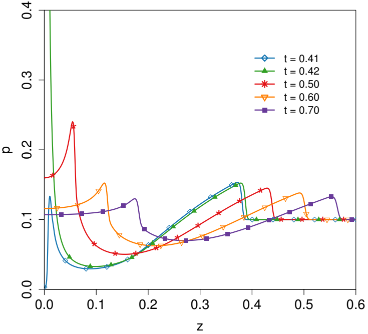
Relativistic shocks in non-Cartesian geometries



Robert Blaga, Victor E. Ambruș

TIM17 Physics Conference

Timișoara

27th May 2017

Overview

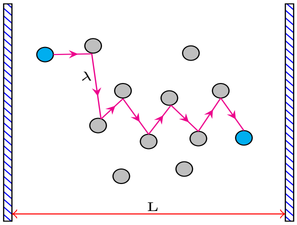
1. What is this Boltzmann?
   - Of fluids and gases
   - The equation
2. Relativistic Lattice Boltzmann
   - The expansion of $f^{(\mathrm{eq})}$
   - Quadrature relations
3. Applications
   - Shock heating
   - Sod problem

Of fluids and gases

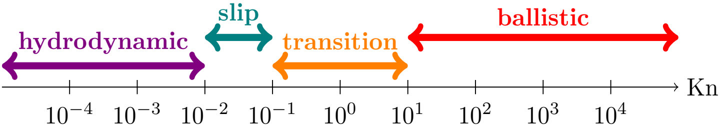
Fluid: physical system with large number of components

Levels of description:

1. **macroscopic**: hydrodynamics/Navier-Stokes eqs.
2. **mesoscopic**: kinetic theory/Boltzmann eq.
3. **microscopic**: molecular dynamics/Newton's eqs.



$$\mathrm{Kn} = \frac{\lambda}{\mathrm{L}}.$$



The equation

- Relativistic Boltzmann equation:

$$\left(p^{\hat{\mu}}\partial_{\hat{\mu}} - \Gamma^{\hat{i}}_{\hat{\mu}\hat{\nu}} p^{\hat{\mu}} p^{\hat{\nu}} \partial_{p^{\hat{i}}}\right) f(t, \vec{x}, \vec{p}) = C[f] \tag{1}$$

- Anderson-Witting collision operator:

$$C[f] = \frac{u^{\hat{\alpha}} p_{\hat{\alpha}}}{\tau} \left(f - f^{(\mathrm{eq})}\right) \tag{2}$$

- Equilibrium distribution function:

$$f^{(\mathrm{eq})} = \frac{n}{8\pi T^3} \exp\left(\frac{u^{\hat{\alpha}} p_{\hat{\alpha}}}{T}\right) \tag{3}$$

Macroscopic profiles

Macroscopic variables are defined as moments of the distribution function.

- Particle four-flow:
$$N^{\hat{\alpha}} = \int \frac{d^3p}{p^{\hat{t}}} \, f \, p^{\hat{\alpha}} \tag{4}$$
- Energy-momentum tensor:
$$T^{\hat{\alpha}\hat{\beta}} = \int \frac{d^3p}{p^{\hat{t}}} \, f \, p^{\hat{\alpha}} p^{\hat{\beta}} \tag{5}$$
- Higher order moments:
$$T^{\hat{\alpha}...\hat{\omega}} = \int \frac{d^3p}{p^{\hat{t}}} \, f \, p^{\hat{\alpha}} ... p^{\hat{\omega}} \tag{6}$$

The expansion of $f^{(\mathrm{eq})}$

We project the equilibrium distribution function onto complete sets of orthogonal polynomials, with respect each momentum-space coordinates[1]:

$$f^{(\mathrm{eq})}(t, \vec{x}, \vec{p}) = \frac{e^{-p/T_0}}{T_0^3} \sum_{\ell=0}^{N_L} \sum_{s=0}^{N_\Omega} a_{\ell,s}^{(\mathrm{eq})}(\vec{x}, t)\, P_s(\xi)\, L_\ell^{(2)}(p/T_0) \tag{7}$$

The series is truncated at the finite values: N_L, N_Ω. In the infinite limit the expansion is exact.

[1] We have introduced spherical coordinates in the momentum space: $\vec{p} = (p, \theta, \phi)$, with the notation $\xi = \cos\theta$.

Quadrature relations

The second ingredient for our lattice Boltzmann model is the discretisation of the momentum space. For massless particles we have $p^{\hat{t}} = |\vec{p}| \equiv p$ and $v^{\hat{\alpha}} = p^{\hat{\alpha}}/p$. We convert the integrals in the particle four-flow and energy momentum tensor into finite sums, as follows[2,3]

$$
\begin{aligned}
N^{\hat{\alpha}} &= \int_0^\infty dp\, p^2 \int_{-1}^{1} d\xi \int_0^{2\pi} d\varphi\, f\, v^{\hat{\alpha}} &&= \sum_{a=1}^{Q_L} \sum_{b=1}^{Q_\xi} \sum_{c=1}^{Q_\varphi} f_{abc}\, v^{\hat{\alpha}}_{bc}, \\
T^{\hat{\alpha}\hat{\beta}} &= \int_0^\infty dp\, p^3 \int_{-1}^{1} d\xi \int_0^{2\pi} d\varphi\, f\, v^{\hat{\alpha}} v^{\hat{\beta}} &&= \sum_{a=1}^{Q_L} \sum_{b=1}^{Q_\xi} \sum_{c=1}^{Q_\varphi} f_{abc}\, p_a v^{\hat{\alpha}}_{bc} v^{\hat{\beta}}_{bc},
\end{aligned} \tag{8}
$$

$$
f_{abc} = \frac{2\pi w^L_a w^\xi_b}{Q_\varphi e^{-p_a/T_0}/T_0^3} f(p_a, \xi_b, \varphi_c).
$$

[2] V.E. Ambruș and V. Sofonea, *Phys. Rev. E* **86** (2012).

[3] R. Blaga and V.E. Ambruș, *preprint arXiv:1612.01287* (2016).

Shock heating[4]

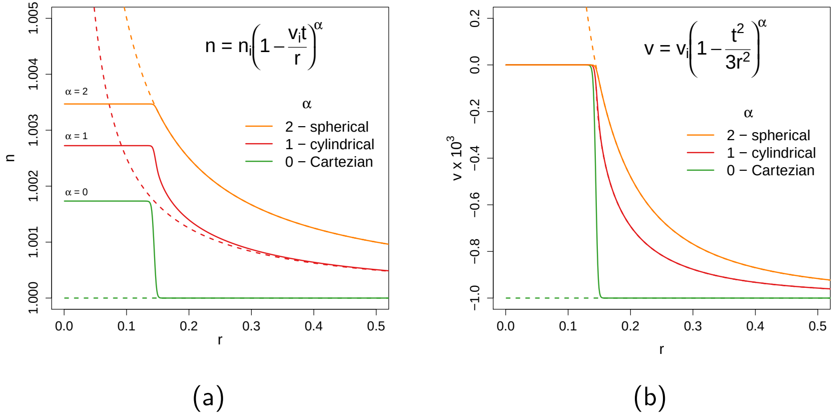


Figure: Density (a) and velocity (b) profiles of fluid in the shock heating setup at time $t = 0.4$, superimposed for the three cases.The initial velocity is $v = -0.001$. The dotted curves represent the approximate solutions to the hydrodynamic equations, at small velocities.

[4] J.M. Martí, et al. , *Astrophys. J.* **479** (1997).

Sod problem - hydrodynamic regime[5]

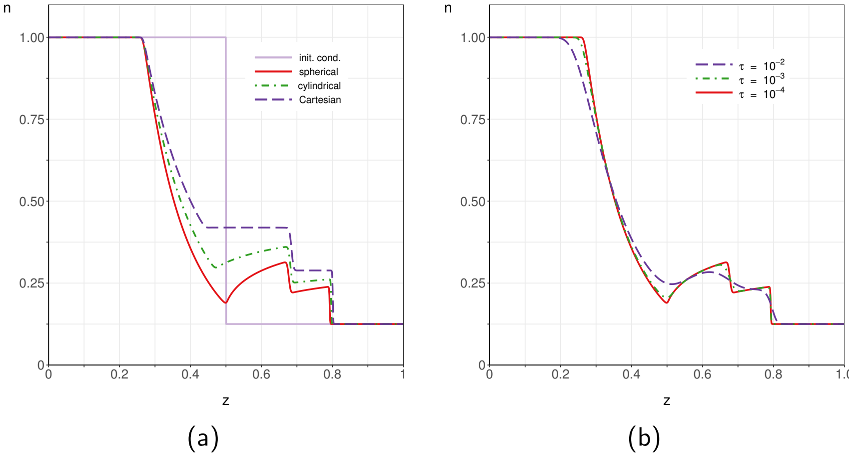


Figure: (a) Density profile of the fluid in Sod problem setup for the three cases, at time $t = 0.4$. A shock wave and contact discontinuity forms and travels outwards, while a rarefaction wave propagates towards the origin. (b) The density profile for the spherical geometry, at different relaxation times.

[5] L. Rezzolla and O. Zanotti, *Relativistic hydrodynamics*. Oxford University Press, Oxford, UK, 2013.

Sod problem - ballistic regime

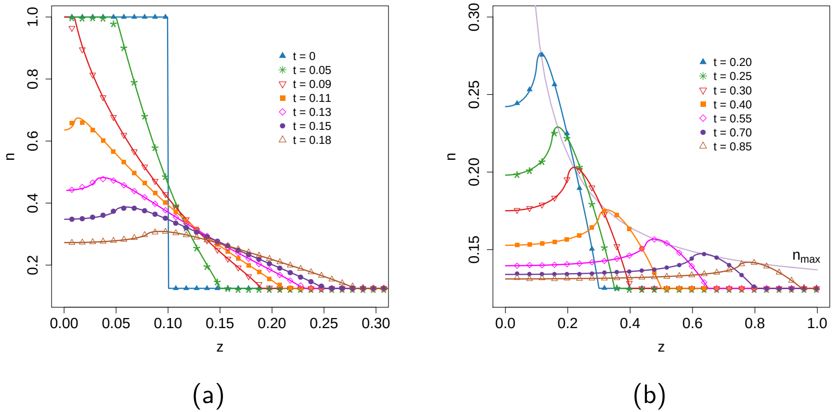


Figure: Density profile of the fluid in the spherically symmetric Sod problem, in the ballistic regime. Panel (a) shows instances of time around the moment when the rarefaction wave arrives at the origin, while panel (b) shows for larger times, as a shell of high density fluid propagates outwards.

Conclusion

- We have developed lattice Boltzmann models, based on Gauss-quadratures, adapted for problems with azimuthal and spherical symmetry
- The models have been tested on two problems of flows containing shock (shock heating and the Sod problem) with very good agreement compared to the analytical solutions
- Our results recommend the models developed here, to be applied in the astrophysical arena

Acknowledgement

- No student was harmed in obtaining these results.
- Everybody who contributed to this work is an author or has been formally acknowledged.

- This work was supported by a grant from the Romanian National Authority for Scientific Research and Innovation, CNCS-UEFISCDI, project number PN-II-RU-TE-2014-4-2910.