x^\mu} - \Gamma^i{}_{\mu\nu} p^\mu p^\nu \frac{\partial f}{\partial p^i} = J[f].$$

- $J[f]$ is the collision term governing the relaxation towards thermodynamic equilibrium $f = f^{(\mathrm{eq})}$.
- For Fermi-Dirac statistics, $f^{(\mathrm{eq})}$ is:

$$f^{(\mathrm{eq})} = \frac{Z}{\exp(-\beta p^\mu u_\mu) + 1}.$$

- Global thermodynamic equilibrium ($f = f^{(\mathrm{eq})}$) attained if βu^μ is a Killing vector field:

$$(\beta u_\mu)_{;\nu} + (\beta u_\nu)_{;\mu} = 0.$$

[3] C. Cercignani, G. M. Kremer, *The relativistic Boltzmann equation: theory and applications*, Birkhäuser Verlag, Basel, Switzerland (2002)

Rigidly-rotating equilibrium states

- Setting $\beta \mathbf{u} = \partial_t + \Omega \partial_\varphi$ gives:

$$\beta = \gamma^{-1} \beta_0, \qquad \mathbf{u} = \gamma(\partial_t + \Omega \partial_\varphi), \qquad \gamma = \frac{1}{\sqrt{1 - \rho^2 \Omega^2}}.$$

- The temperature β^{-1} diverges as the SOL ($\rho\Omega = 1$) is approached.
- The resulting particle flux $N^{\hat{\alpha}}$ and SET are:

$$N^{\hat{\alpha}} = \int \frac{d^3p}{p^{\hat{0}}} f\, p^{\hat{\alpha}} = n u^{\hat{\alpha}},$$

$$T^{\hat{\alpha}\hat{\gamma}} = \int \frac{d^3p}{p^{\hat{0}}} f\, p^{\hat{\alpha}} p^{\hat{\gamma}} = (E + P) u^{\hat{\alpha}} u^{\hat{\beta}} + P \eta^{\hat{\alpha}\hat{\beta}},$$

- Results in the massless case[4]:

$$n = \frac{3\zeta(3)\gamma^3}{\pi^2 \beta_0^3}, \qquad E = \frac{7\pi^2\gamma^4}{60\beta_0^4}, \qquad P = \frac{7\pi^2\gamma^4}{180\beta_0^4}.$$

- The SET diverges as γ^4 as the SOL is approached.

[4] V. E. Ambruș, R. Blaga, Annals of West University of Timisoara - Physics **58** (2015) 89.

Quantum field theory approach

- In co-moving coordinates $\varphi = \varphi_{\rm M} - \Omega t_{\rm M}$, the Minkowski metric reads:

$$ds^2 = -\gamma^{-2} dt^2 + 2\rho^2 \Omega \, dt \, d\varphi + d\rho^2 + \rho^2 d\varphi^2 + dz^2.$$

- The Dirac equation takes the form:

$$\left[\gamma^{\hat{t}}(H + \Omega J_z) - \boldsymbol{\gamma} \cdot \mathbf{P} - \mu\right] \psi(x) = 0.$$

- Mode solutions of Minkowski energy E, z-axis momentum k, z-axis angular momentum $m + \frac{1}{2}$ and helicity λ[5]:

$$U^{\lambda}_{Ekm} = \frac{1}{2\pi} e^{-i\widetilde{E}t + ikz} u^{\lambda}_{Ekm}.$$

- $\widetilde{E} = E - \Omega(m + \frac{1}{2})$ is the eigenvalue of the rotating Hamiltonian $H = i\partial_t$.

[5] V. E. Ambruș and E. Winstanley, Phys. Lett. B **734**, 296 (2014).

Thermal field theory theory

- Thermal expectation values (t.e.v.s) are defined as averages over the Fock space with the weight function $\exp(-\beta_0 H)$[6]:

$$\langle A \rangle_\beta = \mathrm{tr}(e^{-\beta_0 H} A).$$

- The building blocks for the computation of t.e.v.s are[7]:

$$\langle a_k^\dagger a_{k'} \rangle_\beta = \frac{\delta(k,k')}{e^{\beta \widetilde{\omega_k}} - 1}, \qquad \langle b_k^\dagger b_{k'} \rangle_\beta = \langle d_k^\dagger d_{k'} \rangle_\beta = \frac{\delta_{kk'}}{e^{\beta \widetilde{E_k}} + 1}.$$

- The t.e.v. of the SET can be obtained as:

$$\langle : T_{\hat{\alpha}\hat{\gamma}} : \rangle_\beta = \langle T_{\hat{\alpha}\hat{\gamma}} \rangle_\beta - \langle 0 | T_{\hat{\alpha}\hat{\gamma}} | 0 \rangle \,.$$

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[6] N. D. Birrell and P. C. W. Davies, *Quantum fields in curved space* (Cambridge University Press, Cambridge, England, 1982).

[7] A. Vilenkin, Phys. Rev. D **21**, 2260 (1980).

Eckart vs. Landau decompositions

- **Eckart**: $u^{\hat{\alpha}}$ is derived from $N^{\hat{\alpha}}$:

$$N^{\hat{\alpha}} = \int \frac{d^3p}{p^{\hat{0}}} f\, p^{\hat{\alpha}} = n u^{\hat{\alpha}}, \qquad N_{\hat{\alpha}} N^{\hat{\alpha}} = -n^2,$$

$$T^{\hat{\alpha}\hat{\gamma}} = \int \frac{d^3p}{p^{\hat{0}}} f\, p^{\hat{\alpha}} p^{\hat{\gamma}} = E u^{\hat{\alpha}} u^{\hat{\gamma}} + (P + \overline{\omega}) \Delta^{\hat{\alpha}\hat{\gamma}} + 2 q^{(\hat{\alpha}} u^{\hat{\gamma})} + \pi^{\hat{\alpha}\hat{\gamma}},$$

where the energy density E, hydrostatic pressure P, dynamic pressure $\overline{\omega}$, heat flux $q^{\hat{\alpha}}$ and pressure deviator $\pi^{\hat{\alpha}\hat{\gamma}}$ can be computed using:

$$E = u_{\hat{\alpha}} u_{\hat{\gamma}} T^{\hat{\alpha}\hat{\beta}}, \qquad P + \overline{\omega} = \frac{1}{3} \Delta_{\hat{\alpha}\hat{\beta}} T^{\hat{\alpha}\hat{\beta}},$$

$$q^{\hat{\alpha}} = -\Delta^{\hat{\alpha}}{}_{\hat{\beta}} u_{\hat{\gamma}} T^{\hat{\beta}\hat{\gamma}}, \qquad \pi^{\hat{\alpha}\hat{\gamma}} = \left(\Delta^{\hat{\alpha}}{}_{\hat{\gamma}} \Delta^{\hat{\beta}}{}_{\hat{\rho}} - \frac{1}{3} \Delta^{\hat{\alpha}\hat{\beta}} \Delta_{\hat{\gamma}\hat{\rho}} \right) T^{\hat{\gamma}\hat{\rho}},$$

where $\Delta^{\hat{\alpha}\hat{\beta}} = \eta^{\hat{\alpha}\hat{\beta}} + u^{\hat{\alpha}} u^{\hat{\beta}}$ projects on the hypersurface orthogonal to $u^{\hat{\alpha}}$.

- **Landau**: $u_L^{\hat{\alpha}}$ is an eigenvector of $T^{\hat{\alpha}}{}_{\hat{\beta}}$: $T^{\hat{\alpha}}{}_{\hat{\gamma}} u_L^{\hat{\gamma}} = -E_L u_L^{\hat{\alpha}}$:

$$T^{\hat{\alpha}\hat{\gamma}} = E_L u_L^{\hat{\alpha}} u_L^{\hat{\gamma}} + (P_L + \overline{\omega}_L) \Delta_L^{\hat{\alpha}\hat{\gamma}} + \pi_L^{\hat{\alpha}\hat{\gamma}}.$$

- No heat flux in the Landau frame.

Eckart decomposition

- The Eckart decomposition reveals non-zero heat flux and pressure deviator ($P + \overline{\omega} = E/3$):

$$
\begin{aligned}
E =& \frac{7\pi^2\gamma^4}{60\beta^4} + \frac{\Omega^2}{8\beta^2}\left(\frac{4\gamma^6}{3} - \frac{\gamma^4}{3}\right), \\
q^{\hat{\alpha}} =& \frac{\rho\Omega^3\gamma^7}{18\beta^2}\left(\rho\Omega, 0, 1, 0\right), \qquad \gamma = \frac{1}{\sqrt{1-\rho^2\Omega^2}}, \\
\pi^{\hat{\alpha}\hat{\beta}} =& 0.
\end{aligned}
$$

- The classical (kinetic theory) part is subleading as $\rho\Omega \to 1$.

Landau decomposition

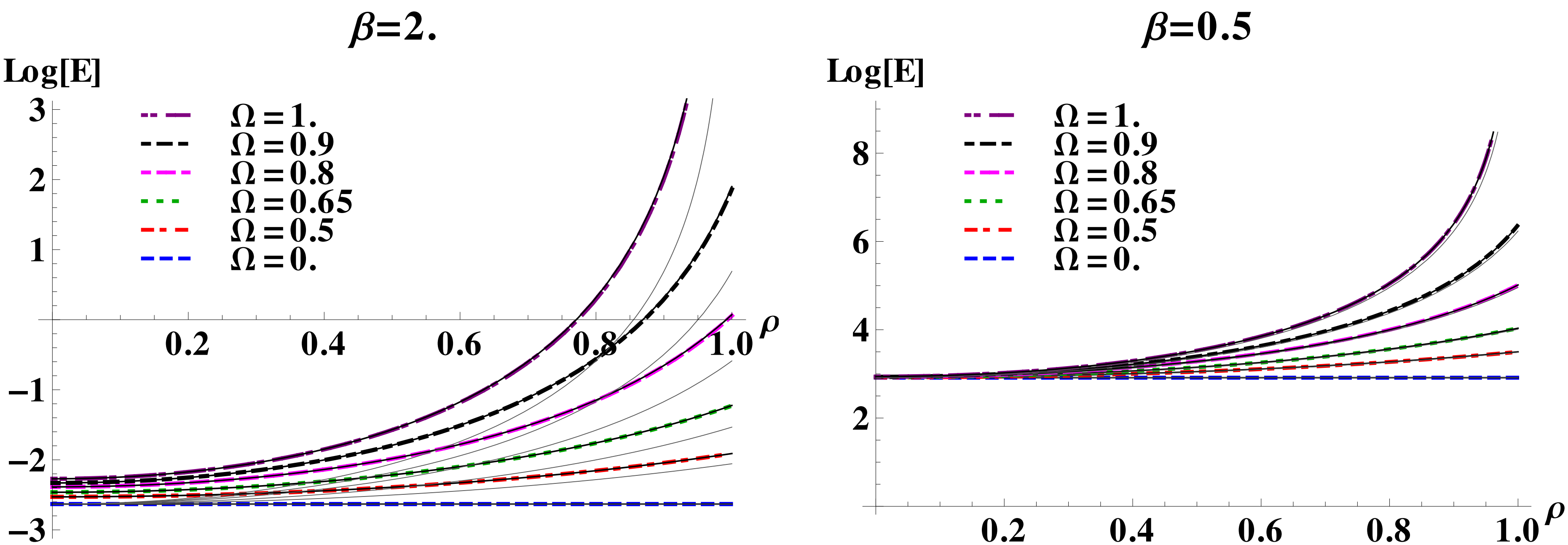
- The Landau energy and velocity have the same form as in the scalar case, with vanishing pressure deviator:

$$E_L = \frac{E}{3} + \sqrt{\frac{4E^2}{9} - q^2},$$
$$u_L^{\hat{\alpha}} = \sqrt{\frac{E_L + \frac{1}{3}E}{2(E_L - \frac{1}{3}E)}} \left(u^{\hat{\alpha}} + \frac{q^{\hat{\alpha}}}{E_L + \frac{1}{3}E} \right).$$

- The pressure deviator in the Landau frame is given by:

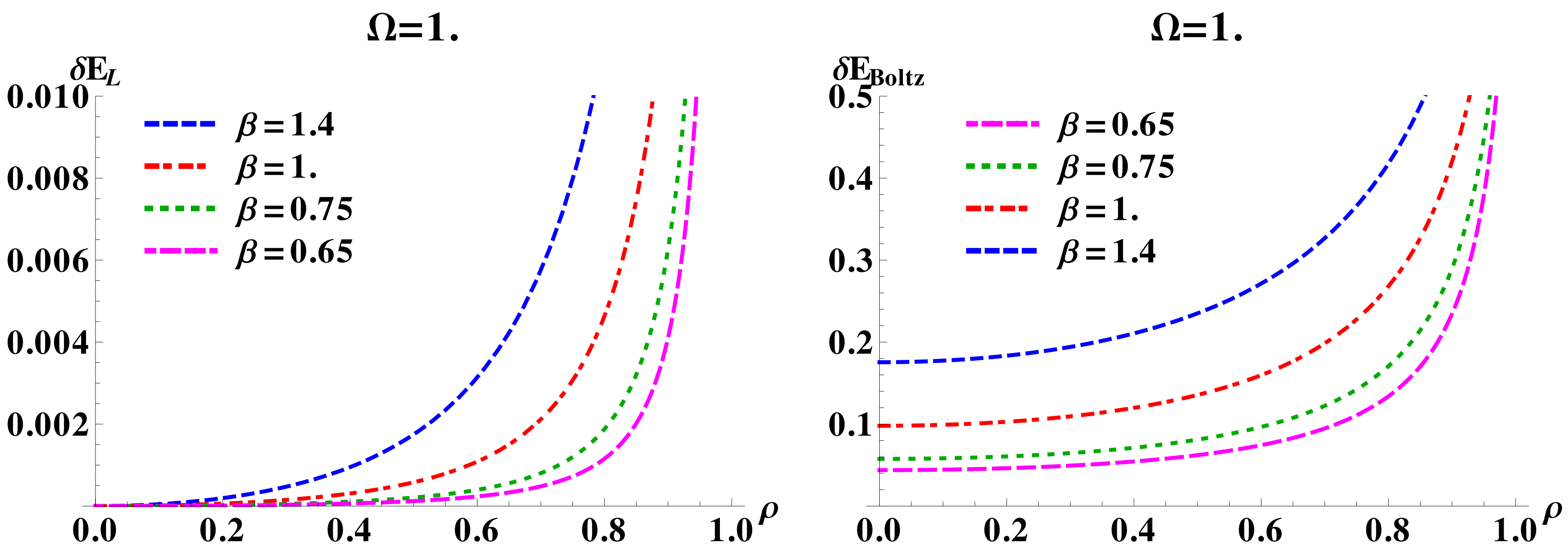
$$\pi_L^{\hat{\alpha}\hat{\beta}} = \frac{2(E - E_L)(3E_L - 2E)}{3(3E_L - E)} u^{\hat{\alpha}} u^{\hat{\beta}} + \frac{E - E_L}{3} \eta^{\hat{\alpha}\hat{\beta}} - \frac{E - E_L}{3E_L - E}(u^{\hat{\alpha}} q^{\hat{\beta}} + u^{\hat{\beta}} q^{\hat{\alpha}}) - \frac{6E_L}{(3E_L - E)(3E_L + E)} q^{\hat{\alpha}} q^{\hat{\beta}}.$$

Comparison of Landau, Eckart and Boltzmann energies



- The Landau energy (thin, continuous dark lines) presents negligible deviations from the Eckart energy (discontinuous, coloured lines).
- The difference between the Boltzmann (thin, continuous light gray lines) and Eckart energies increase for fixed ρ and β as Ω increases; as well as for fixed Ω as ρ is increased.

Comparison to the Eckart energy

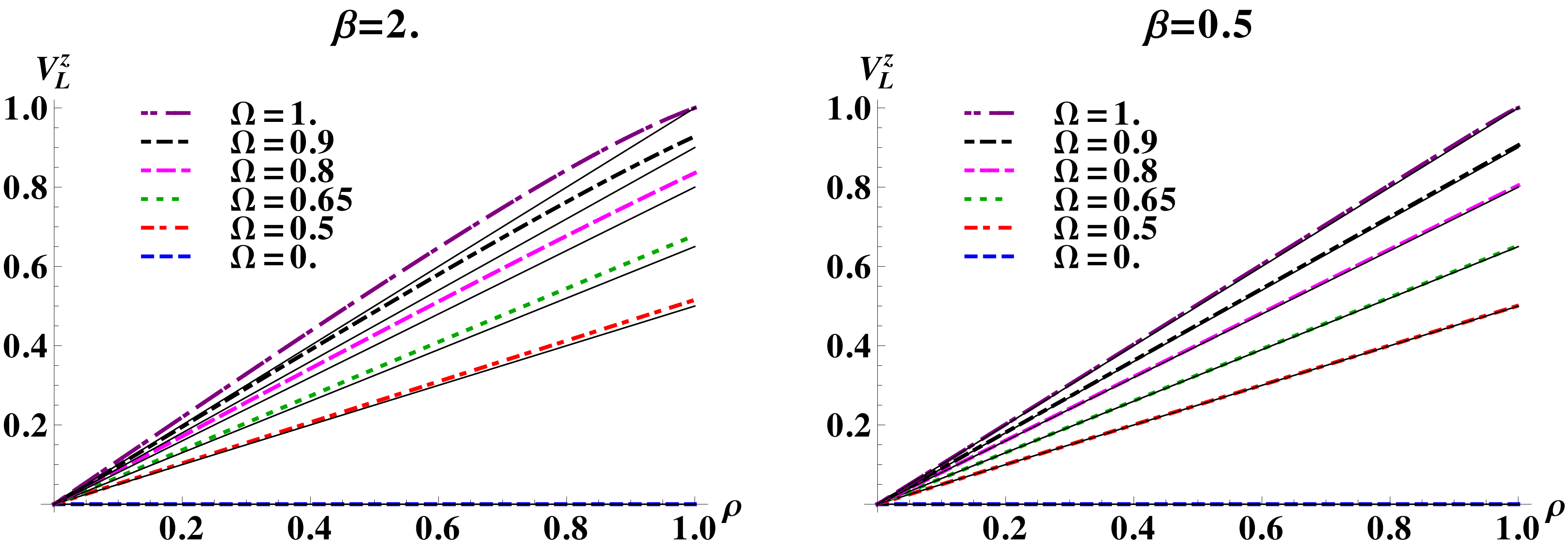


- To better asses the departure of the Landau and Boltzmann energies from the Eckart energy, the following quantities are represented above:

$$\delta E_L = 1 - \frac{E_L}{E}, \qquad \delta E_{\rm Boltz} = 1 - \frac{E_{\rm Boltz}}{E}.$$

- For all β, δE_L and $\delta E_{\rm Boltz}$ increase as the SOL is approached.
- δE_L and $\delta E_{\rm Boltz}$ decrease as the temperature β^{-1} increases.

Landau velocity



- The above plots compare the azimuthal velocity $v_L^z = u_L^3/u_L^0$ in the Landau frame (coloured, discontinuous lines) to the velocity of rigidly-rotating observers $v^z = \rho\Omega$.
- The Landau velocity increases monotonically, but non-linearly, with ρ and Ω.
- $v_L^z \geq v^z$ everywhere.
- v_L^z approaches v^z as β is increased.

Rotating Dirac particles inside a cylindrical boundary

- MIT bag boundary conditions[8]:

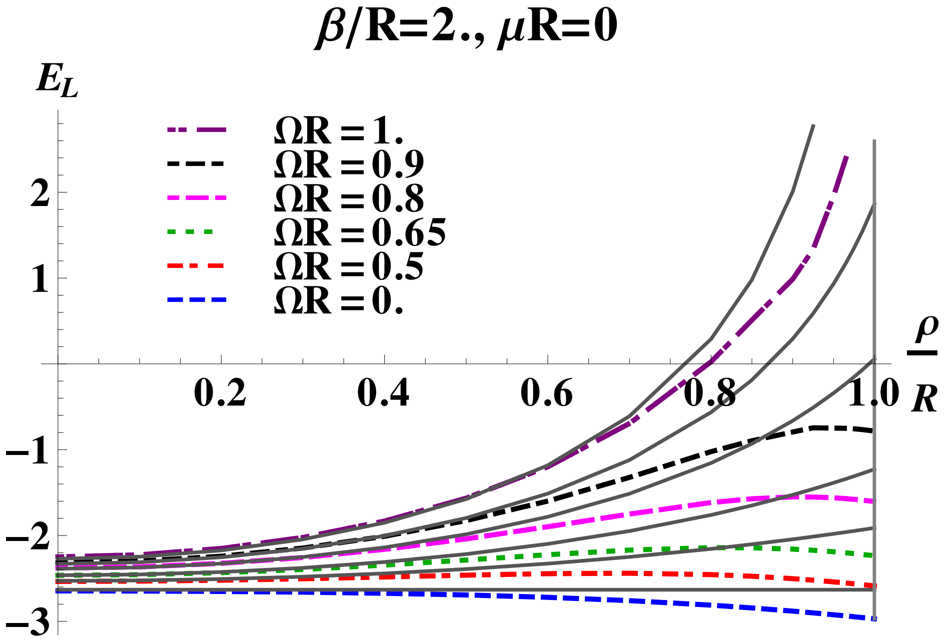
$$i\gamma^{\hat{\rho}}\psi(x_{\rm b}) = -\psi(x_{\rm b}).$$

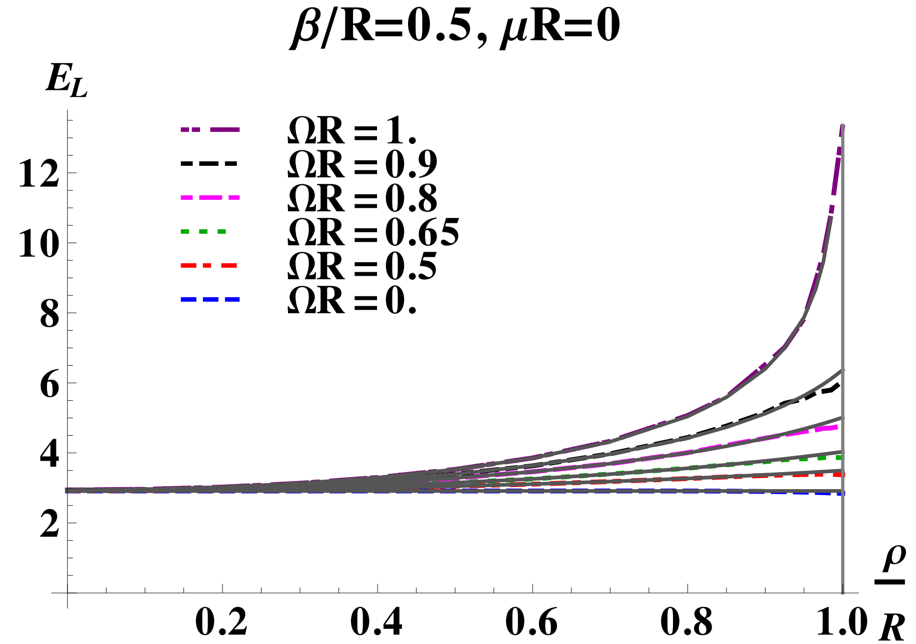
- The boundary conditions imply the quantisation of the transverse momentum q:[9]

$$\mathfrak{j}_{m,\ell}^2 - \frac{2\mu}{q_{m\ell}}\mathfrak{j}_{m,\ell} - 1 = 0, \qquad \mathfrak{j}_{m\ell} = \frac{J_m(q_{m,\ell}R)}{J_{m+1}(q_{m,\ell}R)}.$$

[8] A. Chodos et al., Phys. Rev. D **9** (1974) 3471.
[9] V. E. Ambruș, E. Winstanley, Phys. Rev. D **93** (2016) 104014.

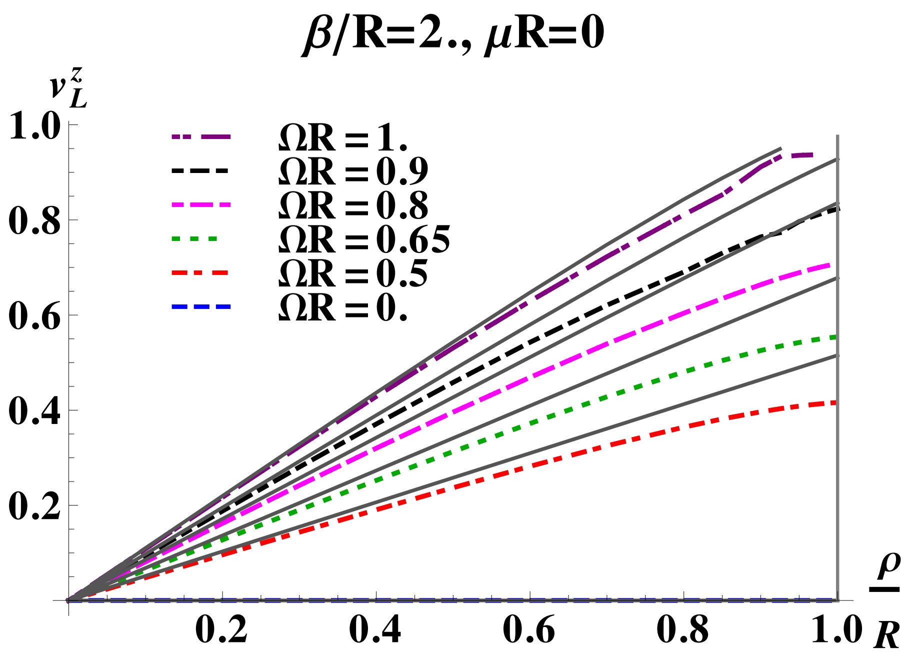
Landau energy of the bounded system

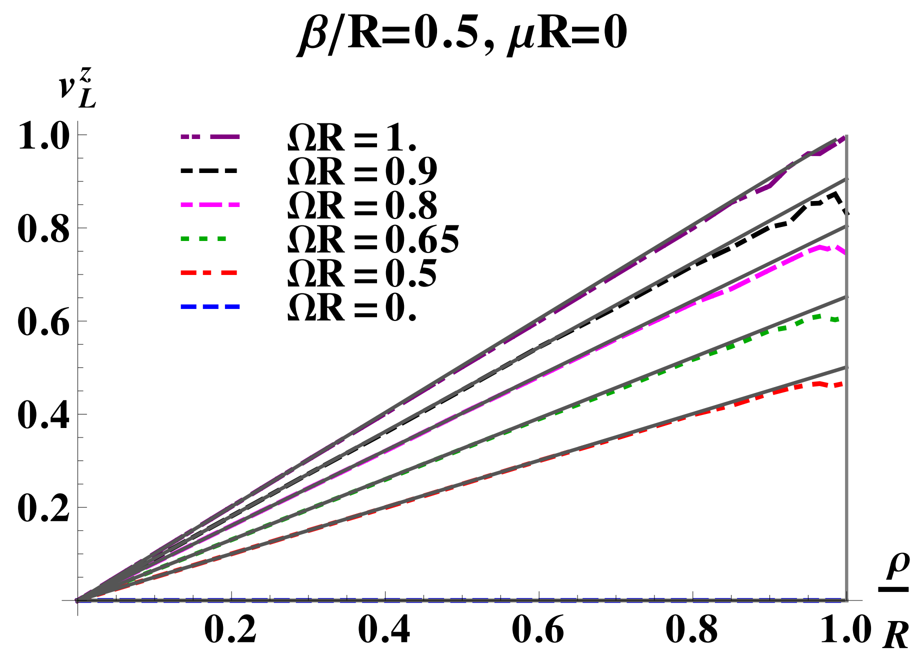




- Comparison between the Landau energies of the bounded (coloured, discontinuous lines) and unbounded systems (thin, continuous lines).
- E_L is always greater in the unbounded system than in the presence of the boundary.
- E_L approaches the unbounded form as β is increased.

Landau velocity





- The Landau velocity $v_L^z = u_L^z / u_L^0$ is lower in the presence of a boundary.
- v_L^z approaches the unbounded profile as the temperature β^{-1} is increased.

Massless scalar field – Eckart decomposition

- The Eckart decomposition reveals non-zero heat flux and pressure deviator ($P + \overline{\omega} = E/3$):

$$
\begin{aligned}
E =& \frac{\pi^2\gamma^4}{30\beta^4} + \frac{\Omega^2\gamma^6}{36\beta^2}, \qquad\qquad \gamma = \frac{1}{\sqrt{1-\rho^2\Omega^2}} \\
q^{\hat{\alpha}} =& \frac{\rho\Omega^3\gamma^7}{18\beta^2}\left(\rho\Omega, 0, 1, 0\right), \\
\pi^{\hat{\alpha}\hat{\beta}} =& \frac{\Omega^2\gamma^8}{54\beta^2}\begin{pmatrix} \rho^2\Omega^2 & 0 & \rho\Omega & 0 \\ 0 & \gamma^{-2} & 0 & 0 \\ \rho\Omega & 0 & 1 & 0 \\ 0 & 0 & 0 & -2\gamma^{-2} \end{pmatrix}.
\end{aligned}
$$

- The classical (kinetic theory) part is subleading as $\rho\Omega \to 1$.

Massless scalar field – Landau decomposition

- The Landau energy and velocity can be found by solving the eigenvalue problem:

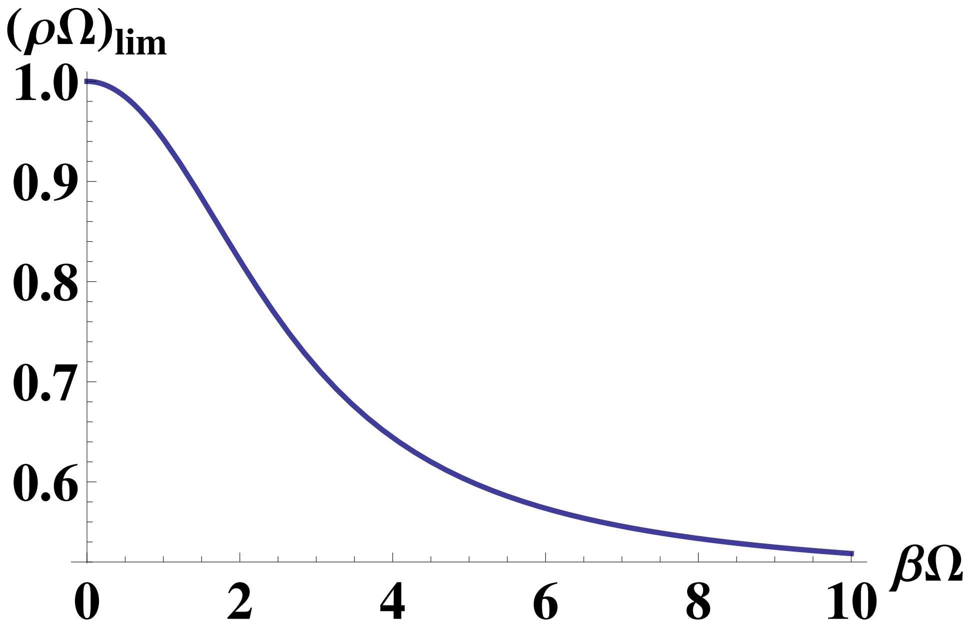
$$E_L = \frac{E}{3} - \frac{1}{2}\hat{\boldsymbol{q}} \cdot \boldsymbol{\pi} \cdot \hat{\boldsymbol{q}} + \sqrt{\left(\frac{2E}{3} + \frac{1}{2}\hat{\boldsymbol{q}} \cdot \boldsymbol{\pi} \cdot \hat{\boldsymbol{q}}\right)^2 - q^2},$$

$$u_L^{\hat{\alpha}} = \sqrt{\frac{E_L + \frac{1}{3}E + \hat{\boldsymbol{q}} \cdot \boldsymbol{\pi} \cdot \hat{\boldsymbol{q}}}{2(E_L - \frac{1}{3}E + \frac{1}{2}\hat{\boldsymbol{q}} \cdot \boldsymbol{\pi} \cdot \hat{\boldsymbol{q}})}} \left(u^{\hat{\alpha}} + \frac{q^{\hat{\alpha}}}{E_L + \frac{1}{3}E + \hat{\boldsymbol{q}} \cdot \boldsymbol{\pi} \cdot \hat{\boldsymbol{q}}}\right).$$

- The pressure deviator in the Landau frame is given by:

$$\pi_L^{\hat{\alpha}\hat{\beta}} = \pi^{\hat{\alpha}\hat{\beta}} + \frac{E - E_L}{3}\eta^{\hat{\alpha}\hat{\beta}} + \frac{2(E - E_L)(E_L - \frac{2}{3}E + \hat{\boldsymbol{q}} \cdot \boldsymbol{\pi} \cdot \hat{\boldsymbol{q}})}{3(E_L - \frac{1}{3}E + \frac{1}{2}\hat{\boldsymbol{q}} \cdot \boldsymbol{\pi} \cdot \hat{\boldsymbol{q}}} u^{\hat{\alpha}} u^{\hat{\beta}}$$
$$+ \frac{E_L - E + \frac{3}{2}\hat{\boldsymbol{q}} \cdot \boldsymbol{\pi} \cdot \hat{\boldsymbol{q}}}{3(E_L - \frac{1}{3}E + \frac{1}{2}\hat{\boldsymbol{q}} \cdot \boldsymbol{\pi} \cdot \hat{\boldsymbol{q}}}(u^{\hat{\alpha}} q^{\hat{\beta}} + u^{\hat{\beta}} q^{\hat{\alpha}})$$
$$- \frac{\frac{2}{3}E_L}{(E_L + \frac{1}{3}E + \hat{\boldsymbol{q}} \cdot \boldsymbol{\pi} \cdot \hat{\boldsymbol{q}})(E_L - \frac{1}{3}E + \frac{1}{2}\hat{\boldsymbol{q}} \cdot \boldsymbol{\pi} \cdot \hat{\boldsymbol{q}})} q^{\hat{\alpha}} q^{\hat{\beta}}.$$

Validity of the Landau decomposition

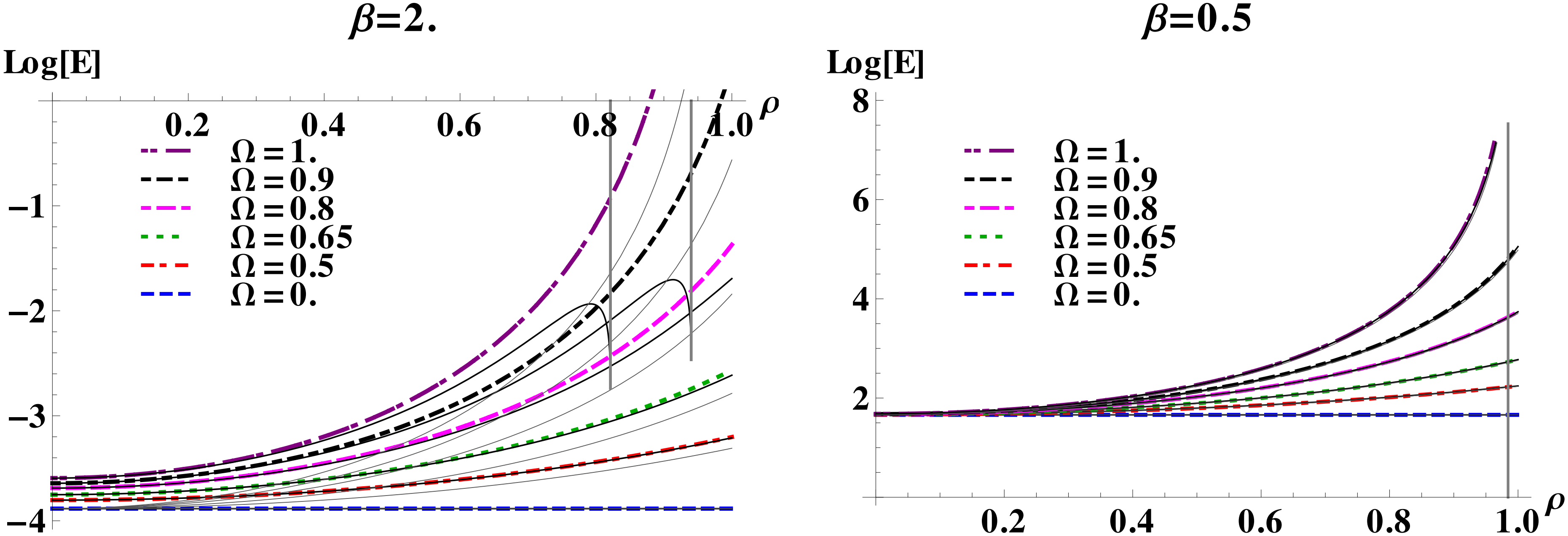


- The Landau frame is well-defined only where E_L is real and positive, requiring that $\rho\Omega < (\rho\Omega)_{\rm lim}$:

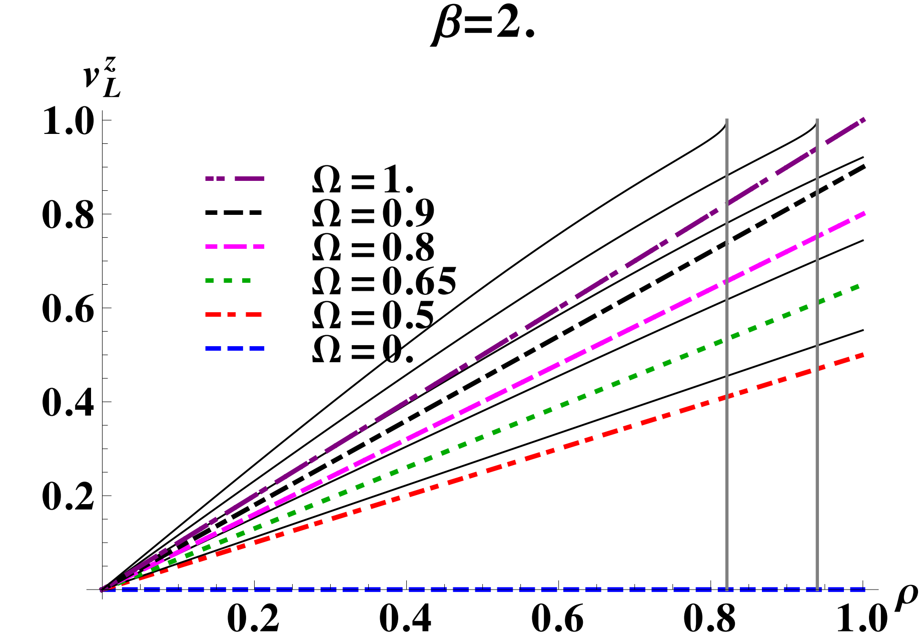
$$(\rho\Omega)_{\rm lim} = \sqrt{x^2 + x + 1} - x, \qquad x = \frac{5}{4\pi^2}(\beta\Omega)^2.$$

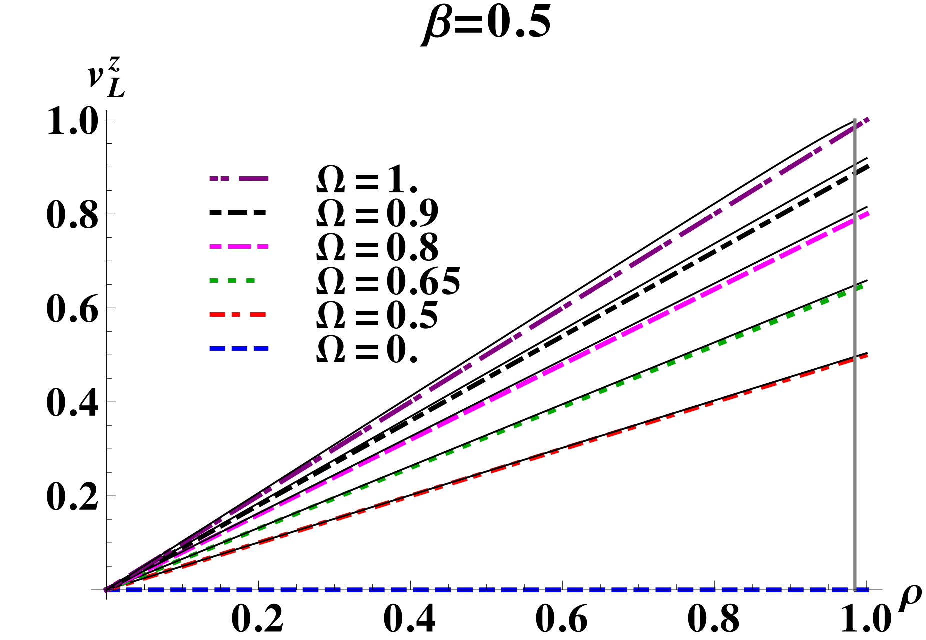
- At $\beta\Omega \to 0$, $(\rho\Omega)_{\rm lim} \to 1$ and the Landau frame is well-defined up to the SOL;
- As $\beta\Omega \to \infty$, $(\rho\Omega)_{\rm lim} \to 0.5$.

Klein-Gordon: comparison of energy density



Klein-Gordon: comparison of azimuthal velocity





Conclusion

- Thermal field theory reveals a non-vanishing heat flux in the thermal expectation value of the stress-energy tensor for a rigidly-rotating massless Dirac field.
- The quantum corrections dominate over the classical (kinetic theory) result in the vicinity of the speed-of-light surface (SOL).
- The velocity in the Landau frame is always larger than the co-rotating velocity ($\rho\Omega$).
- Boundary effects become important at low temperatures.
- This work was supported by a grant of the Romanian National Authority for Scientific Research and Innovation, CNCS-UEFISCDI, project number PN-II-RU-TE-2014-4-2910.