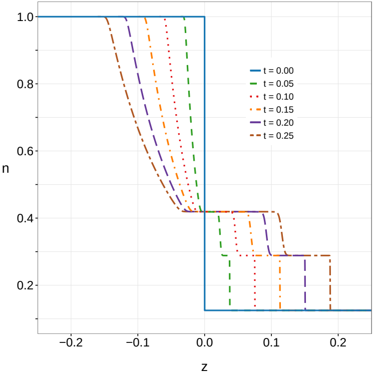
Simulating relativistic flows with a Lattice Boltzmann model based on quadratures



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Outline

1. **Relativistic Boltzmann equation**
2. **Our Lattice Boltzmann model**
3. **Results: The Riemann problem**

Boltzmann equation

- Tool for investigating fluids at a mesoscopic level
- Different levels
  - Macroscopic: continuous fields (hydrodynamics)
  - Mesoscopic: distribution function (*Boltzmann equation*)
  - Microscopic: individual molecules (molecular dynamics)
- Provides a good description of fluids from the hydrodynamic to the ballistic regime

Relativistic Boltzmann equation

- Boltzmann equation for 1D flow in flat spacetime:

$$(\partial_t + v^z\partial_z)\, f = -\tfrac{u^0 - v^z u^z}{\tau}\,(f - f^{\mathrm{eq}})\,,$$

$$v^z \equiv \xi = \tfrac{p^z}{p^0}, \qquad p^0 = p \equiv |\vec{p}|^2. \tag{1}$$

- The relativistic Maxwell-Jüttner (M-J) distribution function for massless particles:

$$f^{eq} = \frac{n}{8\pi T^3}\exp\left(\frac{p^{\hat{\alpha}}u_{\hat{\alpha}}}{T}\right). \tag{2}$$

Quadrature method + finite difference

- We project the distribution functions onto a set of orthogonal polynomials, and truncate the series

$$f^{eq}(x,\vec{p},t) = e^{-p} \sum_{k=0}^{N_p-1} \sum_{n=0}^{N_v-1} a^{i_1\ldots i_n}_{(nk)}(x,t) P^{(n)}_{i_1..i_n}(\vec{v})\, L^{(1)}_j(p) \qquad (3)$$

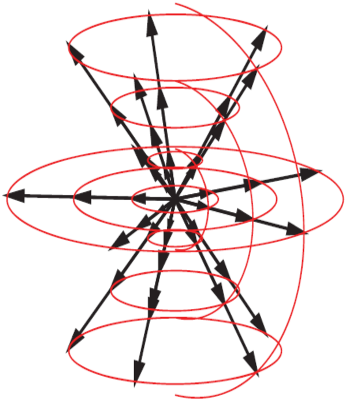
- Use Gauss-quadrature method to write integrals as sums[1,2]

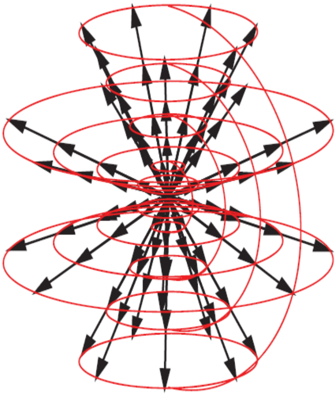
$$\begin{aligned}
\mathcal{Q}(p,\theta) &= \int_0^{2\pi} d\phi\, f^{eq} A_n(p,\xi,\phi) = \sum\nolimits_{i=0}^{Q_\phi-1} \frac{2\pi}{Q_\phi} f^{eq}_i A_n(p,\theta,\phi_i) \\
\mathcal{E}(p) &= \int_{-1}^{1} d(\cos\theta)\, \mathcal{Q}(p,\theta) = \sum\nolimits_{j=0}^{Q_\xi-1} w^\xi_j\, \mathcal{Q}(p,\theta_i) \\
\mathcal{M} &= \int_0^\infty dp\, p\, e^{-p}\, \mathcal{E}(p) = \sum\nolimits_{k=0}^{Q_p-1} w^p_k\, \mathcal{E}(p_k) \qquad (4)
\end{aligned}$$

[1] V.E. Ambruş, V. Sofonea, Phys. Rev. E **86**, 016708 (2012)

[2] P. Romatschke, M. Mendoza, and S. Succi, Phys. Rev. C **84**, 034903 (2012)

- Examples for sets of momenta:[3]

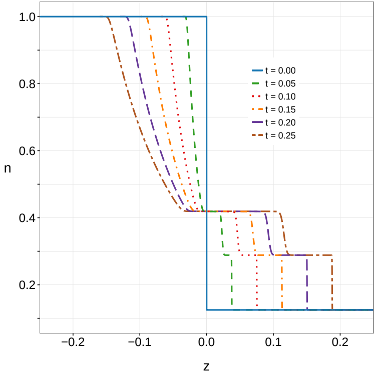


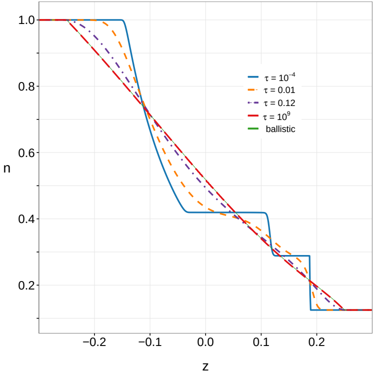


$$\vec{p} \to \{p_i, \xi_j, \phi_k\} \tag{5}$$

[3] V.E. Ambruş, V. Sofonea, Phys. Rev. E **86**, 016708 (2012)

The Riemann problem



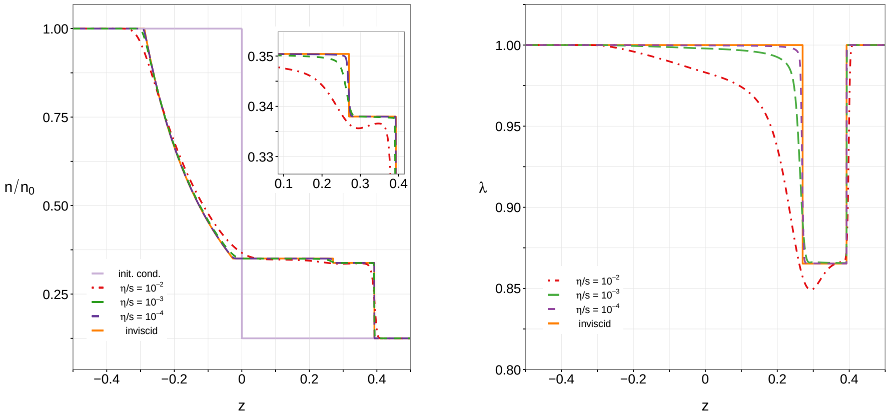


- Time evolution of the Sod shock tube problem[4]: formation of a shock-wave, contact discontinuity and rarefaction wave[5]
- As τ increases, the features are smoothed, while finally in the ballistic regime they do no longer form.

[4] R. Blaga and V. Ambrus, arXiv:1608.03004 (2016)

[5] L. Rezzolla and Z. Olindo, *Relativistic hydrodynamics*, Oxford University Press, 2013.

Inviscid regime ($\tau \to 0$): density and fugacity

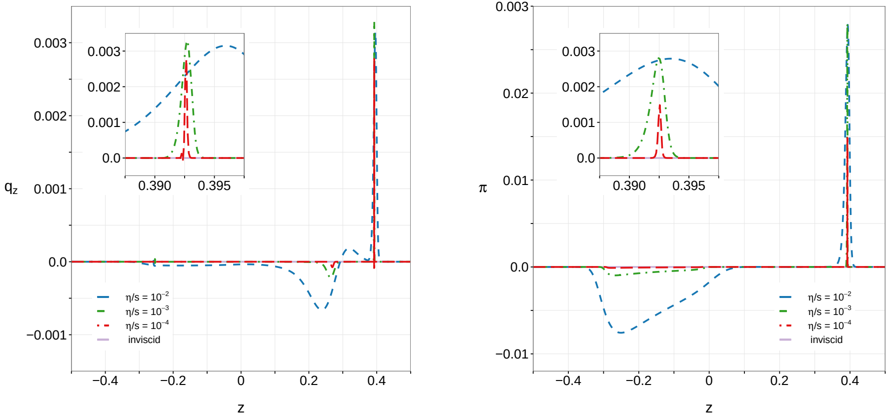


- density (n) and fugacity (λ) profiles for different relaxation times[6]

$$\frac{\eta}{s} \simeq \frac{1}{6}\tau T, \qquad \lambda = e^{\mu_E/T} \equiv \frac{1}{d_G}\frac{n}{8\pi T^3}. \tag{6}$$

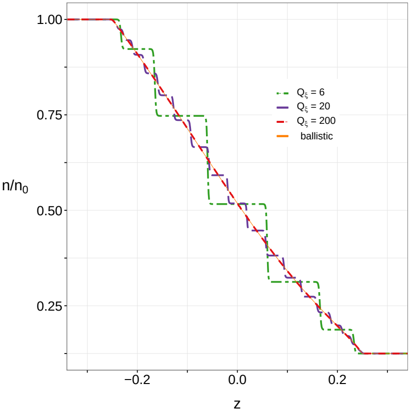
[6] Initial conditions taken from: I. Bouras et al. , Phys. Rev. Lett. **103** (2009).

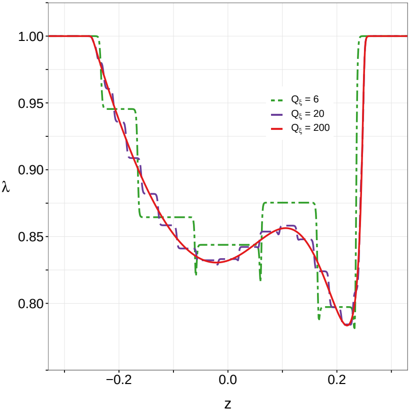
Inviscid regime ($\tau \to 0$): heat flow and shear pressure



- parallel component of the heat flux (q_z) and shear pressure (π), for decreasing values of the relaxation time

[6] I. Bouras et al. , Phys. Rev. Lett. **103** (2009).

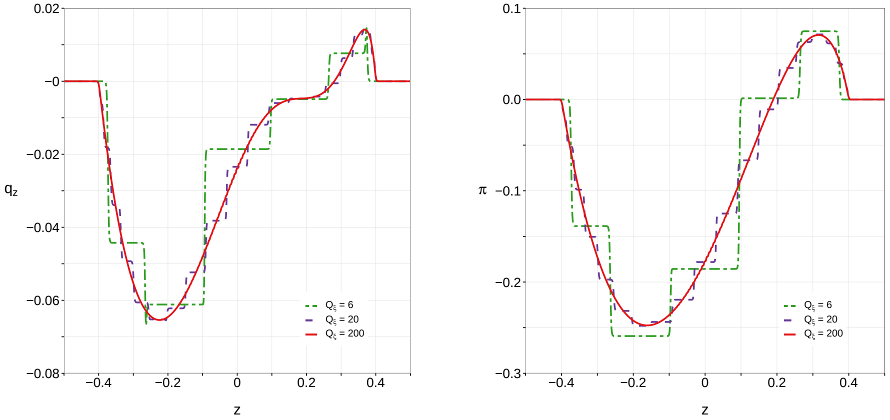




- density (n) and fugacity (λ) profiles in the ballistic regime, for increasing order of the polar quadrature

[6] I. Bouras et al. , Phys. Rev. Lett. **103** (2009).

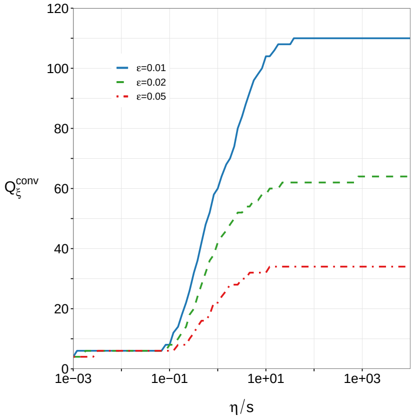
Inviscid regime ($\tau \to \infty$): heat flow and shear pressure



- parallel component of the heat flux (q_z) and shear pressure (π) profiles in the ballistic regime, for increasing order of the polar quadrature

[6] I. Bouras et al. , Phys. Rev. Lett. **103** (2009).

Convergence test



- order of the polar quadrature necessary for convergence at a fixed accuracy

$$\varepsilon = \max_z \left[\frac{n(z) - n_{\text{ref}}(z)}{\Delta n} \right]. \tag{7}$$

- nr. of velocities : $Q = Q_p * Q_\phi * Q_\xi$

$$\begin{aligned} \text{inviscid} : Q &= 3 * 2 * 4 = 24 \\ \text{ballistic} : Q &= 3 * 2 * 110 = 660. \end{aligned} \tag{8}$$

Summary

- we have developed a R-SLaB model based on Laguerre and Gauss quadratures
- numerical tests show good results across all regimes of the relaxation time (from hydro, through viscous, up to the ballistic regime)
- convergence test shows we need:
  - 24-36 velocities in the inviscid regime
  - ~ 1000 velocities in the ballistic regime
  - transition to higher orders around $\tau \sim 0.01 - 0.1$.

THANK YOU!

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