Lattice Boltzmann models for the simulation of rarefied flows in general relativity using tetrad fields

Victor E. Ambruș

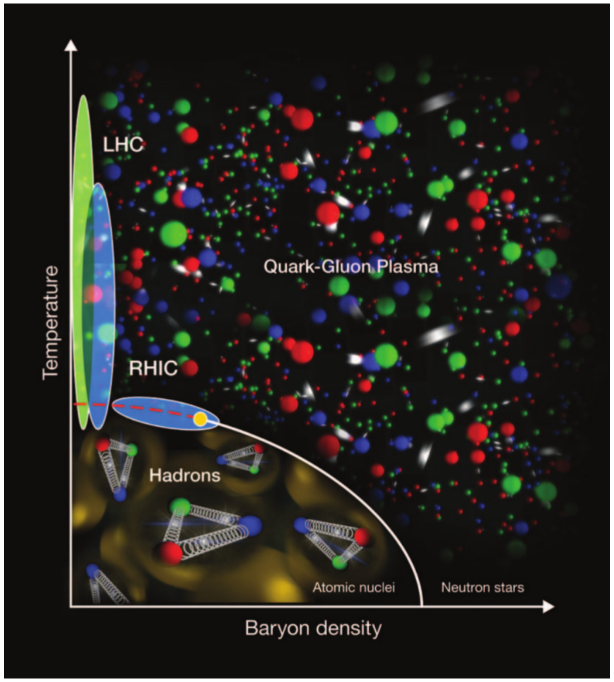
Departement of Physics, West University of Timișoara
Bd. Vasile Pârvan 4, R – 300223 Timișoara, Romania

DSFD 2017, 13/07/2017

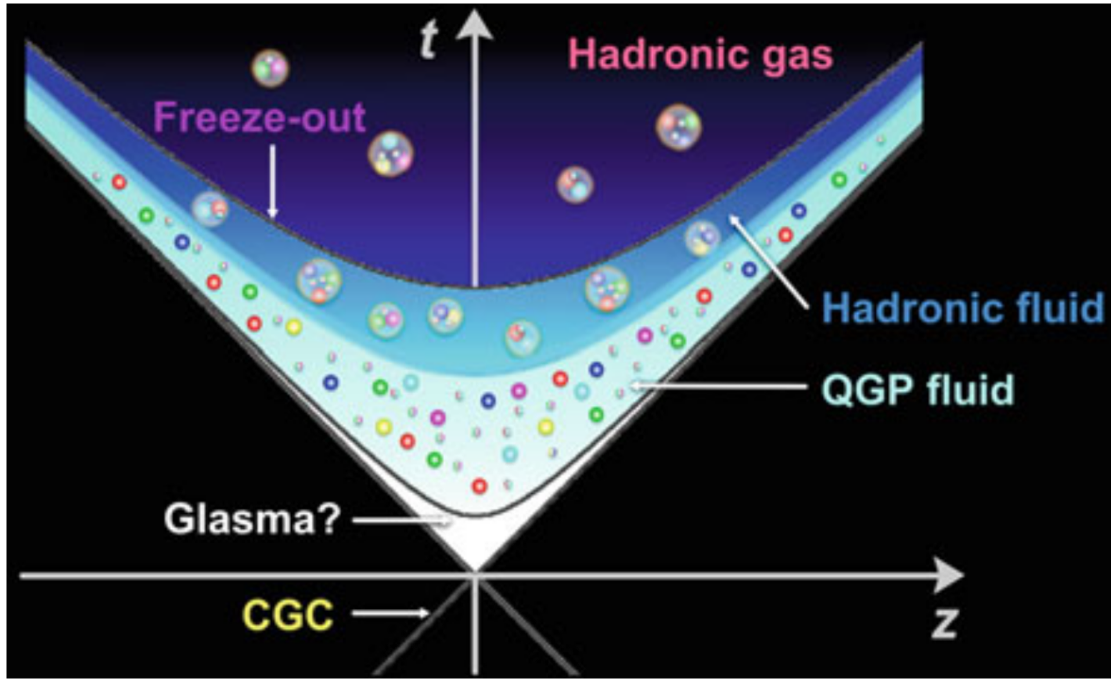
Outline

Relativistic heavy ion collisions

Quark-gluon plasma: hydrodynamic phase



B. V. Jacak, B. Muller, Science **337** (2012) 310.

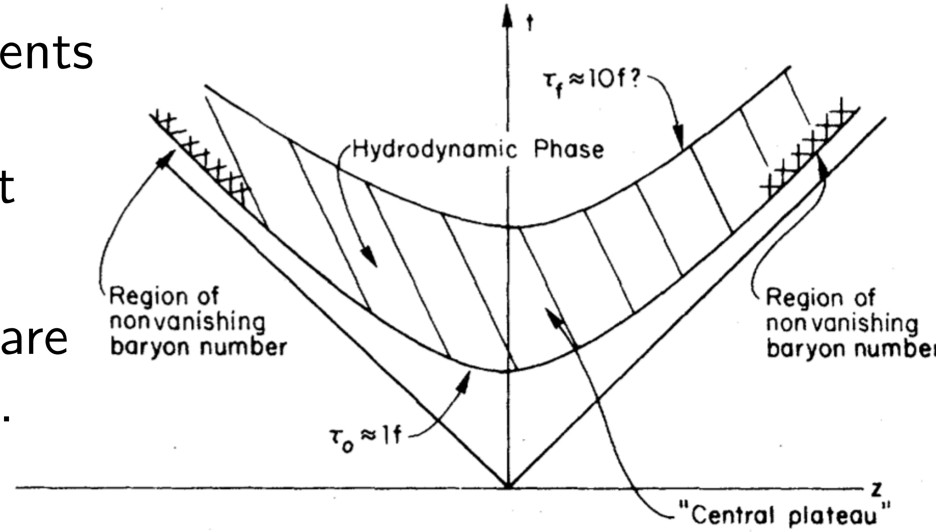


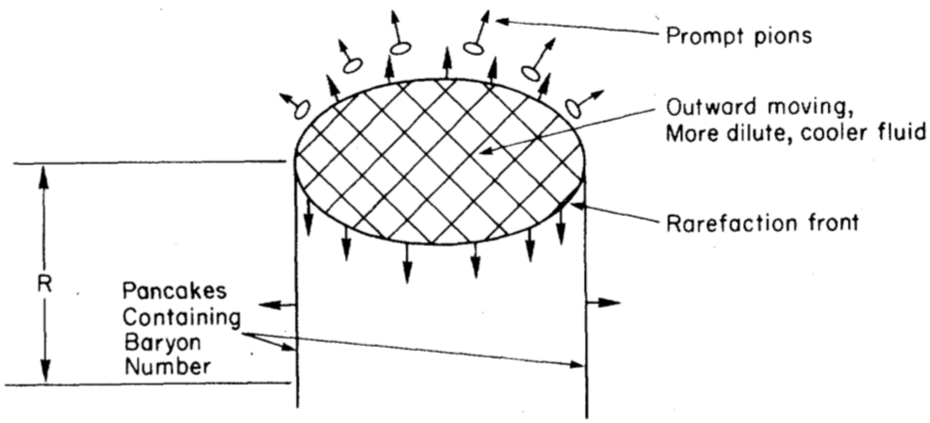
A. Monnai, PhD Thesis (Tokyo, 2014).

- Until $\sim 10\,\mu\mathrm{s}$ after the Big-Bang, the quarks and gluons where deconfined into the QGP.
- At RHIC and LHC, this phase of matter was achieved.
- The strong coupling of the quarks and gluons in the QGP allows such systems to be described using an effective theory: relativistic fluid dynamics.

Central plateau: Bjorken flow

- Bjorken[1] argued that the constituents of colliding nuclei travel longitudinally at the speed of light from the point of collision.
- Between the two nuclei, particles are produced which travel at $v = z/t$.
- The "boost invariance" of this system is ensured by imposing $E \simeq 1 - -10\mathrm{GeV}/\mathrm{fm}^3$ at $\tau = \sqrt{t^2 - z^2} \simeq 1\,\mathrm{fm/c}$
- In the transverse direction, a shock wave propagates outwards, while a rarefaction wave propagates inwards at $c_s \simeq 1/\sqrt{3}$.
- The fluid unaffected by the rarefaction is homogeneous in the transverse directions.

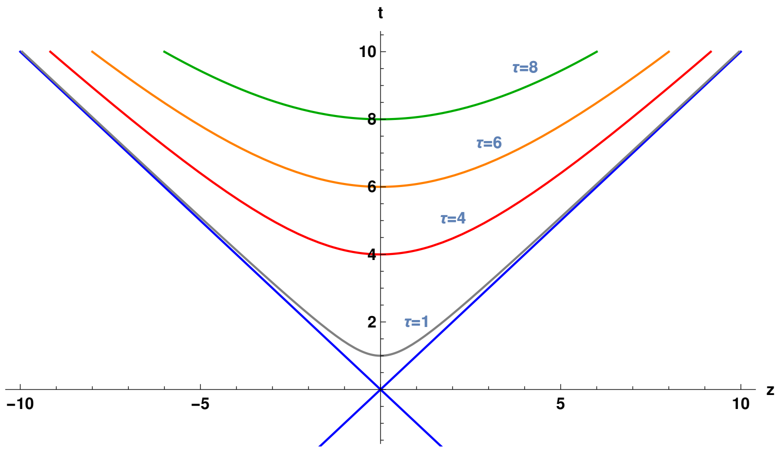




Pictures taken from Bjorken (1983).

[1] J. D. Bjorken, Phys. Rev. D **27** (1983) 140.

Bjorken flow: Milne coordinates



- Boost invariance of the QGP along z prompts the use of τ and w (Milne coordinates):

$$\tau = \sqrt{t^2 - z^2}, \qquad w = \frac{1}{2} \ln \frac{t+z}{t-z} = \mathrm{arctanh}\frac{z}{t}. \tag{1}$$

- The Minkowski line element becomes:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 = -d\tau^2 + dx^2 + dy^2 + \tau^2 dw^2. \tag{2}$$

- With respect to Milne coordinates, $u^\mu_{\rm Milne} = (1, 0, 0, 0)$. In Minkowski:

$$u^\mu_{\rm Mink} = \left(\frac{t}{\tau}, 0, 0, \frac{z}{\tau}\right)^T \Rightarrow v_z = \frac{z}{t}. \tag{3}$$

Gubser flow: transverse propagation

- Bjorken's assumption that transverse dynamics are negligible is not realistic, since the diameter of a nucleus $\sim 10\,\mathrm{fm}$.
- Gubser flow undergoes simultaneously boost-invariant longitudinal and azimuthally symmetric ("radial") transverse expansion.
- Gubser flow is symmetric under the Gubser group $SO(3)_q \otimes SO(1,1) \otimes Z_2$:[2]

$$\begin{aligned} \xi_1 &= \partial_x + q^2 \left[2x(x^\nu \partial_\nu) - (x^\mu x_\mu)\partial_x\right], \\ \xi_2 &= \partial_y + q^2 \left[2y(x^\nu \partial_\nu) - (x^\nu x_\nu)\partial_y\right], \\ \xi_3 &= \partial_\varphi, \end{aligned} \tag{4}$$

  where q is an inverse length scale.
- The resulting velocity has components:

$$u = \cosh\kappa\partial_\tau + \sinh\kappa\partial_R, \qquad \kappa = \mathrm{arctanh}\frac{2\mathrm{q}^2\tau\mathrm{R}}{1+\mathrm{q}^2\tau^2+\mathrm{q}^2\mathrm{R}^2}. \tag{5}$$

- In Minkowski coordinates:

$$u^t = \frac{t}{\sqrt{t^2 - z^2}}\cosh\kappa, \qquad u^R = \sinh\kappa, \qquad u^z = \frac{z}{\sqrt{t^2 - z^2}}\cosh\kappa. \tag{6}$$

[2] S. S. Gubser, Phys. Rev. D **82** (2010) 085027.

Gubser flow: $\mathrm{dS}_3 \otimes \mathbb{R}^3$

- Invariance under the Gubser group is hard to implement in Minkowski space.
- A conformal transformation from Minkowski to $\mathrm{dS}(3) \otimes \mathbb{R}$ makes these symmetries manifest.
- First, a conformal transformation is performed on the Minkowksi metric:

$$d\overline{s}^2 = \frac{ds^2}{\tau^2} = \frac{-d\tau^2 + dR^2 + R^2 d\varphi^2}{\tau^2} + dw^2. \tag{7}$$

- The new coordinates ρ and θ are introduced through:

$$\sinh\rho = -\frac{1 - q^2(\tau^2 - R^2)}{2q\tau}, \qquad \tan\theta = \frac{2qR}{1 + q^2(\tau^2 - R^2)}. \tag{8}$$

- The resulting metric describes the $dS_3 \otimes \mathbb{R}$:

$$d\hat{s}^2 = -d\rho^2 + \cosh^2\rho(d\theta^2 + \sin^2\theta d\varphi^2) + dw^2. \tag{9}$$

- Now the four-velocity of Gubser flow is simply $\overline{u}^\rho = 1$.

[3] S. S. Gubser, A. Yarom, Nucl. Phys. B **846** (2011) 469.

LB models for flat space (Cartesian coordinates)

Relativistic Boltzmann equation

- The Boltzmann equation on Minkowski space reads:

$$p^t \partial_t f + \mathbf{p} \cdot \nabla f \equiv p^\mu \partial_\mu f = J[f], \tag{10}$$

where $p^2 = -(p^t)^2 + \mathbf{p}^2 = 0$ for massless (ultrarelativistic) particles.

- The macroscopic quantities N^μ and $T^{\mu\nu}$ are obtained using:

$$N^\mu = \int \frac{d^3p}{p^t} f\, p^\mu, \qquad T^{\mu\nu} = \int \frac{d^3p}{p^t} f\, p^\mu p^\nu. \tag{11}$$

- Multiplying Eq. (10) by the collision invariants $\psi \in \{1, p^\mu\}$ and integrating w.r.t. d^3p/p^t yields the conservation equation:

$$\partial_\mu N^\mu = 0, \qquad \partial_\nu T^{\mu\nu} = 0. \tag{12}$$

Landau frame

- In the Landau (energy) frame, E and u^μ are obtained from:

$$T^\mu{}_\nu u^\nu = -E\, u^\mu, \qquad u^2 = -1. \tag{13}$$

- In general, N^μ and $T^{\mu\nu}$ can be decomposed into equilibrium and non-equilibrium terms:

$$N^\mu = n u^\mu - \frac{n}{E+P} q^\mu, \qquad T^{\mu\nu} = E\, u^\mu u^\nu + (P + \varpi)\Delta^{\mu\nu} + \pi^{\mu\nu}, \tag{14}$$

  where $\Delta^{\mu\nu} = u^\mu u^\nu + \eta^{\mu\nu}$ and $\varpi = 0$ for massless particles
- The Landau frame is used for the Anderson-Witting approximation of $J[f]$:

$$J_{\text{A-W}}[f] = \frac{p \cdot u}{\tau}[f - f^{(\text{eq})}]. \tag{15}$$

- For non-degenerate gases, $f^{(\text{eq})}$ is the Maxwell-Jüttner distribution:

$$f^{(\text{eq})} = \frac{n}{8\pi T^3} \exp\left(\frac{p \cdot u}{T}\right). \tag{16}$$

Lattice Boltzmann modelling

- The momentum space is replaced by p_k $(k = 1, 2, \dots Q)$:

$$\partial_t f_k + \mathbf{v}_k \cdot \nabla f_k = \frac{v_k \cdot u}{\tau}[f_k - f_k^{(\text{eq})}], \tag{17}$$

wher $v_k^\mu = p_k^\mu / p_k^t$ and $p_k^t = \sqrt{\mathbf{p}^2}$ for massless particles.

- The macroscopic quantities are obtained using quadrature sums:

$$N^\mu = \sum_{k=1}^{Q} \frac{p_k^\mu}{p_k} f_k, \qquad T^{\mu\nu} = \sum_{k=1}^{Q} \frac{p_k^\mu p_k^\nu}{p_k} f_k. \tag{18}$$

- The construction of an LB model typically requires 3 steps:

1. Choice of p_k (quadrature);
2. Polynomial truncation of $f_k^{(\text{eq})}$;
3. Implementation of time evolution and advection.

Quadrature

- A moment of order n is computed as:

$$M_n^{\mu_1\dots\mu_n} = \int \frac{d^3p}{p^t} f\, p^{\mu_1}\cdots p^{\mu_n} = \sum_{k=1}^{Q_L}\sum_{j=1}^{Q_\xi}\sum_{i=1}^{Q_\varphi} \frac{p_{ijk}^{\mu_1}\cdots p_{ijk}^{\mu_n}}{p_k} f_{ijk}, \qquad (19)$$

where p_{ijk} are determined by p_k, ξ_j and φ_i:

$$L_{Q_L}^{(2)}(p_k/T_0) = 0, \qquad P_{Q_\xi}(\xi_j) = 0, \qquad \varphi_i = \frac{2\pi}{Q_\varphi}(i - 1/2). \qquad (20)$$

- f_{ijk} is related to f through:

$$f_{ijk} = \frac{2\pi w_j^\xi w_k^L T_0^3}{Q_\varphi e^{-p_k/T_0}} f(p_k, \xi_j, \varphi_i), \qquad (21)$$

where w_j^ξ and w_k^L are the Gauss-Legendre and Gauss-Laguerre quadrature weights:

$$w_j^\xi = \frac{2(1-\xi_j^2)}{[(Q_\xi+1)P_{Q_\xi+1}(\xi_j)]^2}, \qquad w_k^L = \frac{(Q_L+1)(Q_L+2)p_k/T_0}{[(Q_L+1)L_{Q_L+1}^{(2)}(p_k/T_0)]^2}. \qquad (22)$$

Truncation of $f^{(\mathrm{eq})}$

- The moments of $f^{(\mathrm{eq})}$ are also recovered using quadrature sums:

$$M^{\mu_1\dots\mu_n}_{n,(\mathrm{eq})} = \sum_{k=1}^{Q_L}\sum_{j=1}^{Q_\xi}\sum_{i=1}^{Q_\varphi} f^{(\mathrm{eq})}_{ijk} \frac{p^{\mu_1}_{ijk}\cdots p^{\mu_n}_{ijk}}{p_k}, \qquad f^{(\mathrm{eq})}_{ijk} = \frac{2\pi w^\xi_j w^L_k T_0^3}{Q_\varphi e^{-p_k/T_0}} f^{(\mathrm{eq})}(p_k, \xi_j, \varphi_i), \tag{23}$$

- The quadrature rule can be applied only if $f^{(\mathrm{eq})}_{ijk}$ is a polynomial in p_k, ξ_j and $(\cos\varphi_i, \sin\varphi_i)$:

$$f^{(\mathrm{eq})}_{ijk} = \frac{n\, u^0\, w^\xi_j w^L_k}{2Q_\varphi} \sum_{\ell=0}^{N_L} L^{(2)}_\ell(p_k/T_0) \sum_{s=0}^{N_\Omega} \widetilde{a}^{(\mathrm{eq})}_{\ell,s} P_s(\cos\gamma_{u;ij}), \tag{24}$$

where $\mathbf{u}\cdot\mathbf{p}_{ijk} = u p_k \cos\gamma_{u;ij}$.

- The coefficients $\widetilde{a}^{(\mathrm{eq})}_{\ell,s}$ can be obtained analytically ($\theta = T/T_0$):

$$\widetilde{a}^{(\mathrm{eq})}_{0,0} = 1, \qquad \widetilde{a}^{(\mathrm{eq})}_{0,1} = \frac{u}{u^0}, \qquad \widetilde{a}^{(\mathrm{eq})}_{0,2} = \frac{3}{2u^3 u^0}\operatorname{arcsinh} u + 1 - \frac{3}{2u^2},$$

$$\widetilde{a}^{(\mathrm{eq})}_{1,0} = \widetilde{a}^{(\mathrm{eq})}_{0,0} - \frac{\theta}{3u^0}(3+4u^2), \qquad \widetilde{a}^{(\mathrm{eq})}_{1,1} = \widetilde{a}^{(\mathrm{eq})}_{0,1} - \frac{4u\theta}{3}, \qquad \widetilde{a}^{(\mathrm{eq})}_{1,2} = \widetilde{a}^{(\mathrm{eq})}_{0,2} - \frac{4\theta u^2}{3u^0},$$

etc...

Numerical scheme

- In order to implement the advection, we write the Boltzmann equation in the following form:
$$\frac{\partial f}{\partial t} + \xi \frac{\partial f}{\partial z} = S, \tag{25}$$
where $\xi = p^z/p$ and the flow is assumed to be homogeneous along the x and y directions.
- Writing $\partial_t f = L[f]$, the TVD RK-3 scheme consists of the following steps:[4]
$$\begin{aligned} f^{(1)} &= f(t) + \delta t L[f(t)], \\ f^{(2)} &= \frac{3}{4} f(t) + \frac{1}{4} f^{(1)} + \frac{1}{4} \delta t L[f^{(1)}], \\ f(t+\delta t) &= \frac{1}{3} f(t) + \frac{2}{3} f^{(2)} + \frac{2}{3} \delta t L[f^{(2)}]. \end{aligned} \tag{26}$$
- The advection is implemented using WENO-5:[5]
$$\frac{\partial(\xi f)}{\partial z} = \frac{\mathcal{F}_{s+1/2} - \mathcal{F}_{s-1/2}}{z_{s+1/2} - z_{z-1/2}}. \tag{27}$$

[4] C.-W. Shu, S. Osher, J. Comput. Phys. **77** (1988) 439-471.
[5] L. Rezzolla, O. Zanotti, *Relativistic hydrodynamics* (Oxford University Press, Oxford, UK, 2013).

Arbitrary coordinates and General relativity

Arbitrary coordinates

- Let us employ arbitrary coordinates $\{x^{\widetilde{\mu}}\}$ with $p^{\widetilde{\mu}} = \frac{\partial x^{\widetilde{\mu}}}{\partial x^{\mu}} p^{\mu}$.
- Geodesic motion is ensured using the Christoffel symbols ($\tau^* = \tau/m$):[6]

$$\frac{dx^{\widetilde{\mu}}}{d\tau^*} = p^{\widetilde{\mu}}, \qquad \frac{dp^{\widetilde{\imath}}}{d\tau^*} = -\Gamma^{\widetilde{\imath}}{}_{\widetilde{\mu}\widetilde{\nu}} p^{\widetilde{\mu}} p^{\widetilde{\nu}}. \tag{28}$$

- The Boltzmann equation in covariant form reads:

$$\frac{df}{d\tau^*} = p^{\widetilde{\mu}} \frac{\partial f}{\partial x^{\widetilde{\mu}}} - \Gamma^{\widetilde{\imath}}{}_{\widetilde{\mu}\widetilde{\nu}} p^{\widetilde{\mu}} p^{\widetilde{\nu}} \frac{\partial f}{\partial p^{\widetilde{\imath}}} = J[f], \qquad p^{\widetilde{\mu}} = \frac{\partial x^{\widetilde{\mu}}}{\partial x^{\mu}} p^{\mu}. \tag{29}$$

[6] C. Cercignani, G. M. Kremer, *The Relativistic Boltzmann Equation - Theory and Applications* (Birkhäuser Basel, 2002).

General relativity

- In GR, $\eta_{\mu\nu}$(flat space-time) $\rightarrow g_{\mu\nu}$(curved space-time).
- $g_{\mu\nu}$ satisfies the Einstein equations:

$$G_{\mu\nu} = 8\pi T_{\mu\nu}. \tag{30}$$

- The metric tensor is linked to the line element through:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{31}$$

- The formalism presented previously (29) applies unchanged for GR flows:

$$p^\mu \frac{\partial f}{\partial x^\mu} - \Gamma^i{}_{\mu\nu} p^\mu p^\nu \frac{\partial f}{\partial p^i} = J[f], \tag{32}$$

where the Chirstoffel symbols must be computed from the metric:

$$\Gamma^\lambda{}_{\mu\nu} = g^{\lambda\sigma} \Gamma_{\sigma\mu\nu}, \qquad \Gamma_{\sigma\mu\nu} = \frac{1}{2}(g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma}). \tag{33}$$

Tetrad field

- The mass-shell condition $p^2 = g_{\mu\nu} p^\mu p^\nu = -m^2$ implies:

$$p^t = \frac{p_t - g_{ti} p^i}{g_{tt}}, \qquad p_t = -\sqrt{-m^2 g_{tt} - (g_{tt} g_{ij} - g_{ti} g_{tj}) p^i p^j}. \tag{34}$$

- We introduce the tetrad frame $e_{\hat{\alpha}} = e^\mu_{\hat{\alpha}} \partial_\mu$ and the one-form coframe $\omega^{\hat{\alpha}} = \omega^{\hat{\alpha}}_\mu dx^\mu$:

$$g_{\mu\nu} e^\mu_{\hat{\alpha}} e^\nu_{\hat{\beta}} = \eta_{\hat{\alpha}\hat{\beta}}, \qquad \langle \omega^{\hat{\alpha}}, \eta_{\hat{\beta}} \rangle = \omega^{\hat{\alpha}}_\mu e^\mu_{\hat{\beta}} = \delta^{\hat{\alpha}}{}_{\hat{\beta}}. \tag{35}$$

- Defining $p^{\hat{\alpha}} = \omega^{\hat{\alpha}}_\mu p^\mu$, the mass-shell condition becomes:

$$p^2 = -(p^{\hat{0}})^2 + \mathbf{p}^2 = -m^2 \Rightarrow p^{\hat{0}} = \sqrt{\mathbf{p}^2 + m^2}. \tag{36}$$

Spherical coordinates in momentum space

- $p^{\hat{\imath}} \in \{p^{\hat{1}}, p^{\hat{2}}, p^{\hat{3}}\}$ can be expressed using $p^{\widetilde{\imath}} \in \{p, \xi, \varphi\}$:

$$p^{\hat{1}} = p\sqrt{1-\xi^2}\cos\varphi, \qquad p^{\hat{2}} = p\sqrt{1-\xi^2}\sin\varphi, \qquad p^{\hat{3}} = p\xi, \tag{37}$$

  where $\xi = \cos\theta$.
- The mass-shell condition implies $p^{\hat{0}} = \sqrt{m^2 + p^2}$.
- The parametrisation (37) introduces a metric $\lambda_{\widetilde{\imath}\widetilde{\jmath}}$ in the momentum space:

$$d\Phi^2 = \lambda_{\widetilde{\imath}\widetilde{\jmath}} dp^{\widetilde{\imath}} dp^{\widetilde{\jmath}}, \qquad \lambda_{\widetilde{\imath}\widetilde{\jmath}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & p^2/(1-\xi^2) & 0 \\ 0 & 0 & p^2(1-\xi^2) \end{pmatrix}. \tag{38}$$

Boltzmann equation using tetrads

- The Boltzmann equation (29) can be put in conservative form:[7]

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,e^\mu_{\hat\alpha}p^{\hat\alpha}f\right) - \frac{p^{\hat 0}}{\sqrt{\lambda}}\frac{\partial}{\partial p^{\tilde\imath}}\left(P^{\tilde\imath}{}_{\hat\imath}\Gamma^{\hat\imath}{}_{\hat\alpha\hat\beta}\frac{p^{\hat\alpha}p^{\hat\beta}}{p^{\hat 0}}f\sqrt{\lambda}\right) = J[f]. \tag{39}$$

- $P^{\tilde\jmath}_{\hat\imath} = \partial p^{\tilde\jmath}/\partial p^{\hat\imath}$ has components:[8]

$$P^{\tilde\jmath}{}_{\hat\imath} = \begin{pmatrix} \sqrt{1-\xi^2}\cos\varphi & \sqrt{1-\xi^2}\sin\varphi & \xi \\ -\frac{\xi}{p}\sqrt{1-\xi^2}\cos\varphi & -\frac{\xi}{p}\sqrt{1-\xi^2}\sin\varphi & \frac{1}{p}(1-\xi^2) \\ -\frac{\sin\varphi}{p\sqrt{1-\xi^2}} & \frac{\cos\varphi}{p\sqrt{1-\xi^2}} & 0 \end{pmatrix}. \tag{40}$$

[7] C. Y. Cardall, E. Endeve, A. Mezzacappa, Phys. Rev. D **88** (2013) 023011.
[8] V. E. Ambruș, R. Blaga, Annals of West University of Timisoara - Physics **58** (2015) 89–108.

Macroscopic moments

- The macroscopic moments are obtained as on Minkowski space:

$$\begin{pmatrix} N^{\hat{\alpha}} \\ T^{\hat{\alpha}\hat{\beta}} \end{pmatrix} = \int \frac{d^3\hat{p}}{p^{\hat{0}}} f \begin{pmatrix} p^{\hat{\alpha}} \\ p^{\hat{\alpha}} p^{\hat{\beta}} \end{pmatrix}$$
$$= \int_0^\infty dp \begin{pmatrix} p^2 \\ p^3 \end{pmatrix} \int_{-1}^1 d\xi \int_0^{2\pi} d\varphi\, f \begin{pmatrix} v^{\hat{\alpha}} \\ v^{\hat{\alpha}} v^{\hat{\beta}} \end{pmatrix}. \tag{41}$$

- The above integrals can be recovered as on Minkowski space, using the Gauss-Laguerre (p), Gauss-Legendre (ξ) and Mysovskikh (φ) quadratures.
- The conservation equations $\nabla_{\hat{\alpha}} N^{\hat{\alpha}} = 0$ and $\nabla_{\hat{\beta}} T^{\hat{\alpha}\hat{\beta}} = 0$ imply the following conserved quantities:

$$\mathcal{N} = \int d^3x \sqrt{-g} e^t_{\hat{\alpha}} N^{\hat{\alpha}}, \qquad \mathcal{T}_k = \int d^3x \sqrt{-g} e^t_{\hat{\alpha}} T^{\hat{\alpha}\hat{\beta}} k_{\hat{\beta}}, \tag{42}$$

where $k_{\hat{\beta};\hat{\gamma}} + k_{\hat{\gamma};\hat{\beta}} = 0$.

- Our numerical schemes are constructed such that $\mathcal{N}$ and $\mathcal{T}_k$ are exactly preserved.

Global thermodynamic equilibrium

- The Boltzmann equation on curved spaces reads:

$$p^{\hat{\alpha}} e^{\mu}_{\hat{\alpha}} \frac{\partial f}{\partial x^{\mu}} - \Gamma^{\hat{\imath}}{}_{\hat{\alpha}\hat{\beta}} p^{\hat{\alpha}} p^{\hat{\beta}} \frac{\partial f}{\partial p^{\hat{\imath}}} = J[f]. \tag{43}$$

- The collision term $J[f]$ cancels when $f = f^{(\mathrm{eq})}$:

$$f^{(\mathrm{eq})} = \left[\exp\left(-\frac{\mu}{T} - \frac{p \cdot u}{T}\right) - \epsilon\right]^{-1}, \tag{44}$$

where $\epsilon = -1$, 1 and 0 for Fermi-Dirac, Bose-Einstein and Maxwell-Jüttner statistics.

- $f^{(\mathrm{eq})}$ satisfies Eq. (43) when:

$$\nabla_{\hat{\alpha}} \left(\frac{\mu}{T}\right) = 0, \qquad \nabla_{\hat{\alpha}} \left(\frac{u_{\hat{\beta}}}{T}\right) + \nabla_{\hat{\beta}} \left(\frac{u_{\hat{\alpha}}}{T}\right) = 0. \tag{45}$$

- The flow is in global thermal equilibrium if μ/T is constant and when $u^{\hat{\alpha}}/T$ is a Killing vector field.

Conformal transformations

Conformal transformations

- A conformal (Weyl) transformation is $\overline{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu}$.
- Under a Weyl transformation:

$$\overline{e}^\mu_{\hat{\alpha}} = \Omega^{-1} e^\mu_{\hat{\alpha}}, \qquad \overline{p}^{\hat{\alpha}} = \Omega^{-1} p^{\hat{\alpha}}, \qquad \overline{u}^{\hat{\alpha}} = u^{\hat{\alpha}}, \qquad \overline{f} = f, \qquad \overline{T} = \Omega^{-1} T,$$
$$\overline{\Gamma}^{\hat{\gamma}}{}_{\hat{\alpha}\hat{\beta}} \overline{p}^{\hat{\alpha}} \overline{p}^{\hat{\beta}} = \frac{1}{\Omega^2} \left[\Gamma^{\hat{\gamma}}{}_{\hat{\alpha}\hat{\beta}} p^{\hat{\alpha}} p^{\hat{\beta}} + (e^\nu_{\hat{\alpha}} p^2 - p^{\hat{\beta}} e^\nu_{\hat{\beta}} p_{\hat{\alpha}}) \partial_\nu \Omega^{-1} \right],$$
$$(\partial_\mu \overline{f})_{\overline{p}^{\hat{\imath}}} = (\partial_\mu f)_{p^{\hat{\imath}}} - (\partial_\mu \Omega^{-1}) p^{\hat{\imath}} \partial_{p^{\hat{\imath}}} f. \tag{46}$$

- The Boltzmann equation transforms as:

$$\overline{p}^{\hat{\alpha}} \overline{e}^\mu_{\hat{\alpha}} \partial_\mu \overline{f} - \overline{\Gamma}^{\hat{\imath}}{}_{\hat{\alpha}\hat{\beta}} \overline{p}^{\hat{\alpha}} \overline{p}^{\hat{\beta}} \frac{\partial \overline{f}}{\partial \overline{p}^{\hat{\imath}}} - \overline{J}[\overline{f}] =$$
$$\frac{1}{\Omega^2} \left\{ p^{\hat{\alpha}} e^\mu_{\hat{\alpha}} \partial_\mu f - \Gamma^{\hat{\imath}}{}_{\hat{\alpha}\hat{\beta}} p^{\hat{\alpha}} p^{\hat{\beta}} \frac{\partial f}{\partial p^{\hat{\imath}}} - \Omega^2 \overline{J}[f] - p^2 \eta^{\hat{\imath}\hat{\alpha}} e^\nu_{\hat{\alpha}} \partial_{p^{\hat{\imath}}} f \partial_\nu \Omega^{-1} \right\}, \tag{47}$$

- $\overline{J}[f] = \Omega^{-2} J[f]$ is ensured if $\tau \sim T^{-1}$, i.e.:

$$J[f] = \frac{p \cdot u}{\tau} [f - f^{(\text{eq})}], \qquad \tau = \frac{c_\tau}{T}, \tag{48}$$

- The Boltzmann equation is conformally invariant if $p^2 = 0$.[9]

[9] G. Denicol et al, Phys. Rev. D **90** (2014) 125026.

Conformal soliton flow. $\mathrm{AdS}_2 \otimes \mathrm{S}_2$[11]

- First proposed as a spherically symmetric expanding wave.[10]
- To derive the flow velocity, a conformal transformation with $\Omega = r^{-1}$ is performed:

$$d\bar{s}^2 = \frac{ds^2}{r^2} = -\cosh^2\rho\, dT^2 + d\rho^2 + d\theta^2 + \sin^2\theta d\varphi^2, \tag{49}$$

where T and ρ are:

$$\cosh\rho = \frac{1}{2Lr}\sqrt{[L^2 + (r+t)^2][L^2 + (r-t)^2]}, \qquad \tan T = \frac{L^2 + r^2 - t^2}{2Lt}. \tag{50}$$

- The flow is described by $\overline{u}^T = 1/\cosh\rho$ which in Minkowski corresponds to:

$$u^t = \frac{L^2 + r^2 + t^2}{\sqrt{[L^2 + (r+t)^2][L^2 + (r-t)^2]}},$$
$$u^r = \frac{2tr}{\sqrt{[L^2 + (r+t)^2][L^2 + (r-t)^2]}}. \tag{51}$$

---

[10] J. J. Friess, S. S. Gubser, G. Michalogiorgakis, S. S. Pufu, JHEP04(2007)080.
[11] J. Noronha, G. S. Denicol, Phys. Rev. D **92** (2015) 114032.

$\mathrm{AdS}_2 \otimes \mathrm{S}_2$: Boltzmann equation

- The following tetrad can be employed:

$$e_{\hat{T}} = \frac{\partial_T}{\cosh\rho}, \qquad e_{\hat{\rho}} = \partial_\rho, \qquad e_{\hat{\theta}} = \partial_\theta, \qquad e_{\hat{\phi}} = \frac{\partial_\phi}{\sin\theta}. \tag{52}$$

- In the case when the flow is spherically-symmetric, the Boltzmann equation reads:

$$\partial_T f + \xi \partial\rho(f\cosh\rho) - \sinh\rho \left\{ \frac{\xi}{p^2} \frac{\partial(fp^3)}{\partial p} + \frac{\partial[(1-\xi^2)f]}{\partial\xi} \right\} \\ = -\frac{\cosh\rho}{\tau}(u^{\hat{\tau}} - \mathbf{v}\cdot\mathbf{u})[f - f^{(\mathrm{eq})}], \tag{53}$$

where $p^{\hat{\rho}} = p\xi$, $p^{\hat{\theta}} = p\sqrt{1-\xi^2}\cos\varphi$ and $p^{\hat{\varphi}} = p\sqrt{1-\xi^2}\sin\varphi$.

$\mathrm{AdS}_2 \otimes \mathrm{S}_2$: LB model

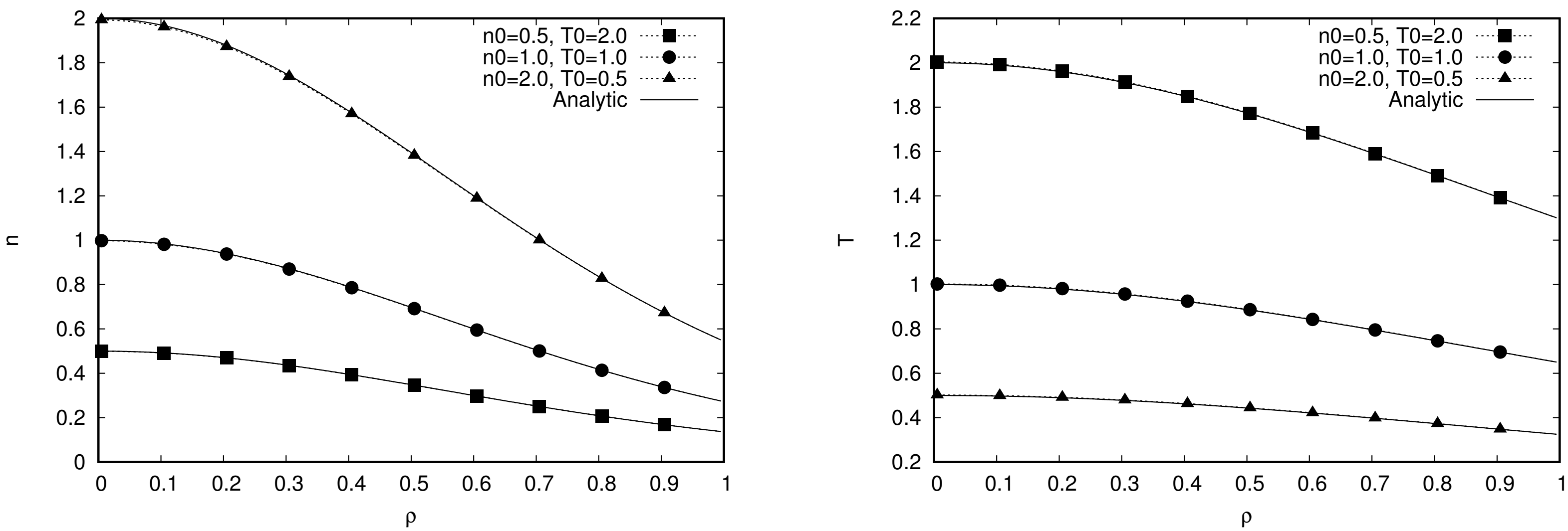
- The discretisation of the momentum space is performed using $Q = Q_L \times Q_\xi \times Q_\varphi$ quadrature points.
- Since the azimuthal degrees of freedom contribute trivially, we set $Q_\varphi = 1$ and perform the φ integrations analytically in deriving macroscopic moments.
- The p degrees of freedom are also trivial when $m = 0$, such that $Q_L = 2$ is sufficient for the exact recovery of the evolution of $T^{\hat{\alpha}\hat{\beta}}$.
- The derivatives w.r.t. ξ and p are computed using:

$$\left[\frac{1}{p^2}\frac{\partial(fp^3)}{\partial p}\right]_{ijk} = \sum_{k'=1}^{Q_L} \mathcal{K}^L_{k,k'} f_{ijk'}, \qquad \left[\frac{\partial[(1-\xi^2)f]}{\partial \xi}\right]_{ijk} = \sum_{j'=1}^{Q_\xi} \mathcal{K}^P_{j,j'} f_{ij'k},$$

$$\mathcal{K}^L_{k,k'} = w^L_k \sum_{\ell=1}^{Q_L-1} \frac{1}{\ell+1} L^{(2)}_\ell(\overline{p}_k) \left[L^{(2)}_{\ell-1}(\overline{p}_{k'}) - \frac{\ell}{\ell+2} L^{(2)}_\ell(\overline{p}_{k'})\right],$$

$$\mathcal{K}^P_{j,j'} = w^\xi_j \sum_{\ell=1}^{Q_\xi-1} \frac{\ell(\ell+1)}{2} P_\ell(\xi_j)\left[P_{\ell+1}(\xi_{j'}) - P_{\ell-1}(\xi_{j'})\right]. \qquad (54)$$

Global equilibrium



- The solution we are interested in is $u^{\hat{\alpha}} = (1, 0, 0, 0)^T$.
- This corresponds to

$$n = \frac{n_0}{\cosh^3 \rho}, \qquad T = \frac{T_0}{\cosh \rho}. \tag{55}$$

- The system is initialised with $n = n_0$ and $T = T_0$.
- On the right boundary $\rho = \rho_{\rm b}$, the condition $f = f^{(\rm eq)}(n_{\rm b}, T_{\rm b})$ is imposed, where $n_{\rm b}$ and $T_{\rm b}$ are given by (55) when $\rho = \rho_{\rm b}$

Conclusion

Conclusion

- Using a conformal transformation, a complicated flow on Minkowski can become simple on a curved space.
- LB modelling can be useful to investigate dissipative processes in curved spaces.
- Using the vielbein (tetrad) formalism, the LB algorithm is identical to that for Minkowski space (same quadrature).
- Quadrature methods coupled with finite difference are versatile tools for simulations in arbitrary coordinate systems.
- This work was supported by a grant of the Romanian National Authority for Scientific Research and Innovation, CNCS-UEFISCDI, project number PN-II-RU-TE-2014-4-2910.