Study of heat transfer in Cartesian, cylindrical and spherical geometries

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Outline

- Boltzmann equation with respect to arbitrary coordinates
- Boltzmann equation with respect to orthonormal vielbein fields
- Gauss-Hermite quadratures
- Heat transfer in cartesian, cylindrical and spherical symmetric geometries
  - Hydrodynamic,transition and free molecular flow regimes

Boltzmann equation

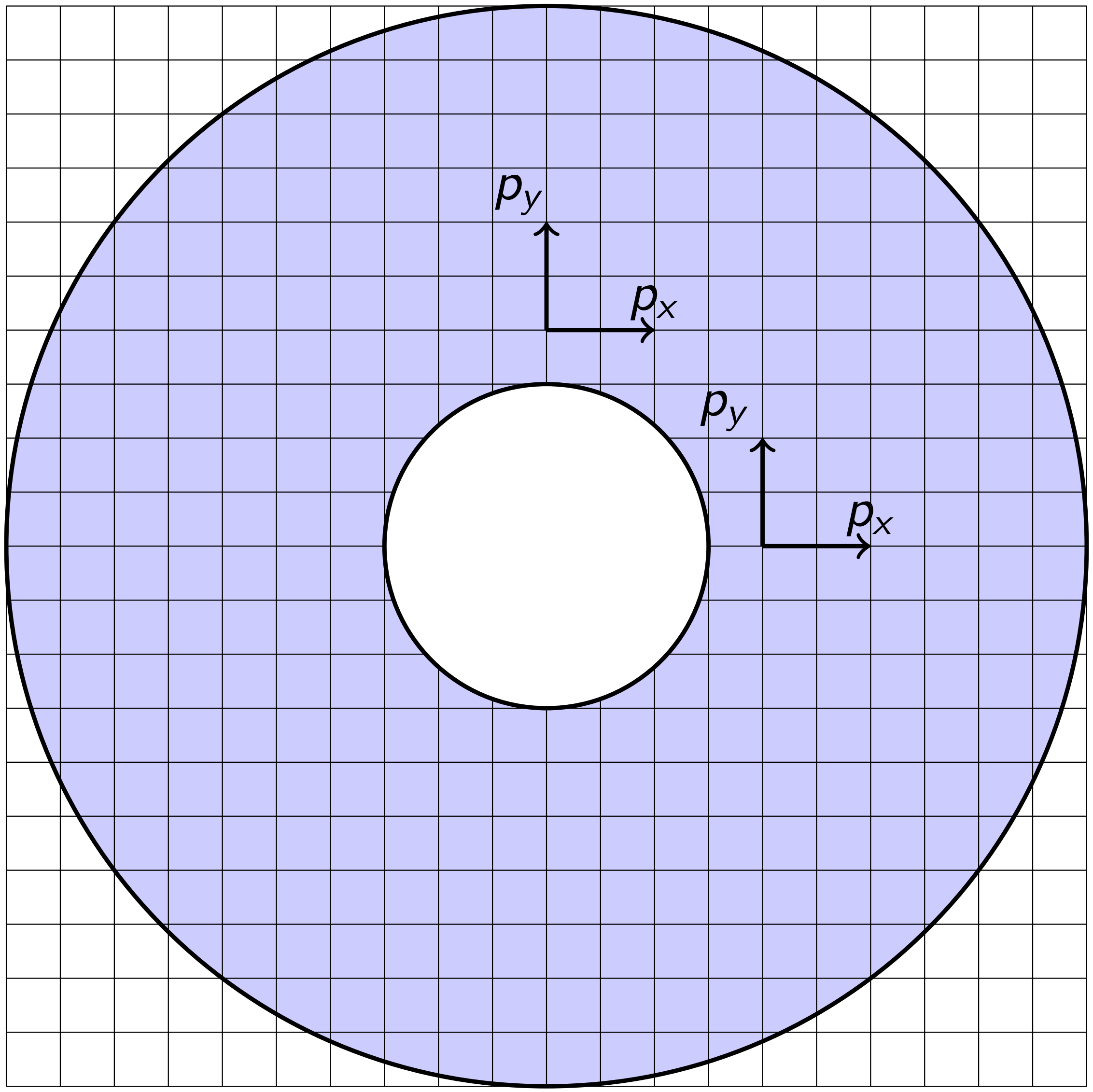
- Evolution equation of the one-particle distribution function f with respect to the Cartesian coordinates:

$$\frac{\partial f}{\partial t} + \frac{p^i}{m}\frac{\partial f}{\partial x^i} + F^i \frac{\partial f}{\partial p^i} = J[f]. \tag{1}$$

- Hydrodynamic moments of order N give macroscopic quantities:

$$\begin{aligned}
&N = 0: && \text{number density:} && n = \int d^3p\, f, \\
&N = 1: && \text{velocity:} && \mathbf{u} = \frac{1}{nm}\int d^3p\, f\, \mathbf{p}, \\
&N = 2: && \text{temperature:} && T = \frac{2}{3n}\int d^3p\, f\, \frac{\boldsymbol{\xi}^2}{2m}, \qquad (\boldsymbol{\xi} = \mathbf{p} - m\mathbf{u}), \\
&N = 3: && \text{heat flux:} && \mathbf{q} = \frac{1}{2m^2}\int d^3p\, f\, \boldsymbol{\xi}^2\, \boldsymbol{\xi}.
\end{aligned}$$

Cartesian grid and momentum space decomposition along the Cartesian axes.



Boltzmann equation - arbitrary coordinates

- Consider a coordinate transformation $x^i \to x^{\widetilde{i}}$, which induces a metric $g_{\widetilde{\imath}\widetilde{\jmath}}$, as follows:

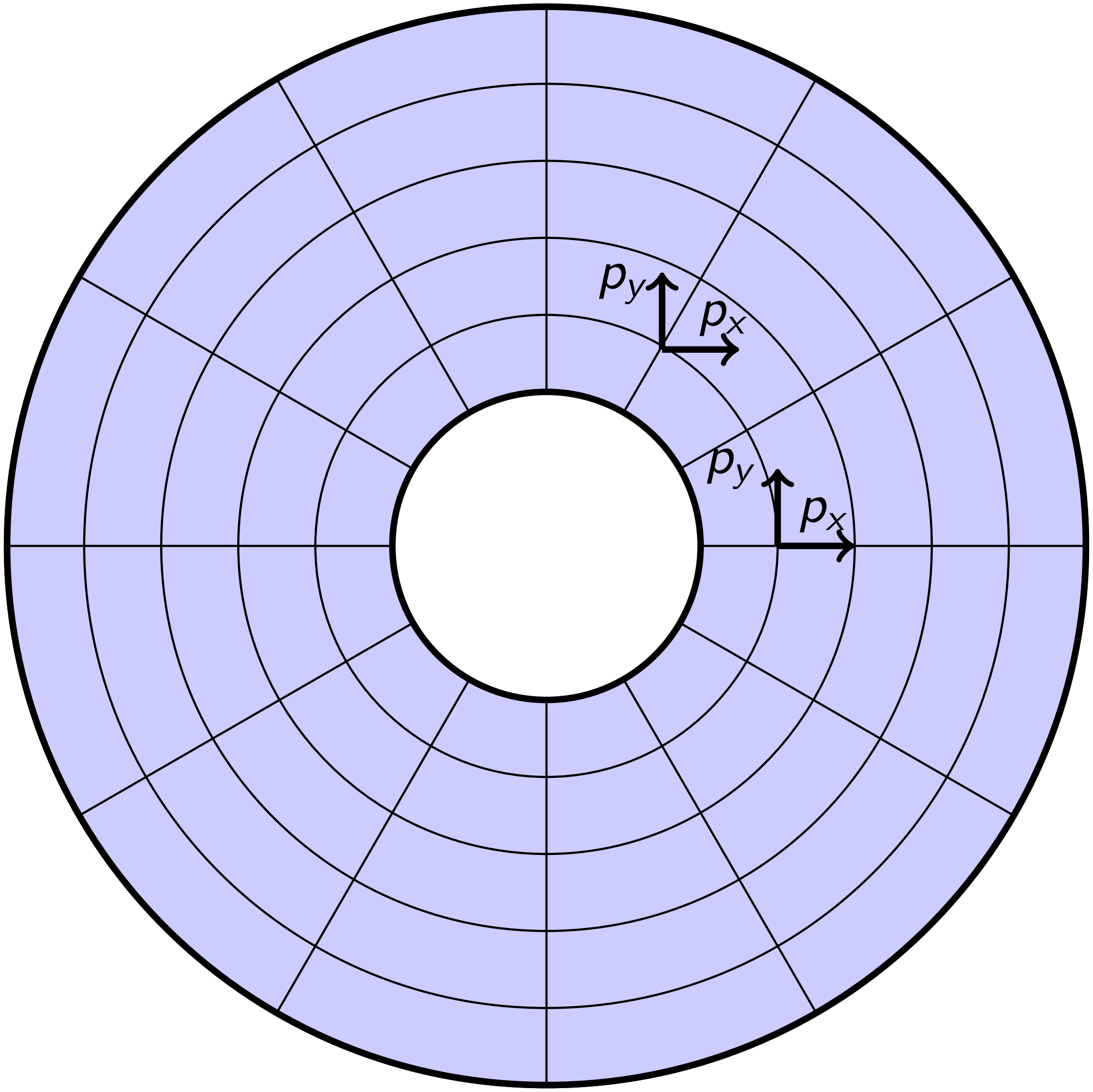
$$ds^2 = \delta_{ij} dx^i dx^j = g_{\widetilde{\imath}\widetilde{\jmath}} dx^{\widetilde{\imath}} dx^{\widetilde{\jmath}},$$

$$g_{\widetilde{\imath}\widetilde{\jmath}} = \delta_{ij} \frac{\partial x^i}{\partial x^{\widetilde{\imath}}} \frac{\partial x^j}{\partial x^{\widetilde{\jmath}}}.$$

- The Boltzmann equation with respect to the new spatial coordinates reads:

$$\frac{\partial f}{\partial t} + \frac{p^i}{m} \frac{\partial x^{\widetilde{\imath}}}{\partial x^i} \frac{\partial f}{\partial x^{\widetilde{\imath}}} + F^i \frac{\partial f}{\partial p^i} = J[f]. \tag{2}$$

Cylindrical grid and momentum space decomposition along the Cartesian axes



Boltzmann equation - orthonormal vielbein fields

- Furthermore, we note that:

$$p^{\widetilde{\imath}} = p^i \frac{\partial x^{\widetilde{\imath}}}{\partial x^i}, \qquad F^{\widetilde{\imath}} = F^i \frac{\partial x^{\widetilde{\imath}}}{\partial x^i}, \qquad \mathbf{p}^2 \equiv g_{\widetilde{\imath}\widetilde{\jmath}} p^{\widetilde{\imath}} p^{\widetilde{\jmath}}$$

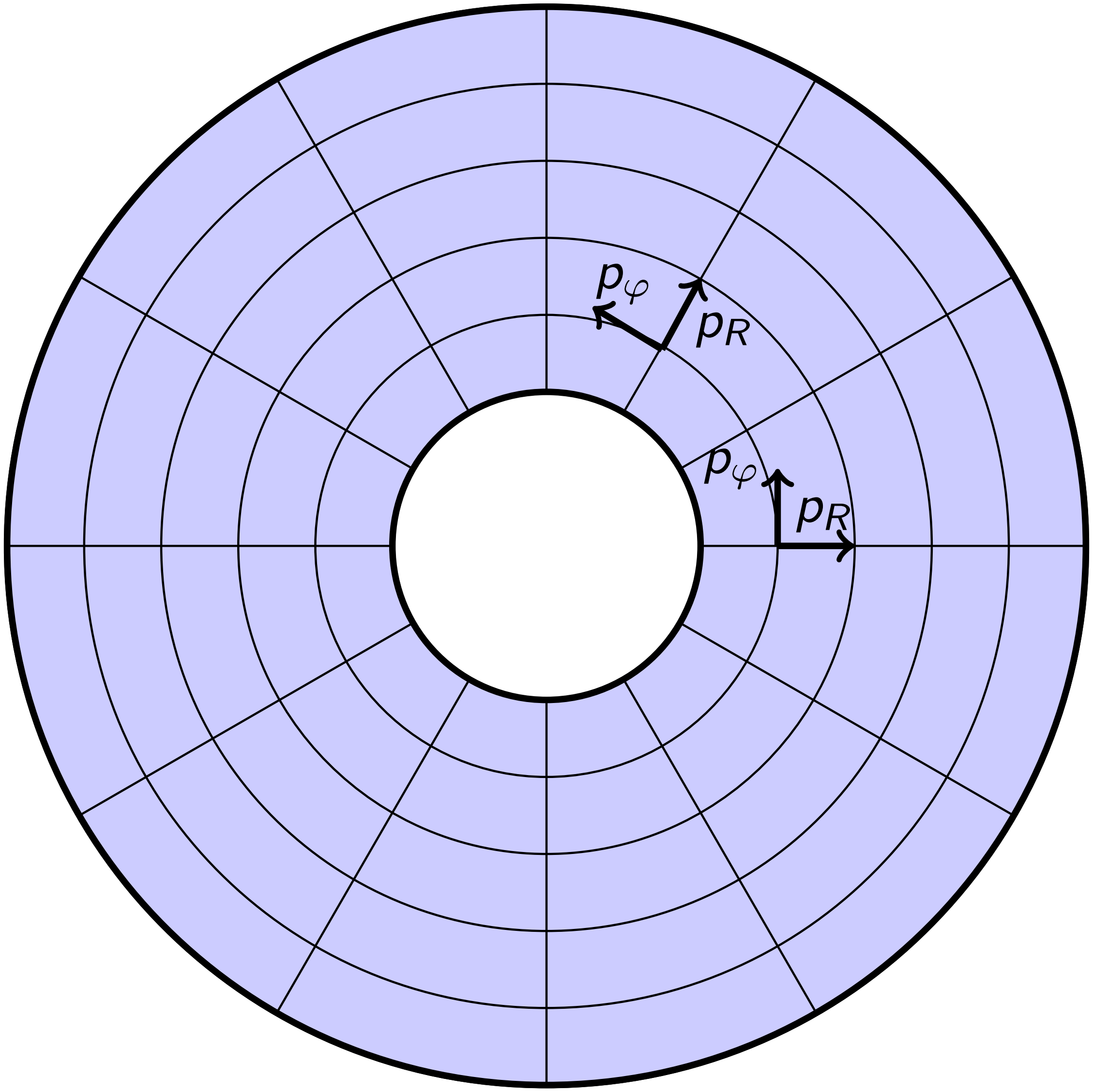
- By introducing the triad vector frame and the associated one-form co-frame:

$$e_{\hat{a}} = e_{\hat{a}}^{\widetilde{\imath}} \partial_{\widetilde{\imath}}, \qquad \omega^{\hat{a}} = \omega^{\hat{a}}_{\widetilde{\imath}} dx^{\widetilde{\imath}}, \qquad g_{\widetilde{\imath}\widetilde{\jmath}} dx^{\widetilde{\imath}} dx^{\widetilde{\jmath}} = \delta_{\hat{a}\hat{b}} \omega^{\hat{a}} \omega^{\hat{b}}$$

  the inner product $\mathbf{p}^2 \equiv \delta_{\hat{a}\hat{b}} p^{\hat{a}} p^{\hat{b}}$ is decoupled from the metric.
- The Boltzmann equation reads:

$$\frac{\partial f}{\partial t} + \frac{p^{\hat{a}}}{m} e_{\hat{a}}^{\widetilde{\imath}} \frac{\partial f}{\partial x^{\widetilde{\imath}}} + \left(F^{\hat{a}} - \frac{1}{m} \Gamma^{\hat{a}}{}_{\hat{b}\hat{c}} p^{\hat{b}} p^{\hat{c}} \right) \frac{\partial f}{\partial p^{\hat{a}}} = J[f], \tag{3}$$

Cylindrical grid and momentum space decomposition adapted to the curvilinear coordinates.



Cylindrical coordinates

- The line element in cylindrical coordinates and the associated triad are:

$$ds^2 = dR^2 + R^2 d\varphi^2 + dz^2, \qquad e_{\hat{R}} = \partial_R, \qquad e_{\hat{\varphi}} = R^{-1}\partial_\varphi, \qquad e_{\hat{z}} = \partial_z. \quad (4)$$

- The Boltzmann equation when the flow is homogeneous w.r.t. φ and z reads:

$$\frac{\partial f}{\partial t} + \frac{2p^{\hat{R}}}{m}\frac{\partial(fR)}{\partial R^2} + \frac{1}{mR}\left[(p^{\hat{\varphi}})^2 \frac{\partial f}{\partial p^{\hat{R}}} - p^{\hat{R}}\frac{\partial(fp^{\hat{\varphi}})}{\partial p^{\hat{\varphi}}}\right] = -\frac{1}{\tau}(f - f^{(\mathrm{eq})}). \quad (5)$$

- The advection term is implemented following Falle et al.[1]

[1] S. A. E. G. Falle and S. S. Komissarov, Mon. Not. R. Astron. Soc. **278**, 586-602 (1996).

Spherical coordinates

- The line element in spherical coordinates and its associated triad are:

$$ds^2 = dR^2 + R^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad e_{\hat{R}} = \partial_R, \qquad e_{\hat{\theta}} = \frac{\partial_\theta}{R}, \quad e_{\hat{\varphi}} = \frac{R^{-1}}{\sin\theta}\partial_\varphi. \tag{6}$$

- Assuming that the flow is homogeneous w.r.t. θ and φ, the Boltzmann equation reads:

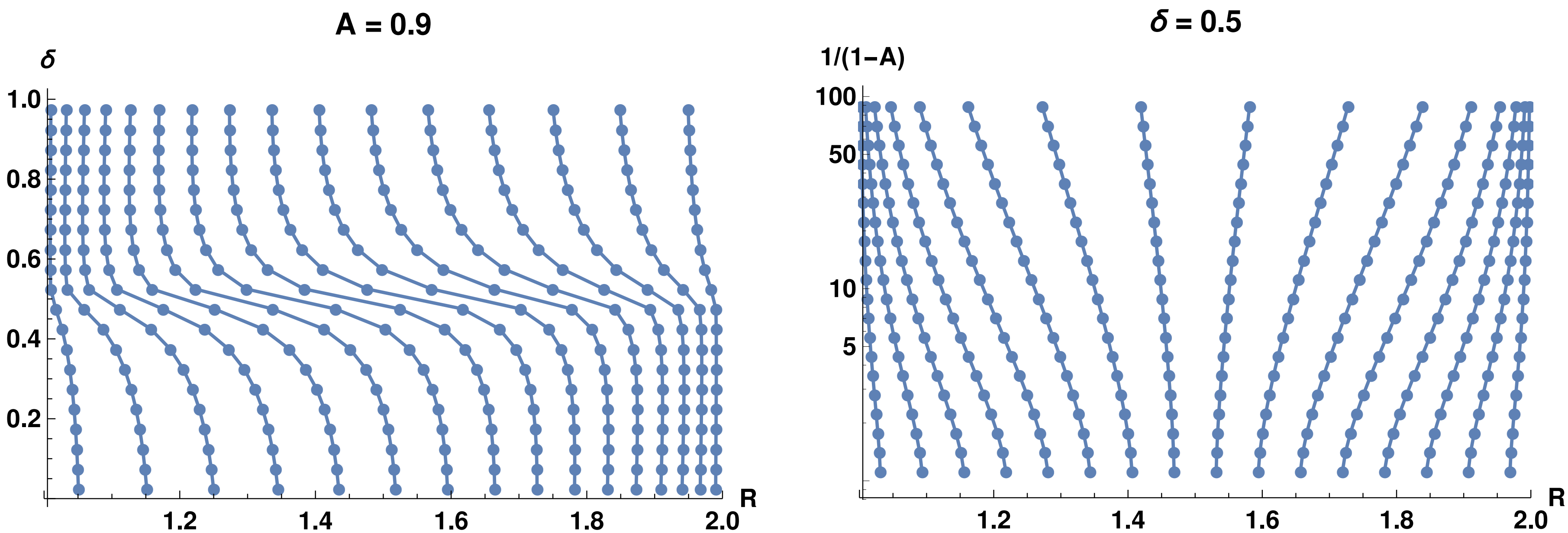
$$\frac{\partial f}{\partial t} + \frac{3p^{\hat{R}}}{m}\frac{\partial(fR^2)}{\partial R^3} + \frac{1}{mR}\left[\left((p^{\hat{\theta}})^2 + (p^{\hat{\varphi}})^2\right)\frac{\partial f}{\partial p^{\hat{R}}} - p^{\hat{R}}\left(\frac{\partial(fp^{\hat{\theta}})}{\partial p^{\hat{\theta}}} + \frac{\partial(fp^{\hat{\varphi}})}{\partial p^{\hat{\varphi}}}\right)\right] = -\frac{1}{\tau}(f - f^{(\mathrm{eq})}). \tag{7}$$

- The advection term is implemented following Downes et al.[2]

[2] T. P. Downes, P. Duffy and S. S. Komissarov, Mon. Not. R. Astron. Soc. 332, 144–154 (2002).

Coordinates streching

$$\text{Let } R(\eta) = R_{\rm in} + (R_{\rm out} - R_{\rm in})\left(\delta + \frac{A_0}{A}\tanh\eta\right), \qquad A_0 = \max(\delta, 1-\delta). \quad (8)$$



(a) (b)

Figure 1: Effect of grid stretching on 16 points between $R_{\rm in} = 1$ and $R_{\rm out} = 2$.
(a) The parameter $\delta \in (0,1)$ controls the positioning of the stretching centre.
(b) The parameter $A \in (0,1)$ contains the amplitude of the stretching.
The points are equidistant in η.

Lattice Boltzmann - Gauss-Hermite quadratures

- The numerical models employed are mixed quadrature lattice Boltzmann[3]. f and $f^{(\mathrm{eq})}$ are expanded with respect to Hermite(HLB) and Half-Range Hermite(HHLB) polynomials, e. g.(1D case):

$$f_i^{eq}(\mathbf{x}, t) = w_i \sum_{\ell=0}^{N} \frac{1}{\ell!} \mathbf{a}^{eq}_{(\ell)}(\mathbf{x}, t) \mathcal{H}^{(\ell)}(p_i)$$

- 3D model $=$ 1D $\times$ 1D $\times$ 1D.
- We will denote such models by: $\mathrm{HHLB}(Q_R) \times \mathrm{HLB}(\mathrm{Q}_\varphi) \times \mathrm{HLB}(Q_z)$.
- The moments of the distribution function are evaluated as:

$$\int d^3\hat{p}\, f\, p^{\hat{a}_1} \cdots p^{\hat{a}_s} = \sum_{i=1}^{\mathcal{Q}_R} \sum_{j=1}^{\mathcal{Q}_\varphi} \sum_{k=1}^{\mathcal{Q}_z} f_{ijk} \prod_{\ell=1}^{s} p_{ijk}^{\hat{a}_\ell}. \qquad (9)$$

Here $Q_\alpha, \alpha \in \{R, \varphi, z\}$ represents the quadrature order.

[3] V.E. Ambruș and V. Sofonea, J. Comput. Phys. **316**, 760-788 (2016)

Force term

Boltzmann equation written for the discrete distributions f_{ijk}:

$$\frac{\partial f_{ijk}}{\partial t} + \frac{2p_i^{\hat{R}}}{m}\frac{\partial(f_{ijk}R)}{\partial R^2} + \frac{1}{mR}\left\{(p_j^{\hat{\varphi}})^2\left(\frac{\partial f}{\partial p^{\hat{R}}}\right)_{ijk} - p_i^{\hat{R}}\left[\frac{\partial(fp^{\hat{\varphi}})}{\partial p^{\hat{\varphi}}}\right]_{ijk}\right\} = -\frac{1}{\tau}(f_{ijk} - f_{ijk}^{(\mathrm{eq})})$$

- We write the terms involving the derivative of f as follows[4] :

$$\left(\frac{\partial f}{\partial p^{\hat{R}}}\right)_{ijk} = \sum_{i'=1}^{\mathcal{Q}_R} \mathcal{F}^R_{i,i'} f_{i'jk},$$

- The matrix $\mathcal{F}^R_{k,j}$ has the following form:

$$\mathcal{F}^R_{k,j} = -w_k^H \sum_{\ell=0}^{Q-1} \frac{1}{\ell!} H_{\ell+1}(p_k) H_\ell(p_j)$$

[4] V. E. Ambruș, V. Sofonea, R. Fournier, and S. Blanco, *Implementation of the force term in half-range lattice Boltzmann models*, in preparation.

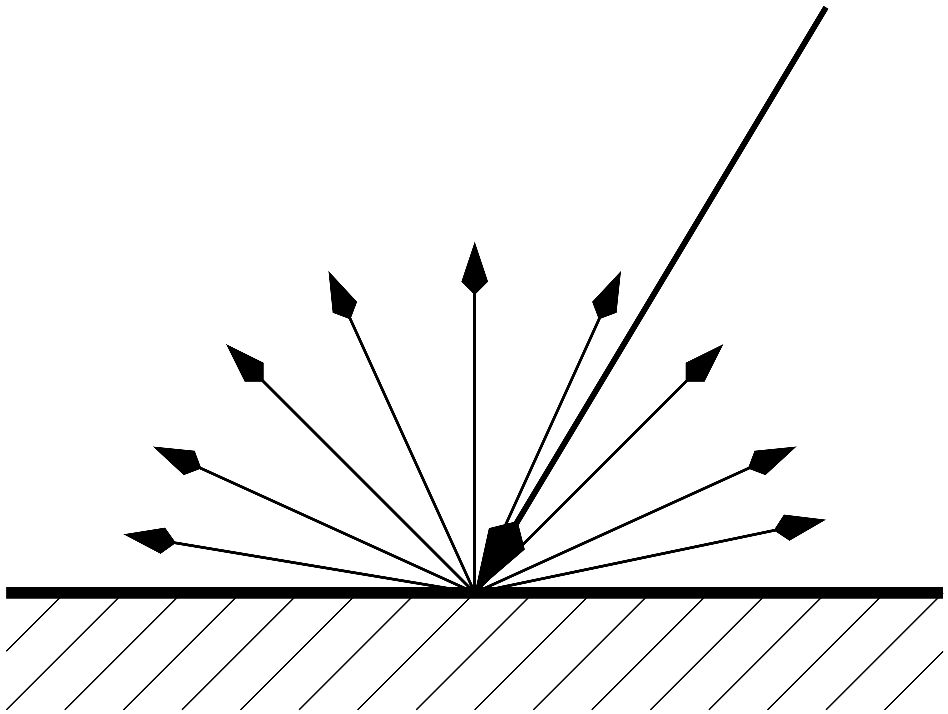
Lattice Boltzmann - Numerical scheme

- Velocity vectors are given by the roots of the Hermite/Half-Range Hermite polynomials.
- The roots are irrational numbers for a quadrature order $Q > 2$ which implies a off-lattice velocity set.
- Time-stepping: TVD RK3[5]
- Advection: The fifth order WENO-5 scheme[6].

[5] C.-W. Shu and S. Osher, J. Comput. Phys. 77, 439–471 (1988).
[6] G. S. Jiang and C. W. Shu, J. Comput. Phys. 126, 202 (1996).

Diffuse reflection boundary conditions

- Reflected particles carry some information that belongs to the wall.

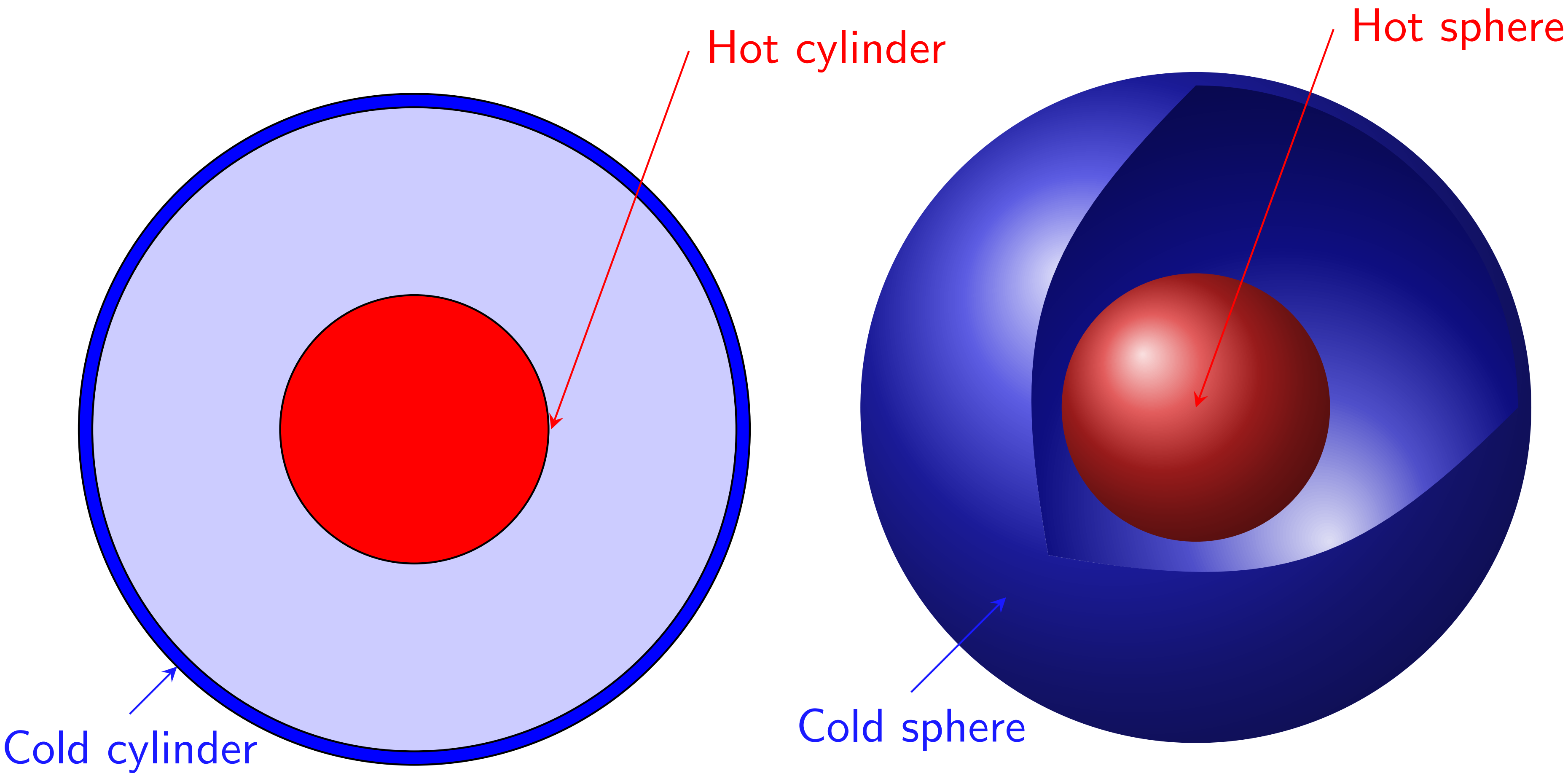


diffuse reflection

- The density $n_{\rm w}$ is fixed by imposing zero flux through the boundary:

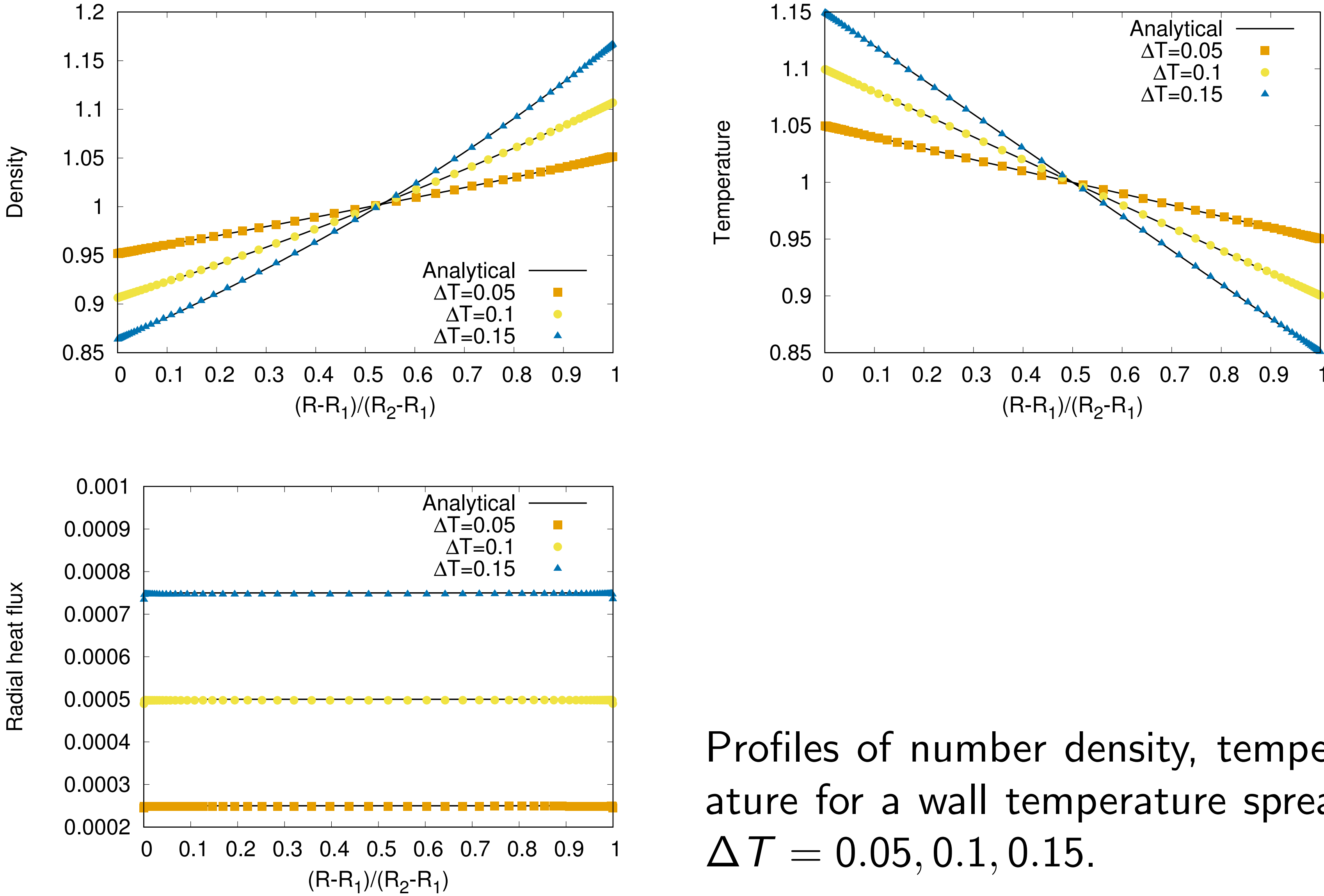
$$\int_{\mathbf{p}\cdot\chi>0} d^3p\, f\,(\mathbf{p}\cdot\chi) = -\int_{\mathbf{p}\cdot\chi<0} d^3p\, f^{(\rm eq)}\,(\mathbf{p}\cdot\chi).$$

Heat transfer - cartesian, cylindrical and spherical symmetry



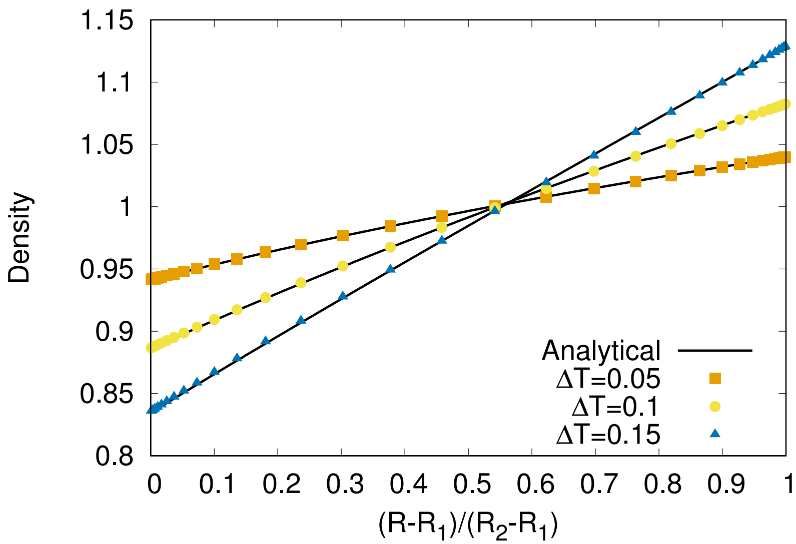
$$T_{hot/cold} = T \pm \Delta T$$

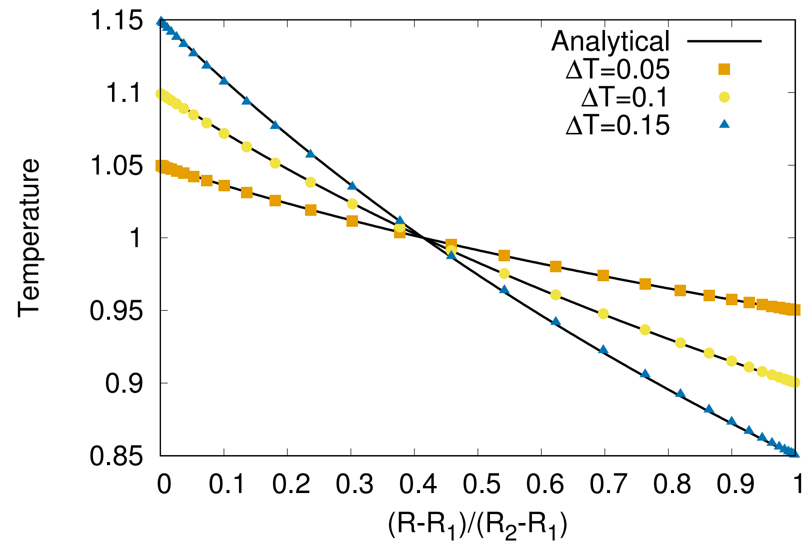
Parallel plates $Kn \to 0$

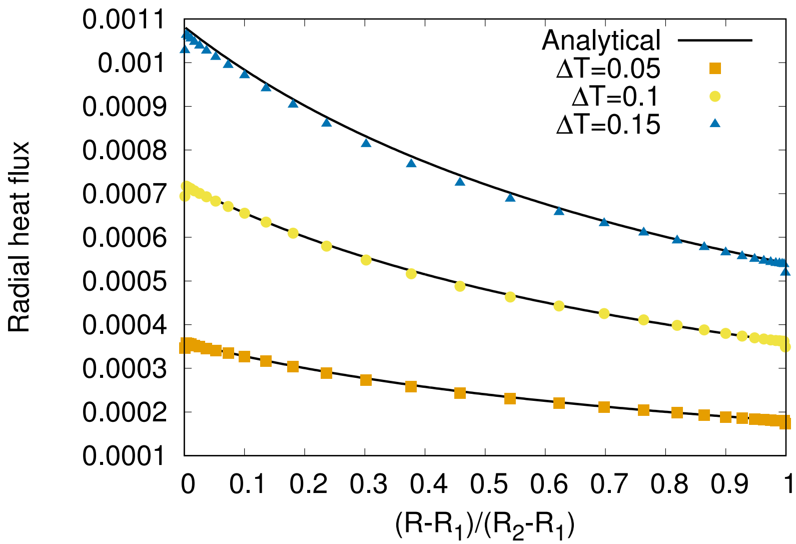


Profiles of number density, temperature for a wall temperature spread $\Delta T = 0.05, 0.1, 0.15$.

Cylindrical symmetry $Kn \to 0$

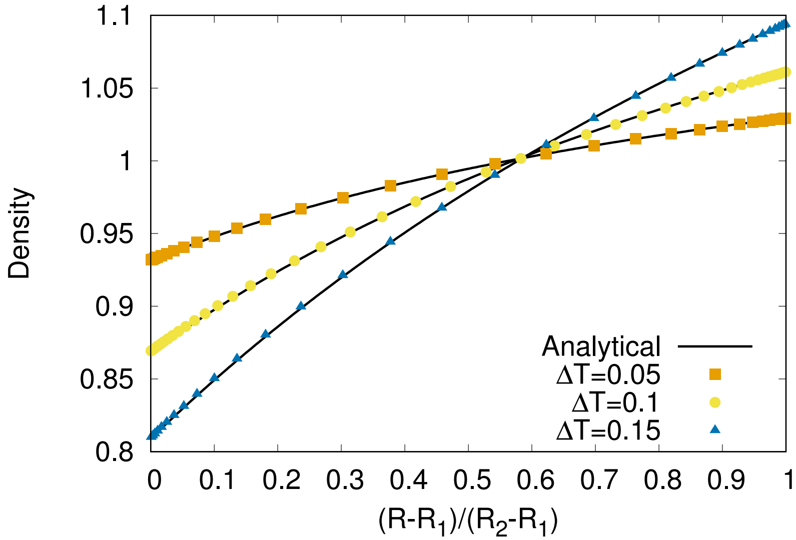


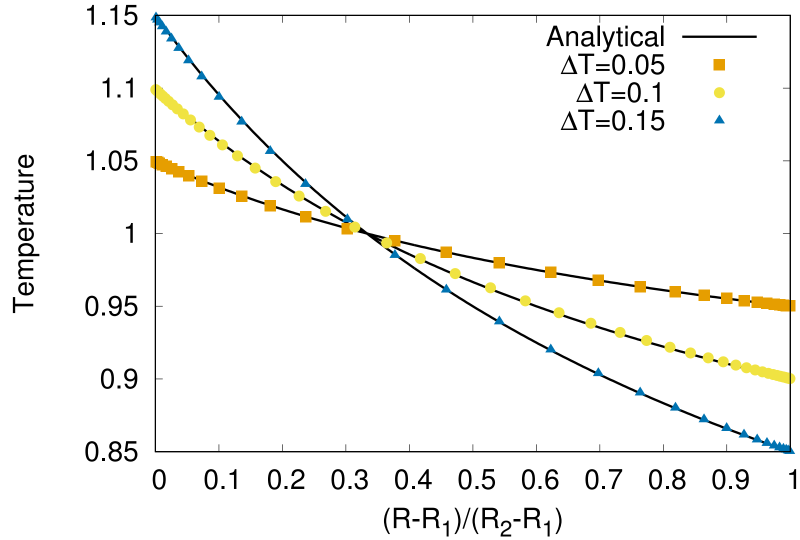


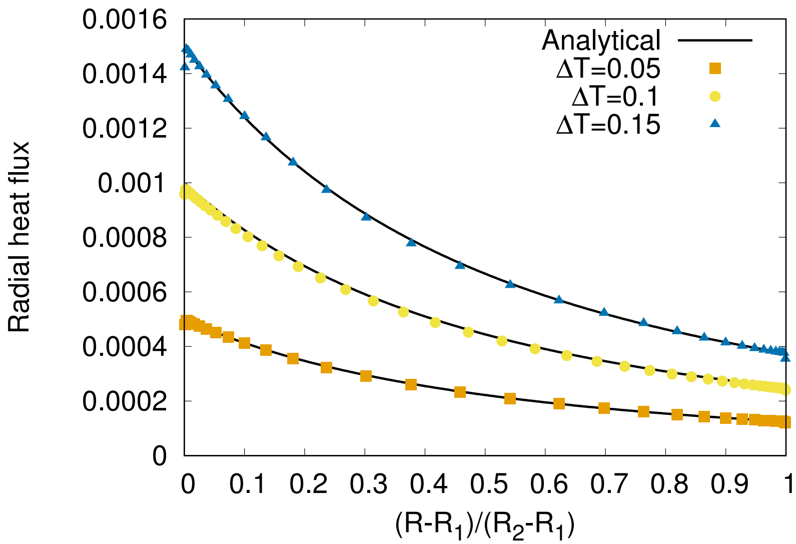


Profiles of number density, temperature for a wall temperature spread $\Delta T = 0.05, 0.1, 0.15$.

Spherical symmetry $Kn \to 0$

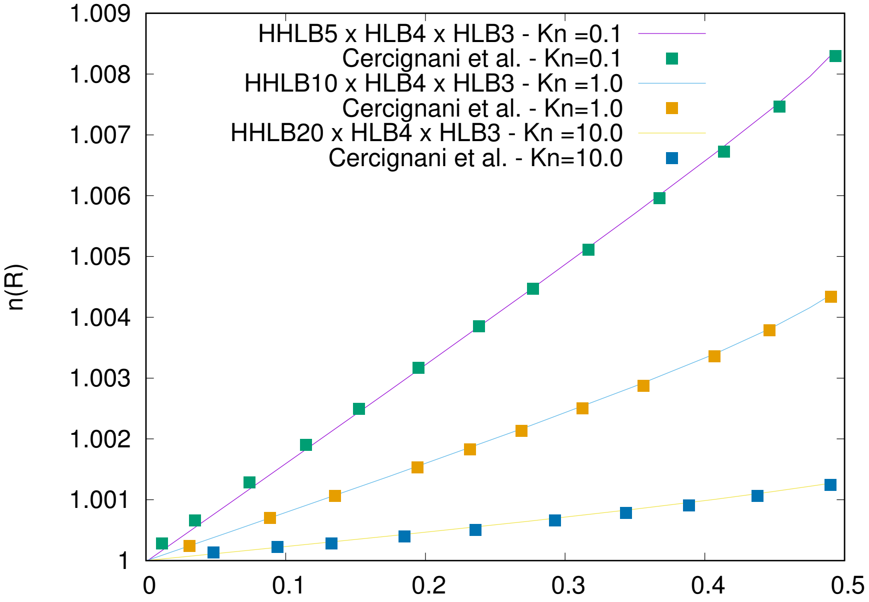






Profiles of number density, temperature for a wall temperature spread $\Delta T = 0.05, 0.1, 0.15$.

Parallel plates - transition regime



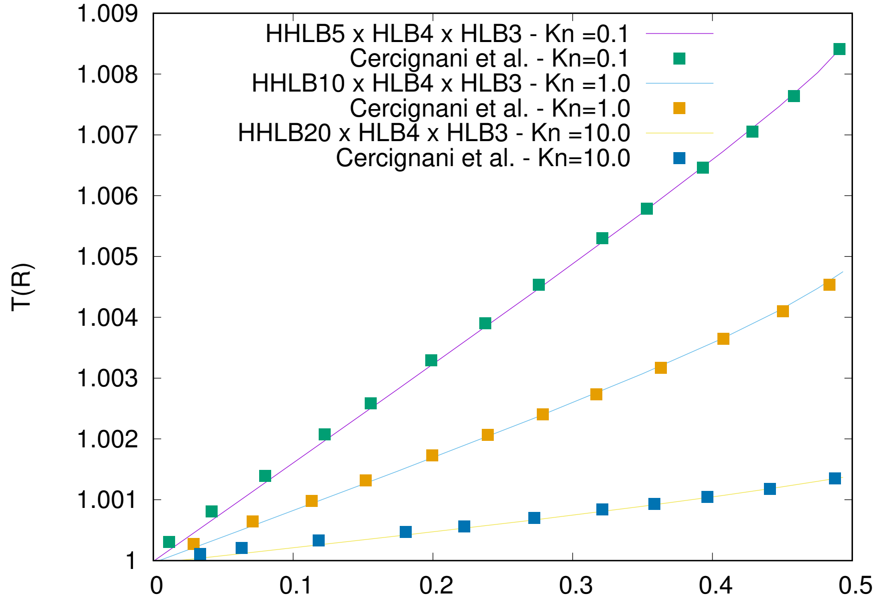


Figure 2: Half domain heat transfer comparison with the study of Bassanini et al.[7]

[7] P. Bassanini, C. Cercignani and C. D. Pagani, Int.J. Heat Transfer, Vol. 10, pp. 447-460. 1967

Parallel plates - transition regime

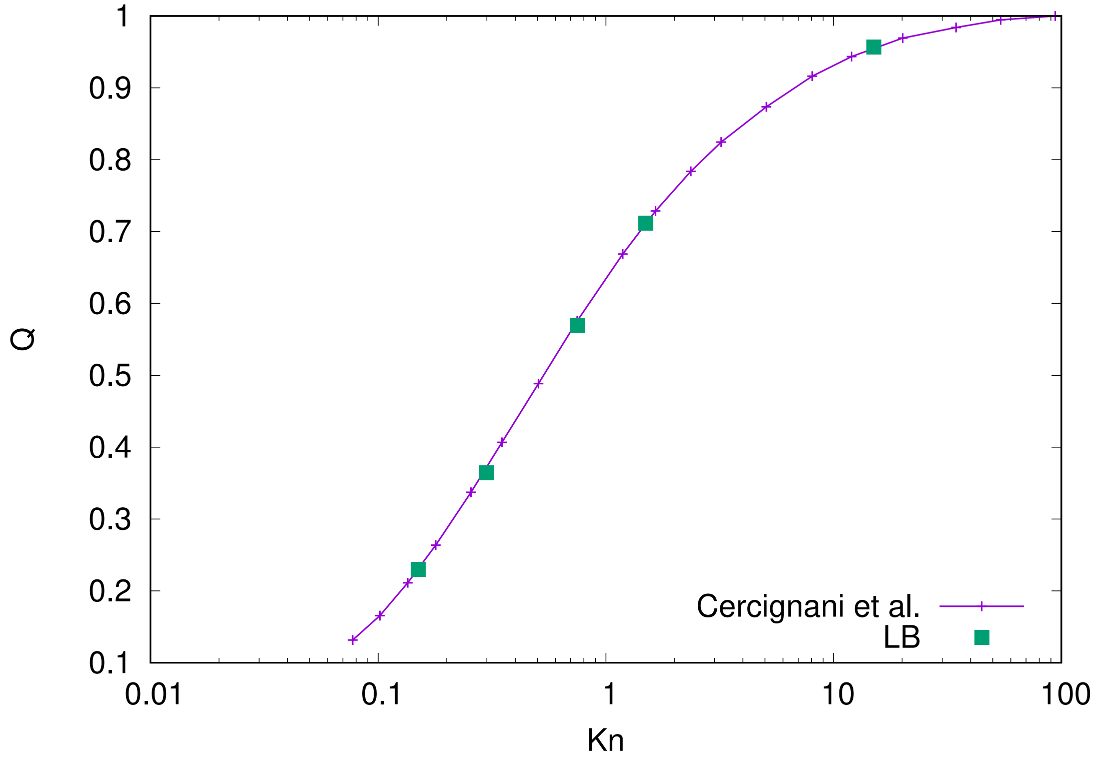
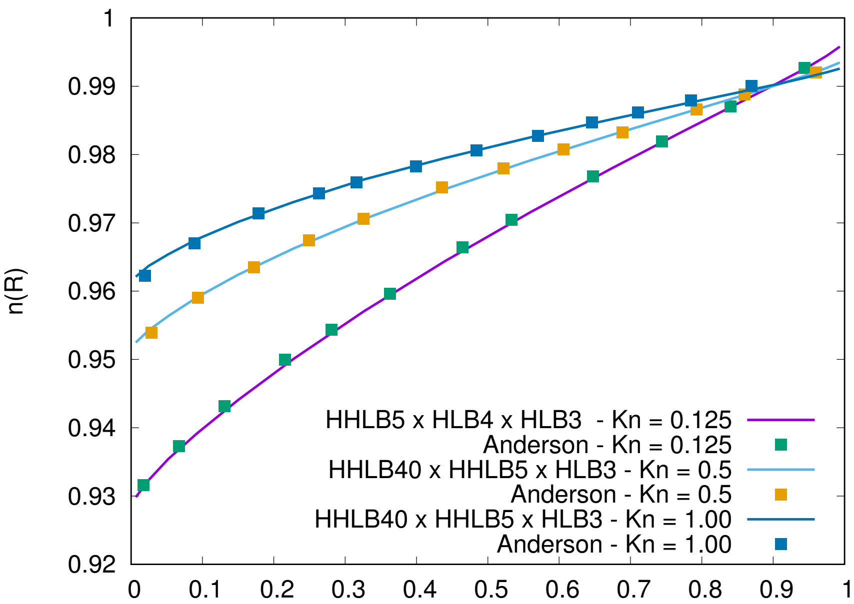


Figure 3: Q constant comparison with the study of Bassanini et al.[8].
Q=q(R)=const.

[8] P. Bassanini, C. Cercignani and C. D. Pagani, Int.J. Heat Transfer, Vol. 10, pp. 447-460. 1967

Cylindrical Symmetry - transition regime



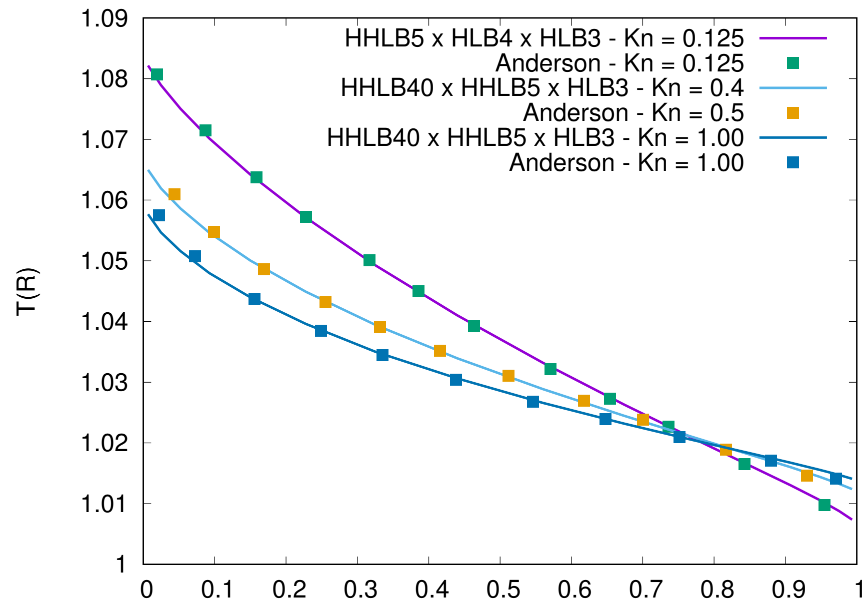


Figure 4: Heat transfer comparison with the study of Anderson[9] for $\Delta T = 0.1$

[9] Donald G. M. Anderson, J. Plasma Physics (1967), vol. 1, part 2, pp. 255-265

Cylindrical Symmetry - transition regime

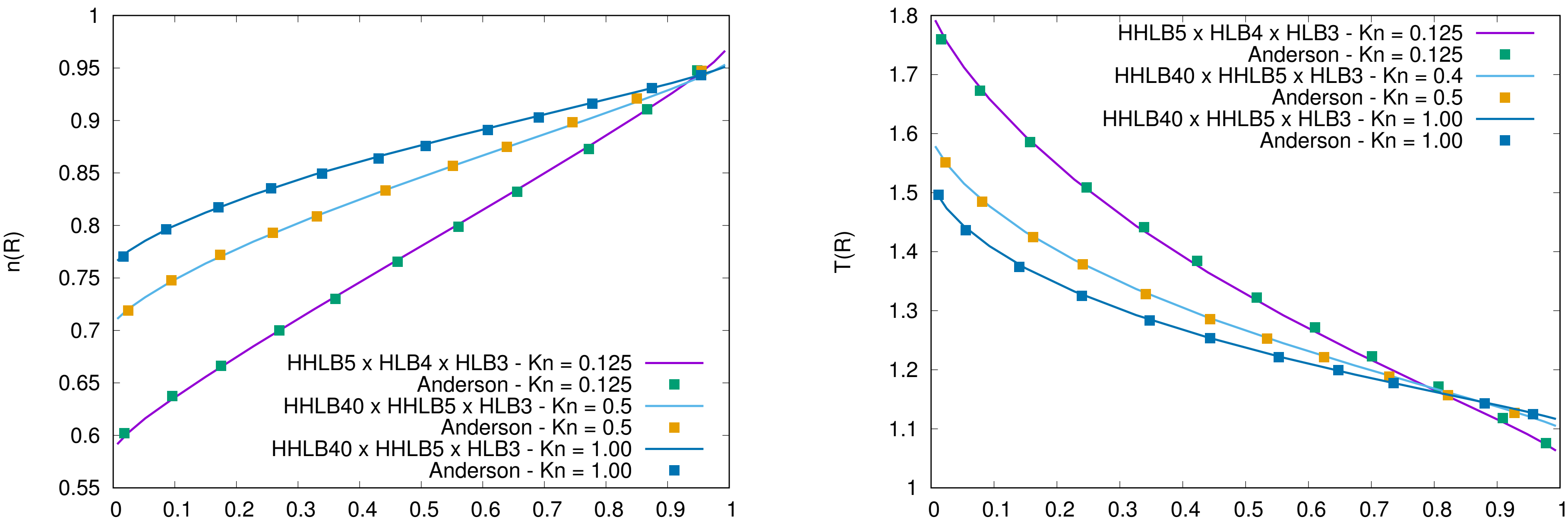


Figure 5: Heat transfer comparison with the study of Anderson[10] for $\Delta T = 1.0$.

[10]Donald G. M. Anderson, J. Plasma Physics (1967), vol. 1, part 2, pp. 255-265

Cylindrical Symmetry - transition regime

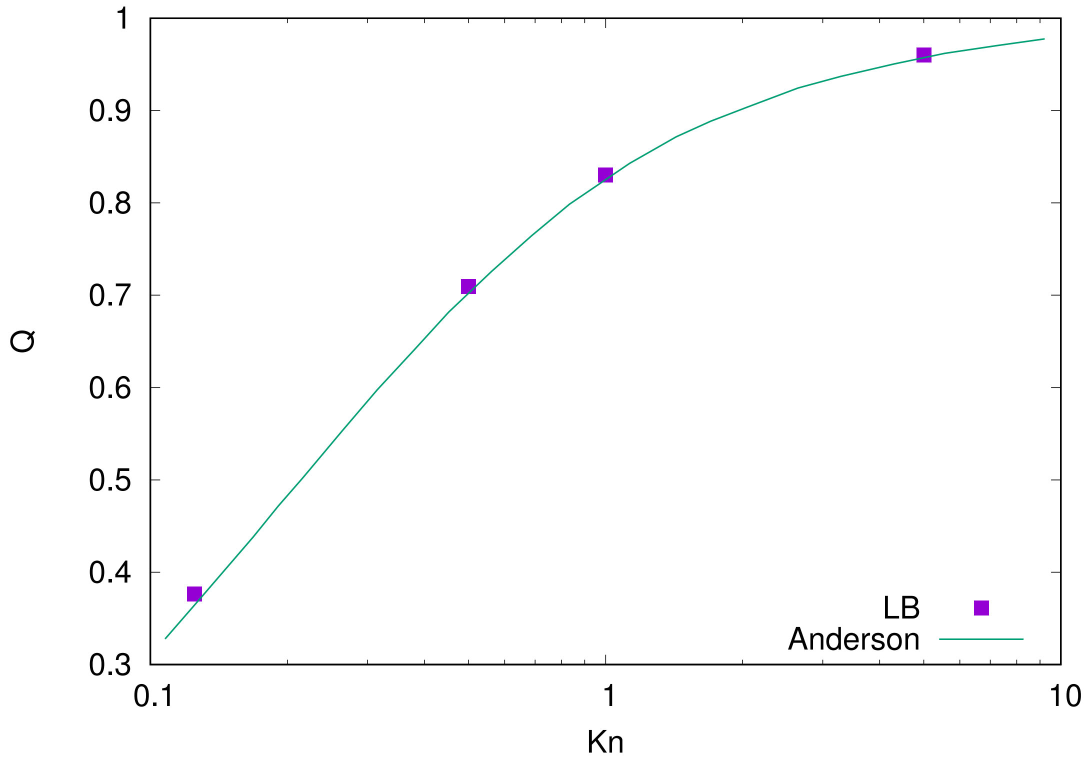


Figure 6: Q constant comparison with the study of Anderson[11].

$$Q = q(R) \cdot R$$

[11] Donald G. M. Anderson, J. Plasma Physics (1967), vol. 1, part 2, pp. 255-265

Spherical Symmetry - transition regime

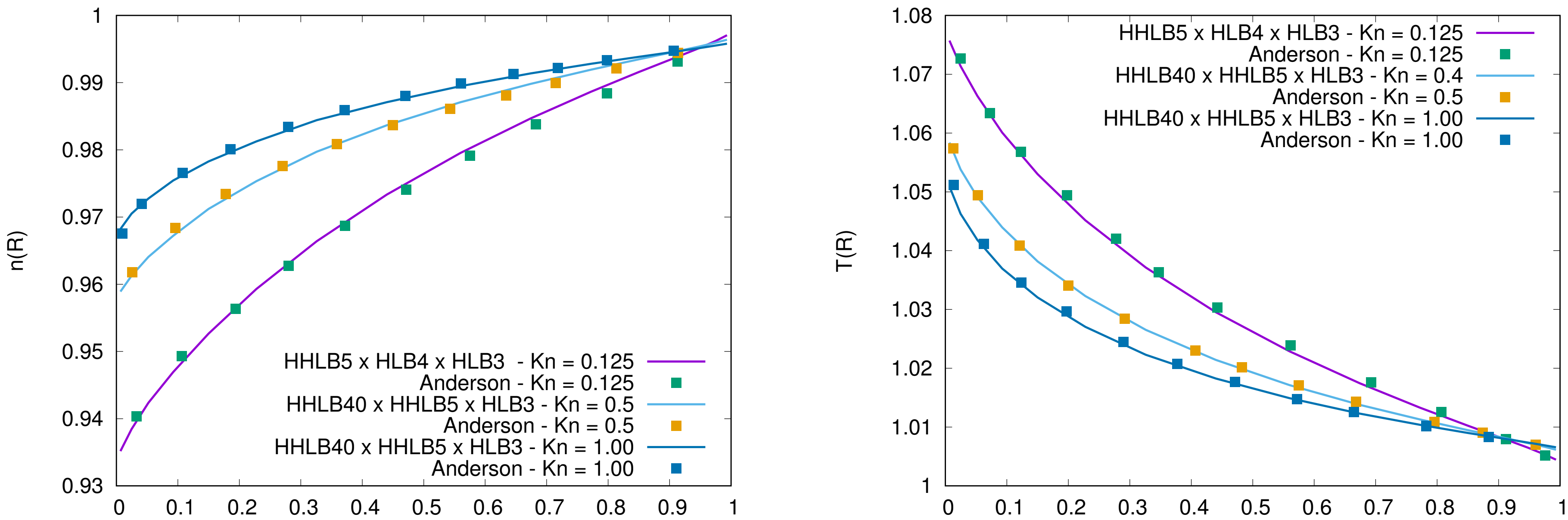


Figure 7: Heat transfer comparison with the study of Anderson[12] for $\Delta T = 0.1$.

[12] Donald G. M. Anderson, J. Plasma Physics (1967), vol. 1, part 2, pp. 255-265

Spherical Symmetry - transition regime

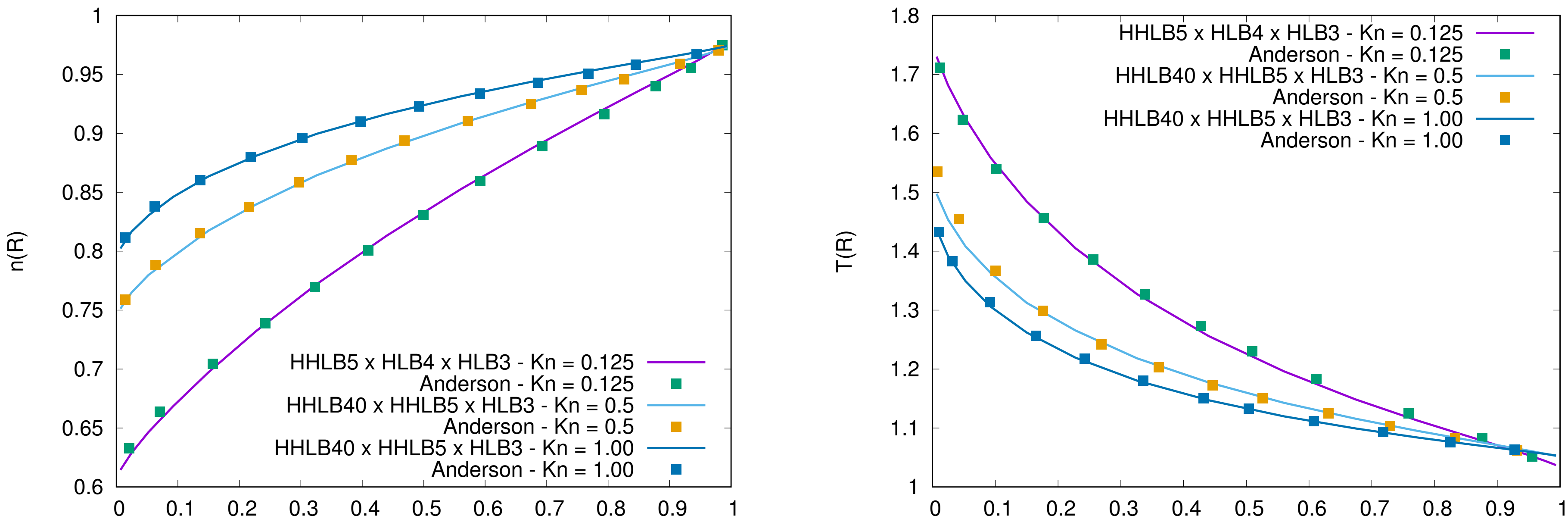


Figure 8: Heat transfer comparison with the study of Anderson[13] for $\Delta T = 1.0$.

[13]Donald G. M. Anderson, J. Plasma Physics (1967), vol. 1, part 2, pp. 255-265

Spherical Symmetry - transition regime

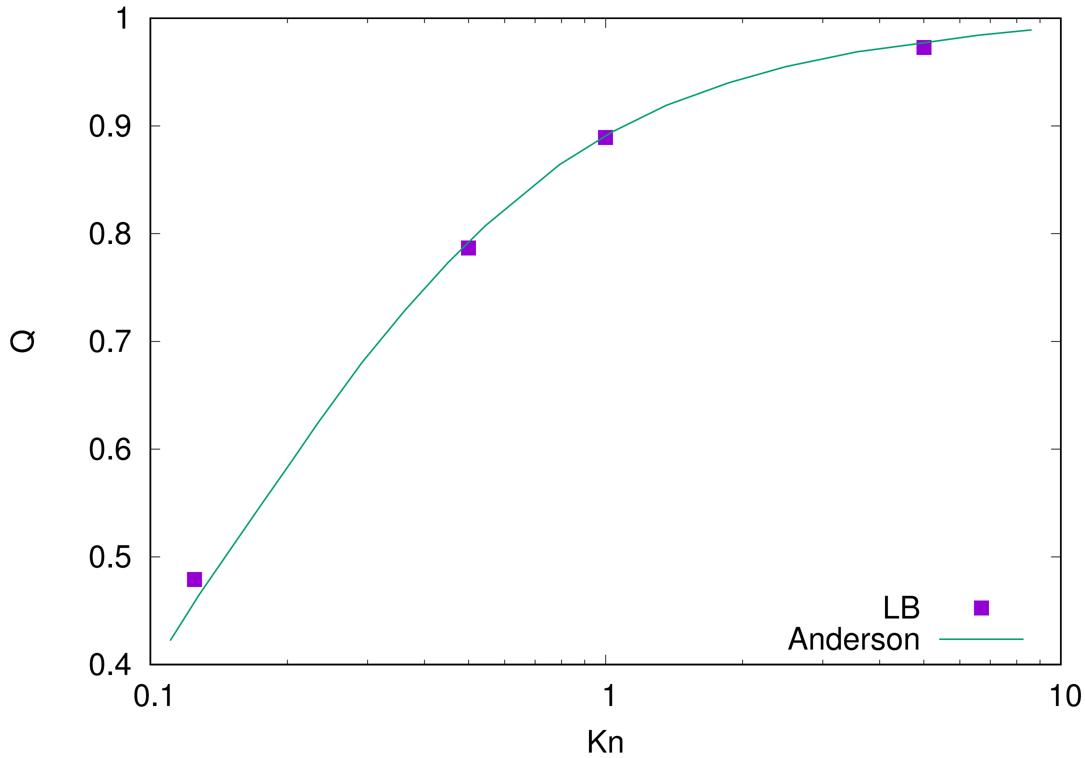
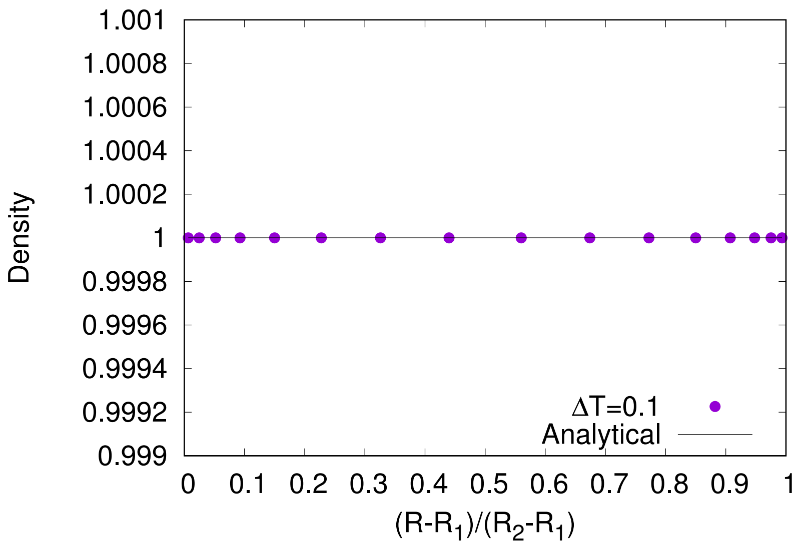


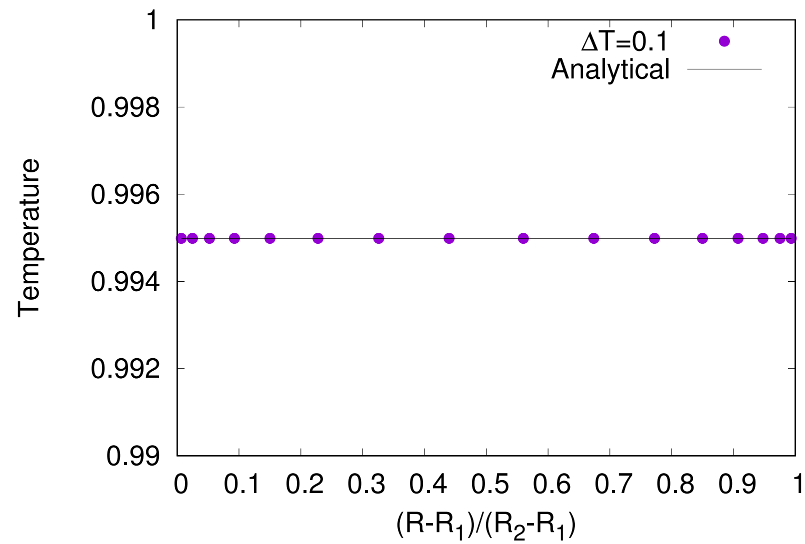
Figure 9: Q constant comparison with the study of Anderson[14].

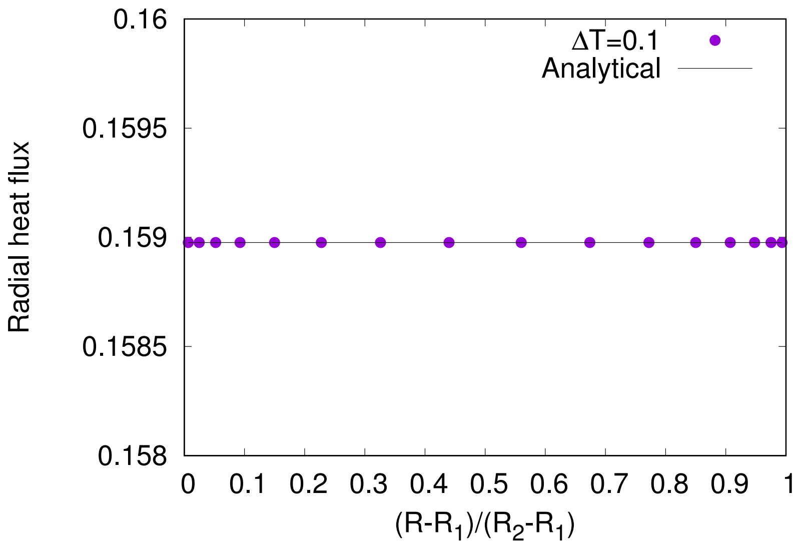
$$Q = q(R) \cdot R^2$$

[14] Donald G. M. Anderson, J. Plasma Physics (1967), vol. 1, part 2, pp. 255-265

Parallel plates $Kn \to \infty$

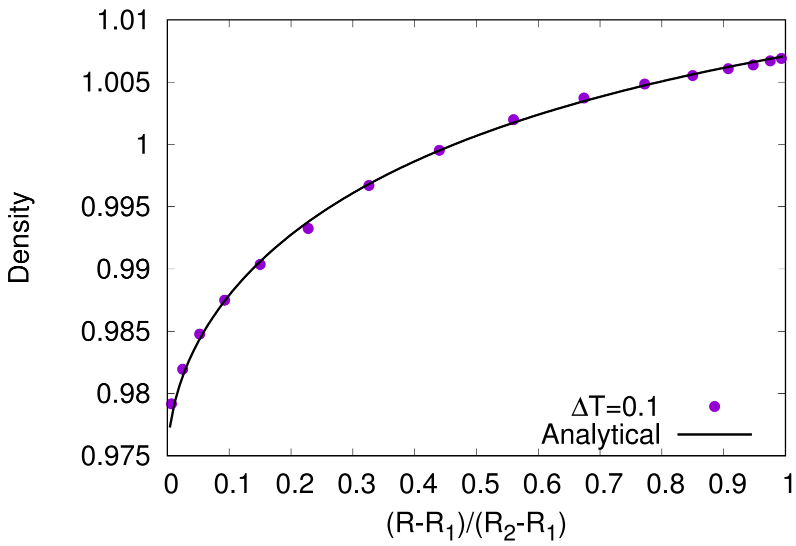


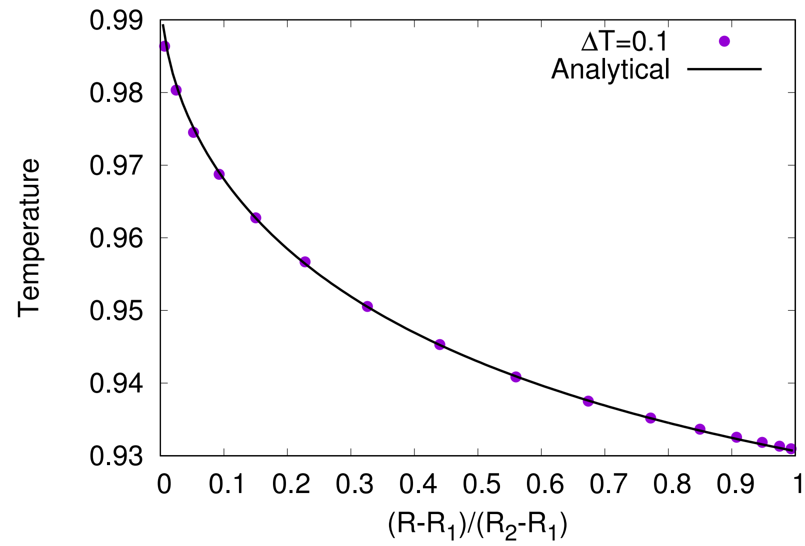


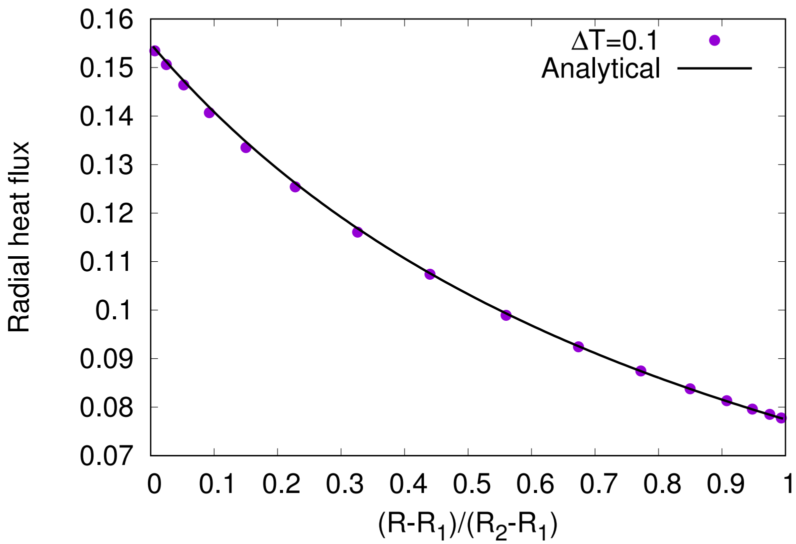


Profiles of number density, temperature for a wall temperature spread $\Delta T = 0.1$.

Cylindrical symmetry $Kn \to \infty$

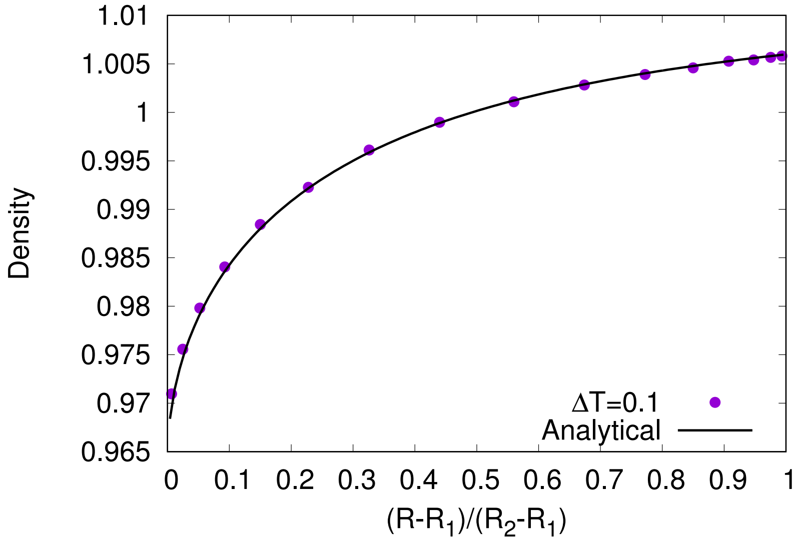


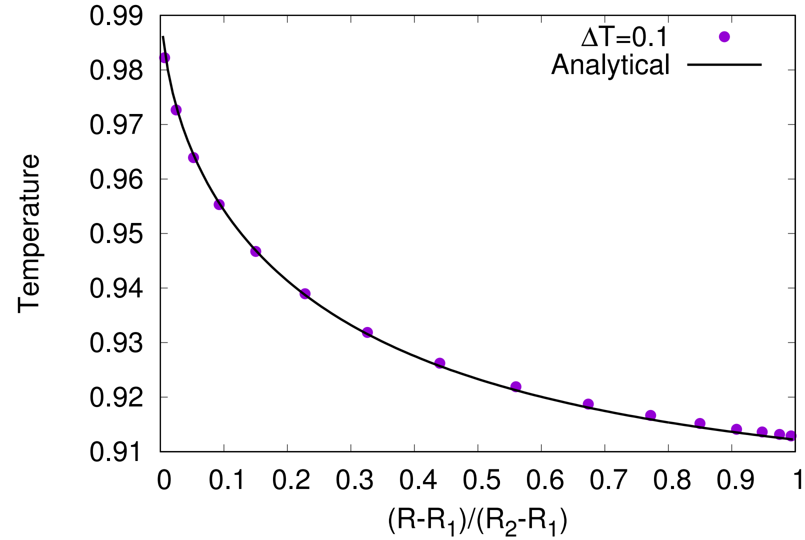


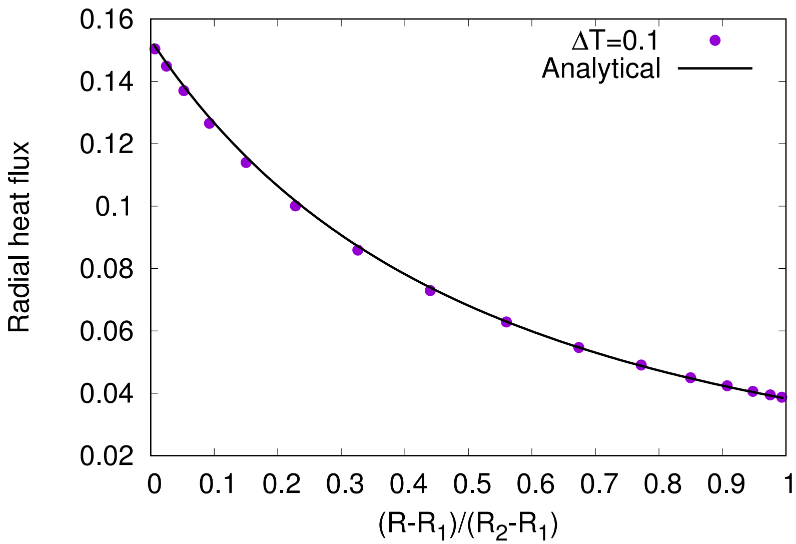


Profiles of number density, temperature for a wall temperature spread $\Delta T = 0.1$.

Spherical symmetry $Kn \to \infty$







Profiles of number density, temperature for a wall temperature spread $\Delta T = 0.1$.

Conclusion

- We introduced the Boltzmann equation written with respect to orthogonal vielbeins.
- The vielbeins allow the decoupling between the momentum space and the coordinate space, i.e. $\mathbf{p}^2$ is coordinate-independent.
- The vielbein permits the momentum space degrees of freedom to be aligned according to the symmetries of the flow.
- We considered the problems of heat transfer between parallel plates, two coaxial cylinders and two concentric spheres which become one-dimensional in space since the vielbein allows the symmetry of the geometry to be transferred to the momentum space.
- Our models give accurate results throughout the whole range of Kn.

Thank you for your attention.

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