Quantum corrections in thermal states of fermions on anti-de Sitter spacetime

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Outline

Section 1

Introduction

Motivation

- The adS/CFT correspondence sparked an explosion of interest in the behaviour of classical and quantum fields on asymptotically adS space-times.
- There has been recent quantitative analyses of quantum corrections to thermal states in various geometries.[1,2,3]
- Our study shows that kinetic theory results represent a certain limit of the semi-classical QFT in curved spaces.
- Similarly, the full theory of quantum gravity may reduce in some limit to QFT in curved spaces.
- We focus on (the covering space of) pure adS, where analytic expressions can be obtained using both kinetic theory and QFT.

[1] F. Becattini and E. Grossi, Phys. Rev. D 92 (2015) 045037.
[2] R. Panerai, Phys. Rev. D **93** (2016) 104021.
[3] V. E. Ambruș, Phys. Lett. B **771** (2017) 151.

Section 2

Anti-de Sitter space

Anti-de Sitter space

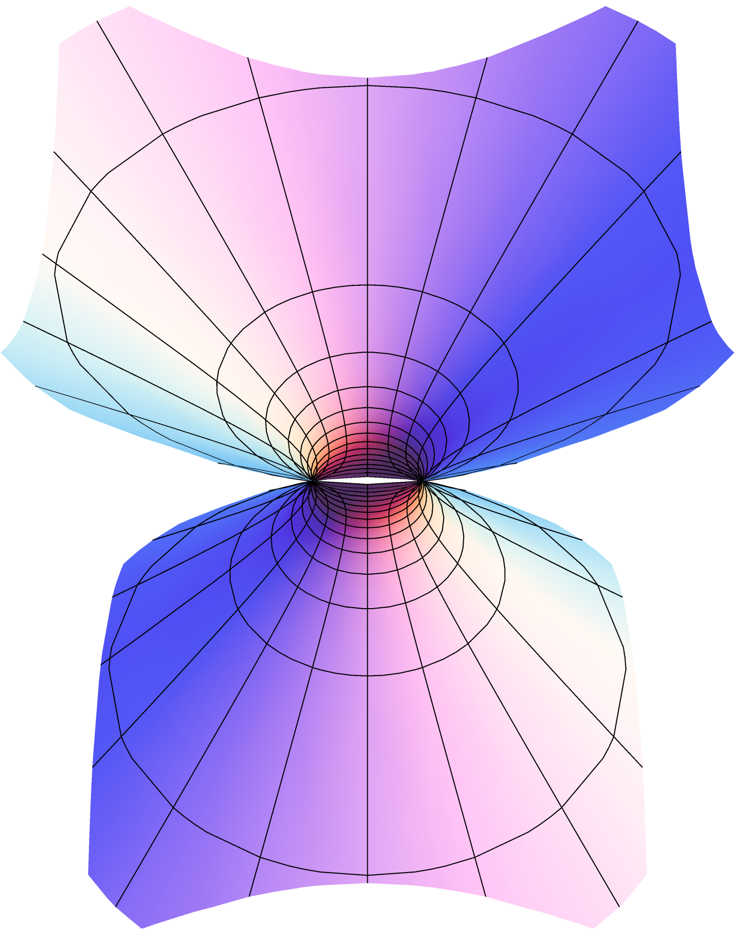
- In the $5D$ embedding space, adS is a hyperboloid:

$$z_0^2 - z_1^2 - z_2^2 - z_3^2 + z_4^2 = \frac{1}{\omega^2},$$

  where ω is the inverse radius of curvature.
- AdS is a maximally symmetric vacuum solution of the Einstein equations with negative cosmological constant $\Lambda = -3\omega^2$.
- Coordinates for static chart:

$$z^0 = \frac{1}{\omega}\frac{\cos\omega t}{\cos\omega r}, \qquad z^5 = \frac{1}{\omega}\frac{\sin\omega t}{\cos\omega r},$$
$$z^i = \frac{\tan\omega r}{\omega r} x^i,$$

  where $t \in (-\infty, \infty)$ and $0 \le \omega r < \frac{\pi}{2}$.



Geometry of adS

- Line element of adS:

$$ds^2 = \frac{1}{\cos^2 \omega r}\left[-dt^2 + dr^2 + \frac{\sin^2 \omega r}{\omega^2}\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right], \tag{1}$$

- Geodetic interval $s(x, x')$ is given through:

$$\cos\omega s = \frac{\cos\omega\Delta t}{\cos\omega r\cos\omega r'} - \cos\gamma\tan\omega r\tan\omega r', \tag{2}$$

where $\Delta t = t - t'$ and $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\Delta\varphi$.

- The metric can be put in Minkowski form $\eta_{\hat\alpha\hat\beta} = g_{\mu\nu}e^\mu_{\hat\alpha}e^\nu_{\hat\beta}$ using the tetrad:[4]

$$\begin{aligned} e_{\hat t} &= \cos\omega r\,\partial_t, & e_{\hat i} &= \cos\omega r\left[\frac{\omega r}{\sin\omega r}\left(\delta_{ij} - \frac{x^i x^j}{r^2}\right) + \frac{x^i x^j}{r^2}\right]\partial_j, \\ \omega^{\hat t} &= \frac{dt}{\cos\omega r}, & \omega^{\hat i} &= \frac{1}{\cos\omega r}\left[\frac{\sin\omega r}{\omega r}\left(\delta_{ij} - \frac{x^i x^j}{r^2}\right) + \frac{x^i x^j}{r^2}\right]dx^j. \end{aligned} \tag{3}$$

[4] I. I. Cotăescu, Rom. J. Phys. **52** (2007) 895.

Section 3

Kinetic theory

Boltzmann equation

- The relativistic Boltzmann equation on a curved space-time reads:[5]

$$p^{\hat{\alpha}} e^{\mu}_{\hat{\alpha}} \frac{\partial f}{\partial x^{\mu}} - \Gamma^{\hat{i}}{}_{\hat{\alpha}\hat{\sigma}} p^{\hat{\alpha}} p^{\hat{\sigma}} \frac{\partial f}{\partial p^{\hat{i}}} = J[f], \tag{4}$$

  where $f \equiv f(x^{\mu}, \boldsymbol{p})$ is the Boltzmann distribution function, while $\eta_{\hat{\alpha}\hat{\beta}} p^{\hat{\alpha}} p^{\hat{\beta}} = -m^2$.
- $J[f]$ drives the system towards thermodynamic equilibrium:

$$f^{(\mathrm{eq})}_{-1}(\beta, u^{\hat{\alpha}}) = \frac{4}{(2\pi)^3} (e^{-\widetilde{\beta} p^{\hat{\alpha}} u_{\hat{\alpha}}} + 1)^{-1}, \tag{5}$$

  where $\widetilde{\beta}$ is the *local* inverse temperature and $u^{\hat{\alpha}}$ is the macroscopic four-velocity ($u^2 = -1$).

[5] C. Cercignani, G. Kremer, *The relativistic Boltzmann equation: theory and applications*, Birkhäuser Verlag, Basel (2002).

Thermodynamic equilibrium

- $f = f^{(\mathrm{eq})}$ satisfies the Bolzmann equation (4) if:

$$\nabla_{\hat{\alpha}}(\beta u_{\hat{\beta}}) + \nabla_{\hat{\beta}}(\beta u_{\hat{\alpha}}) = 0.$$

- We are interested in static solutions $u = e_{\hat{t}}$, such that:[6,7]

$$\widetilde{\beta} = \beta\sqrt{-g_{tt}} = \frac{\beta}{\cos\omega r}, \tag{6}$$

where β is the inverse temperature at $r = 0$.

- The temperature $\widetilde{\beta}^{-1}$ monotonically decreases from β^{-1} at the origin down to 0 on the space-time boundary $\omega r = \pi/2$.

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[6] R. C. Tolman, Phys. Rev. **35** (1930) 904;
R. C. Tolman, P. Ehrenfest, Phys. Rev. **36** (1930) 1791.

[7] V. E. Ambruș, I. I. Cotăescu, Phys. Rev. D **94** (2016) 085022.

Moments of f

- The particle four-flow $N^{\hat{\alpha}}$ and stress-energy tensor $T^{\hat{\alpha}\hat{\beta}}$ can be obtained as moments of $f = f^{(\mathrm{eq})}$:

$$N^{\hat{\alpha}} = \int \frac{d^3p}{p^{\hat{t}}} f^{(\mathrm{eq})}\, p^{\hat{\alpha}} = n u^{\hat{\alpha}},$$

$$T^{\hat{\alpha}\hat{\beta}} = \int \frac{d^3p}{p^{\hat{t}}} f^{(\mathrm{eq})}\, p^{\hat{\alpha}} p^{\hat{\beta}} = (E+P) u^{\hat{\alpha}} u^{\hat{\beta}} + P\, \eta^{\hat{\alpha}\hat{\beta}}. \qquad (7)$$

- For Fermi-Dirac statistics, E and P can be expressed as:[8]

$$P_{-1}(\widetilde{\beta}) = -\frac{2m^2}{\pi^2\beta^2}(\cos\omega r)^2 \sum_{j=1}^{\infty} \frac{(-1)^j}{j^2} K_2\left(\frac{mj\beta}{\cos\omega r}\right)$$

$$\xrightarrow[m=0]{} \frac{7\pi^2}{180\beta^4}(\cos\omega r)^4,$$

$$E_{-1}(\widetilde{\beta}) - 3P_{-1}(\widetilde{\beta}) = -\frac{2m^3\cos\omega r}{\pi^2\beta} \sum_{j=1}^{\infty} \frac{(-1)^j}{j} K_1\left(\frac{mj\beta}{\cos\omega r}\right)$$

$$\xrightarrow[m=0]{} 0. \qquad (8)$$

[8] V. E. Ambruș, R. Blaga, Annals of West University of Timișoara - Physics **58** (2015) 89.

Section 4

Quantum field theory

The Feynman Green's function $S_F(x, x')$

- On maximally symmetric spaces, S_F can be written as:

$$iS_F(x, x') = [\mathcal{A}_F(s) + \mathcal{B}_F(s)\not{n}]\Lambda(x, x').$$

- $\mathcal{A}_F$ and $\mathcal{B}_F$ depend only on s and reduce when $m = 0$ to:

$$\mathcal{A}_F\rfloor_{m=0} = \frac{\omega^3}{16\pi^2}\left(\cos\frac{\omega s}{2}\right)^{-3}, \qquad \mathcal{B}_F\rfloor_{m=0} = \frac{i\omega^3}{16\pi^2}\left(\sin\frac{\omega s}{2}\right)^{-3}. \qquad (9)$$

- $n_{\hat{\alpha}} = \nabla_{\hat{\alpha}} s(x, x')$ is the normalised tangent to the geodesic at x.
- The bi-spinor of parallel transport satisfying $n^{\hat{\alpha}} D_{\hat{\alpha}} \Lambda(x, x') = 0$ is given by:[9]

$$\Lambda(x, x') = \frac{\sec(\omega s/2)}{\sqrt{\cos\omega r \cos\omega r'}}\Bigg[$$
$$\cos\frac{\omega\Delta t}{2}\left(\cos\frac{\omega r}{2}\cos\frac{\omega r'}{2} + \sin\frac{\omega r}{2}\sin\frac{\omega r'}{2}\frac{\mathbf{x}\cdot\boldsymbol{\gamma}}{r}\frac{\mathbf{x}'\cdot\boldsymbol{\gamma}}{r'}\right)$$
$$+\sin\frac{\omega\Delta t}{2}\left(\sin\frac{\omega r}{2}\cos\frac{\omega r'}{2}\frac{\mathbf{x}\cdot\boldsymbol{\gamma}}{r}\gamma^{\hat{t}} + \sin\frac{\omega r'}{2}\cos\frac{\omega r}{2}\frac{\mathbf{x}'\cdot\boldsymbol{\gamma}}{r'}\gamma^{\hat{t}}\right)\Bigg].$$

[9] V. E. Ambruș, E. Winstanley, AIP Conf. Proc. **1634** (2014) 40.

Expectation values from two-point functions

- Consider the Feynman two-point function:

$$iS_F(x,x') = \langle \theta(t-t')\psi(x)\overline{\psi}(x') - \theta(t'-t)\overline{\psi}(x')\psi(x)\rangle\,.$$

- The expectation values of $\overline{\psi}\psi$, J^μ and $T_{\mu\nu}$ can be obtained as:[10]

$$\begin{aligned}
\langle\overline{\psi}\psi\rangle &= -\lim_{x'\to x}\mathrm{tr}[iS_F(x,x')\Lambda(x',x)],\\
\langle J^{\hat{\alpha}}\rangle &= -\lim_{x'\to x}\mathrm{tr}[\gamma^{\hat{\alpha}} iS_F(x,x')\Lambda(x',x)],\\
\langle T_{\hat{\alpha}\hat{\sigma}}\rangle &= -\frac{1}{2}\lim_{x'\to x}\mathrm{tr}\Bigg\{\left[\gamma_{(\hat{\alpha}}D_{\hat{\sigma})}S_F(x,x') - g_{\hat{\alpha}}{}^{\hat{\alpha}'}g_{\hat{\sigma}}{}^{\hat{\sigma}'}S_F(x,x')\overleftarrow{D}_{(\hat{\alpha}'}\gamma_{\hat{\sigma}')}\right]\\
&\qquad\times\Lambda(x',x)\Bigg\}.
\end{aligned}$$

[10] P. B. Groves, P. R. Anderson, E. D. Carlson, Phys. Rev. D **66** (2002) 124017.

Feynman Green's function $S_F^\beta(x,x')$ for a thermal state

- $S_F^\beta(x,x')$ is antiperiodic with respect to the imaginary time:[11]

$$S_F^\beta(\Delta t;\mathbf{x},\mathbf{x}') = \sum_{j=-\infty}^{\infty} (-1)^j S_F(\Delta t + ij\beta;\mathbf{x},\mathbf{x}').$$

- The closed form expression of $S_F(x,x')$ can be used to deduce that the quantum t.e.v. of the SET (relative to the vacuum state) is in ideal form:

$$\langle :T^{\hat\alpha\hat\sigma}:\rangle_\beta = (E_\beta + P_\beta)u^{\hat\alpha}u^{\hat\sigma} + P_\beta\eta^{\hat\alpha\hat\sigma},$$

where E_β reduces in the massless limit to:

$$E_\beta\rfloor_{m=0} = -\frac{3\omega^4}{4\pi^2}(\cos\omega r)^4\sum_{j=1}^{\infty}(-1)^j\frac{\cosh\frac{\omega j\beta}{2}}{(\sinh\frac{\omega j\beta}{2})^4}. \tag{10}$$

- An expansion of Eq. (10) in powers of $\beta\omega$ recovers the BOP approximation:[12]

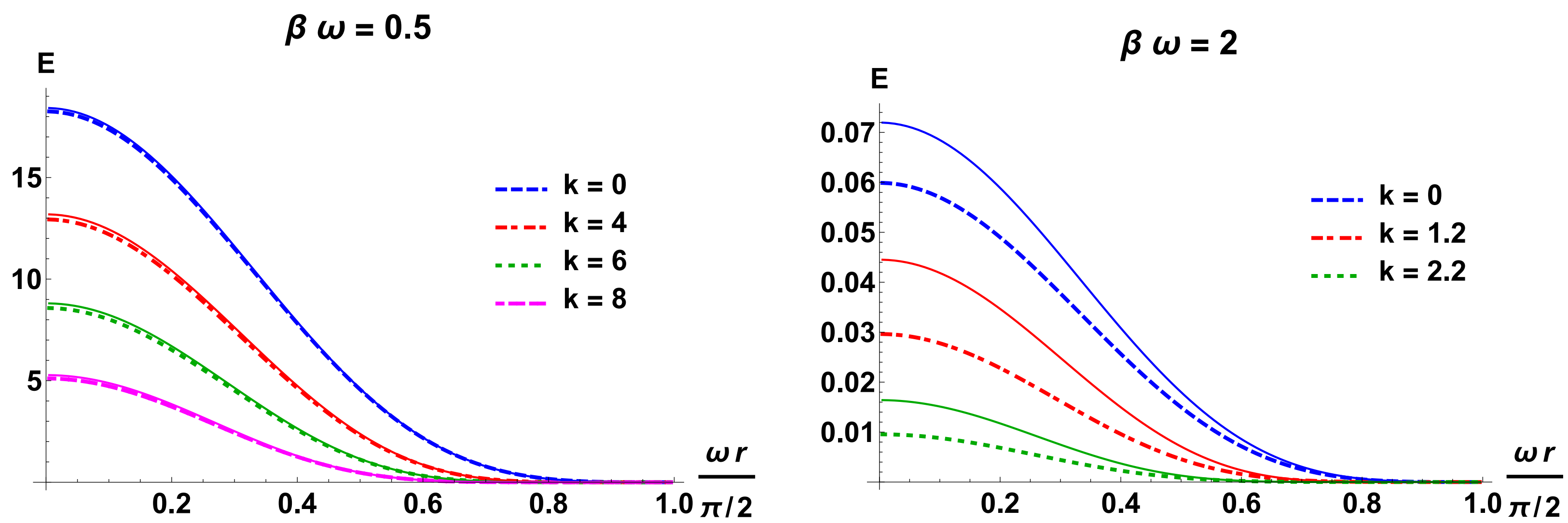
$$E_\beta\rfloor_{m=0} = \frac{7\pi^2}{60\beta^4}(\cos\omega r)^4\left[1 - \frac{5\beta^2\omega^2}{14\pi^2} - \frac{17\beta^4\omega^4}{112\pi^4} + O([\beta\omega]^6)\right]. \tag{11}$$

[11] N. D. Birrell, P. C. W. Davies, *Quantum fields in curved space* (CUP, 1982)
[12] M. R. Brown, A. C. Ottewill, D. N. Page, Phys. Rev. D **33** (1986) 2840.

Section 5

Comparison

Energy density profiles



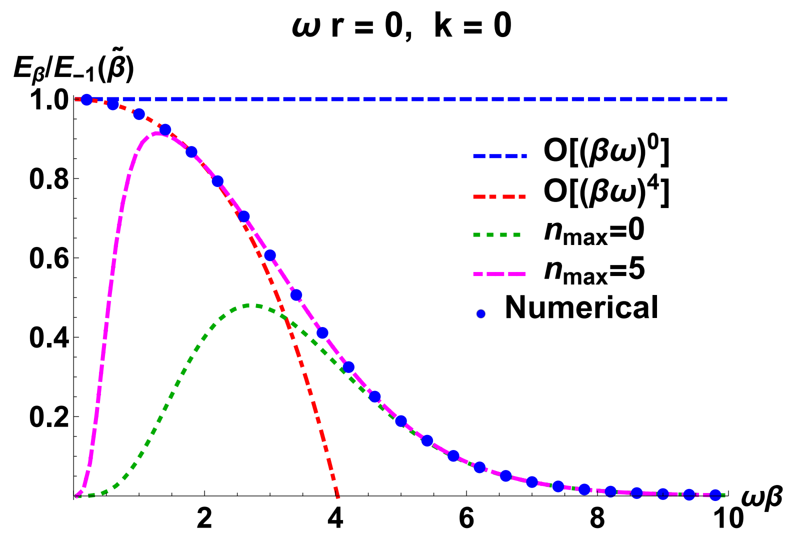
- The kinetic theory and QFT results are almost overlapped at small $\beta\omega$, for a wide range of $k = m/\omega$.
- When $\beta\omega$ is decreased, the QFT energy density becomes significantly lower than the kinetic theory one.

Energy density: massless limit

- In the massless case:

$$\left(\frac{E_\beta}{E_{-1}(\widetilde{\beta})}\right) = \left(\frac{E_\beta}{E_{-1}(\widetilde{\beta})}\right)_{r=0} (\cos\omega r)^4$$

- At $\omega\beta \to 0$, the classical limit is reached and $E_\beta / E_{-1}(\widetilde{\beta}) \to 1$:

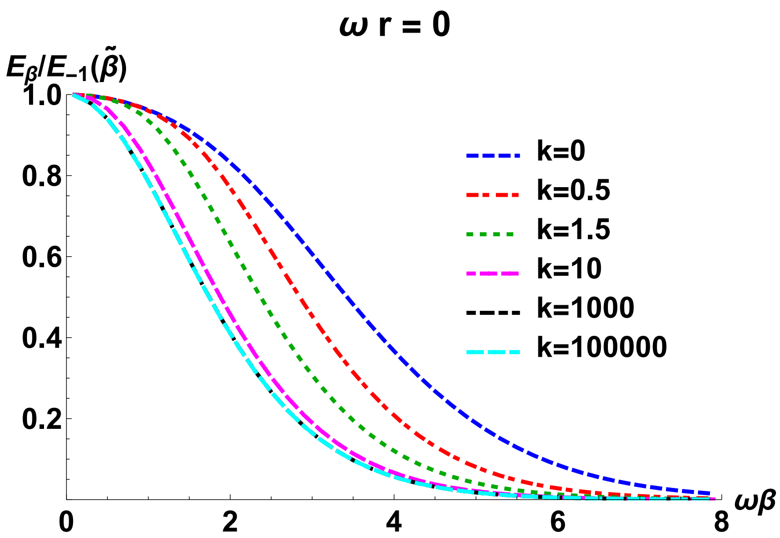


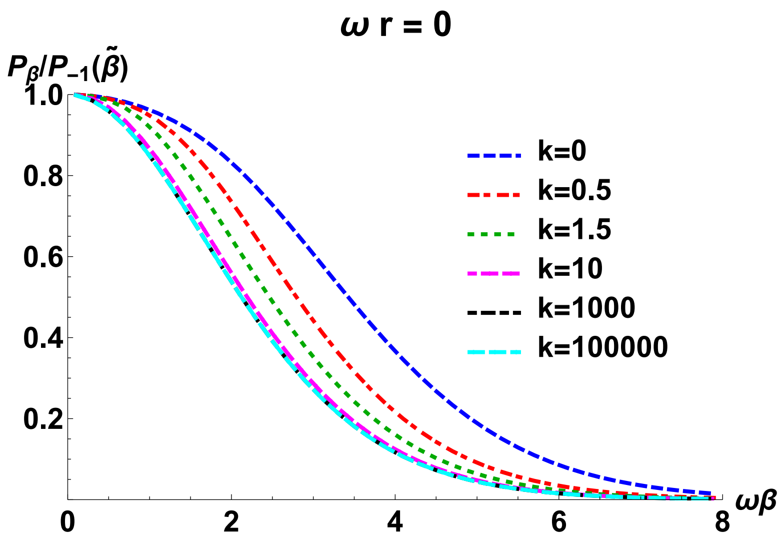
$$E_\beta\rfloor_{m=0} = \frac{7\pi^2}{60\beta^4}(\cos\omega r)^4 \left[1 - \frac{5\beta^2\omega^2}{14\pi^2} - \frac{17\beta^4\omega^4}{112\pi^4} + O([\beta\omega]^6)\right].$$

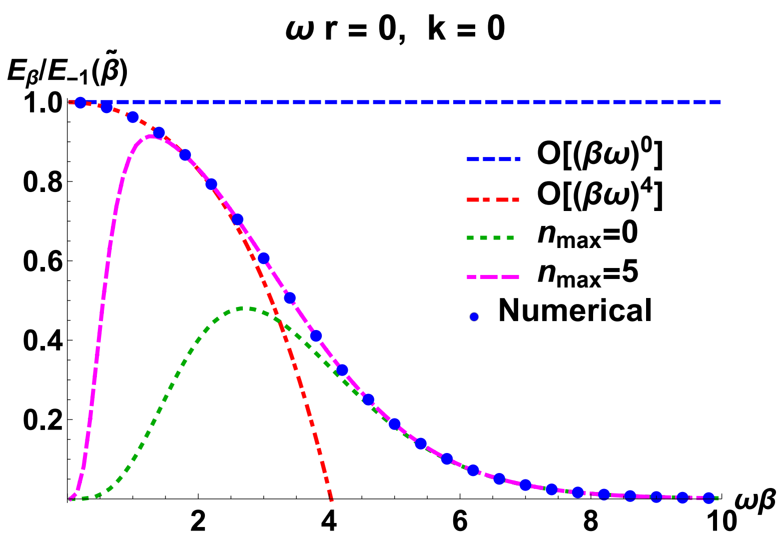
- As $\omega\beta \to \infty$, E_β becomes strongly quenched by quantum corrections:

$$E_\beta = -\frac{6\omega^4}{\pi^2} \frac{(\cos\omega r)^4}{1 + e^{\frac{3}{2}\omega\beta}} \sum_{n=0}^{\infty} e^{-n\omega\beta} \left(1 + \frac{13n}{6} + \frac{3n^2}{2} + \frac{n^3}{3}\right) \frac{1 + e^{-\frac{3}{2}\omega\beta}}{1 + e^{-(\frac{3}{2}+n)\omega\beta}}.$$

Energy density - behaviour at the origin

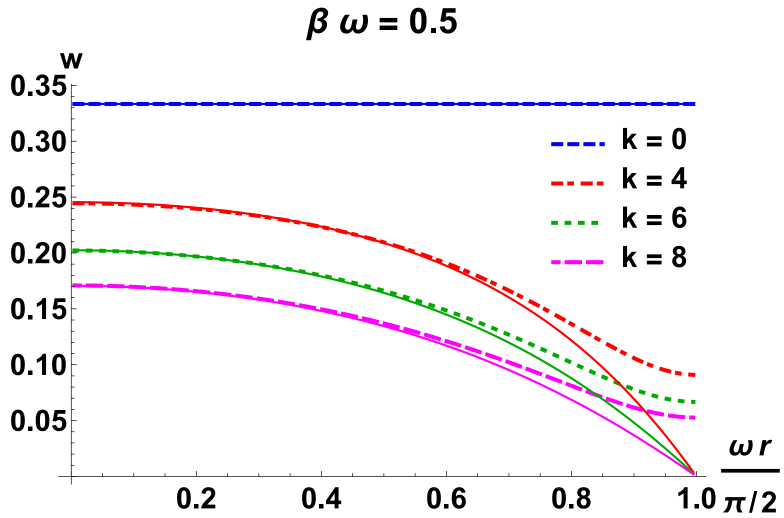






- $E_\beta/E_{-1}(\widetilde{\beta})$ approaches an asymptotic value as $k \to \infty$.
- $P_\beta/P_{-1}(\widetilde{\beta})$ is quenched at a lesser rate.
- $w_\beta/w_{-1}(\widetilde{\beta})$ increases with $\omega\beta$ when $k > 0$.

Equation of state $w = P/E$



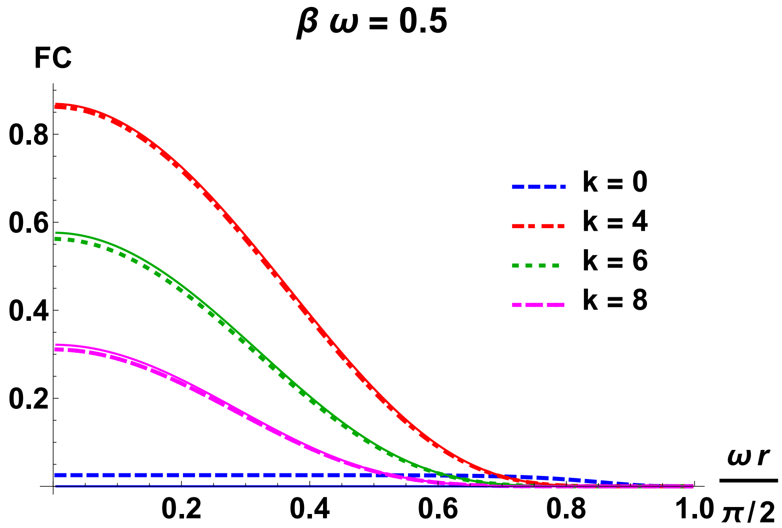
- On the boundary $\omega r = \pi/2$, w_β goes to a finite value, while w_{-1} vanishes for all $k = m/\omega \neq 0$:

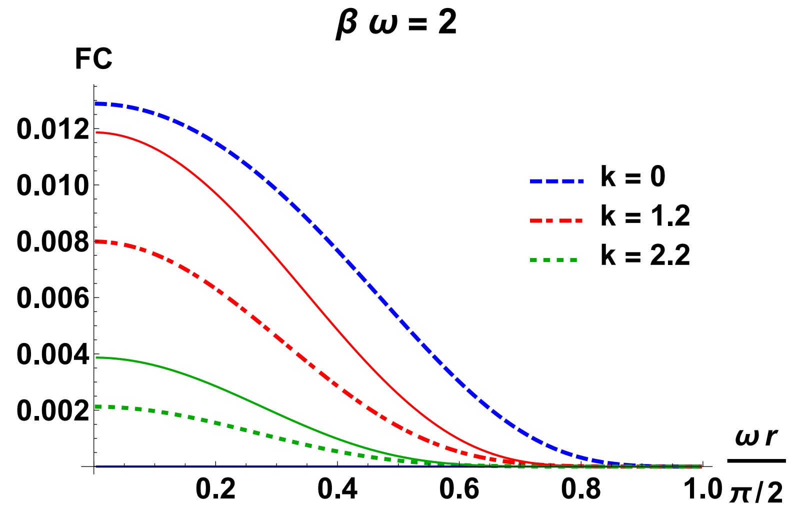
$$w_\beta \simeq \left[3 + 2k - \frac{2k(2+k)}{1+2k}\frac{S_{k+1}}{S_k}\cos^2\omega r\right]^{-1},$$

$$w_{-1}(\widetilde{\beta}) \simeq \left(m\beta + \frac{3}{2}\cos\omega r\right)^{-1}\cos\omega r. \qquad (12)$$

$$[S_\nu = -\sum_{j=1}^{\infty}(-1)^j \frac{\cosh\frac{\omega j \beta}{2}}{(\sinh\frac{\omega j\beta}{2})^{4+2\nu}}]$$

Fermion condensate



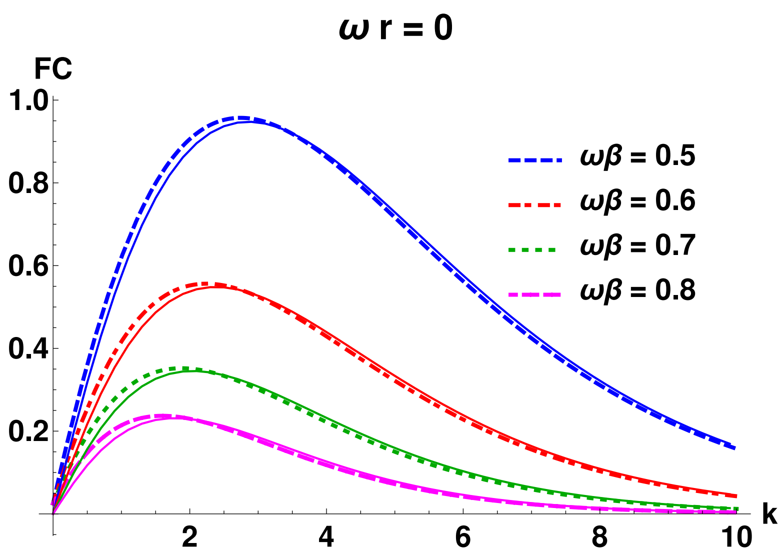


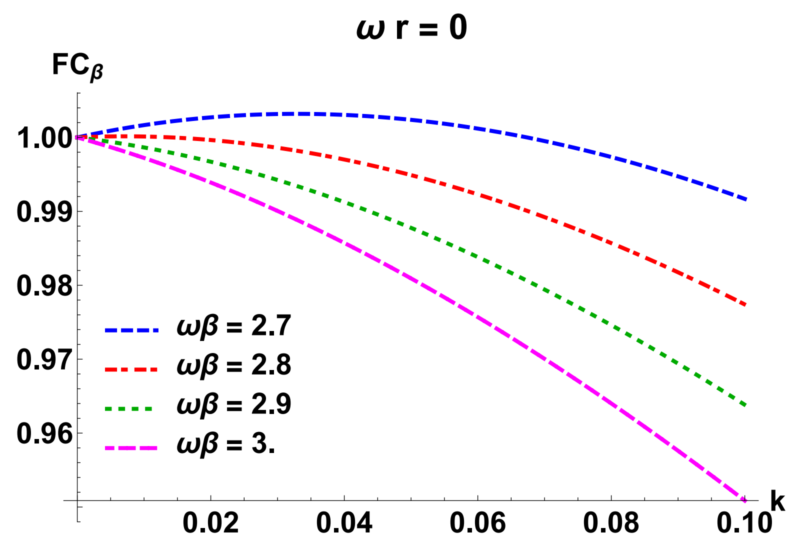
- The fermion condensate satisfies $\langle : \overline{\psi}\psi : \rangle_\beta$:

$$\langle : \overline{\psi}\psi : \rangle_\beta = -\frac{1}{m} \langle : T^{\hat{\alpha}}{}_{\hat{\alpha}} : \rangle_\beta \,.$$

- The kinetic theory FC can be defined similarly: $\mathrm{FC}_{-1} = -\frac{1}{m} T^{\hat{\alpha}}_{-1,\hat{\alpha}}$.
- The quantum FC is non-zero even when $k = 0$, where the kinetic theory FC vanishes.

Fermion condensate - behaviour at the origin





- At the origin, $\mathrm{FC}_{-1}(\widetilde{\beta})$ has the same value as in Minkowski:

$$FC_{-1}(\widetilde{\beta})\Big]_{\omega r=0} = -\frac{2m^2}{\pi^2\beta}\sum_{j=1}^{\infty}\frac{(-1)^j}{j}K_1\left(mj\beta\right).$$

- FC_β no longer exhibits a maximum w.r.t. k when

$$\frac{d(FC_\beta)}{dk}\Big]_{k=0,\omega r=0} = -\frac{\omega^3}{2\pi^2}\sum_{j=1}^{\infty}(-1)^j\left(\frac{1}{\cosh\frac{\omega j\beta}{2}\sinh^2\frac{\omega j\beta}{2}} - \frac{\ln\left[\sinh^2\frac{\omega j\beta}{2}\right]}{\cosh^3\frac{\omega j\beta}{2}}\right) \le 0.$$

- Equality is attained when $\omega\beta \simeq 2.82857$, which is close, but not equal to $\sqrt{8} \simeq 2.82843$.

Section 6

Conclusion

Conclusion

- Using an analytic expression for $S_F(x, x')$ written in terms of $\Lambda(x, x')$, the t.e.v.s of the quantum Dirac field were investigated.
- The quantum SET was shown to describe an ideal (perfect) fluid.
- Quantum corrections were highlighted in comparison to the classical relativistic kinetic theory predictions.
- $w_\beta = P_\beta / E_\beta = (3 + 2k)^{-1}$ on the adS boundary, whereas $w_{-1}(\widetilde{\beta})$ vanishes when $m \neq 0$.
- $\mathrm{FC}_\beta = \langle : \overline{\psi} \psi : \rangle_\beta$ is finite at $k = 0$, whereas $\mathrm{FC}_{-1}(\widetilde{\beta})$ vanishes identically.
- At large $\beta\omega$, $E_\beta / E_{-1}(\widetilde{\beta})$ is strongly quenched by quantum corrections.
- More details can be found in arXiv:1704.00614 [hep-th].

Acknowledgement

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Feynman Green's function

- On maximally symmetric spaces, S_F can be written as:

$$iS_F(x,x') = [\mathcal{A}_F(s) + \mathcal{B}_F(s)\slashed{n}]\Lambda(x,x'),$$

where

$$\begin{aligned}
\mathcal{A}_F =& \frac{\omega^3\Gamma\left(2+k\right)}{16\pi^{\frac{3}{2}}4^k\Gamma\left(\frac{1}{2}+k\right)} \cos\left(\frac{\omega s}{2}\right)\left[-\sin^2\left(\frac{\omega s}{2}\right)\right]^{-2-k} \\
&\times {}_2F_1\left(1+k, 2+k; 1+2k; \mathrm{cosec}^2\left(\frac{\omega s}{2}\right)\right), \\
\mathcal{B}_F =& \frac{i\omega^3\Gamma\left(2+k\right)}{16\pi^{\frac{3}{2}}4^k\Gamma\left(\frac{1}{2}+k\right)} \sin\left(\frac{\omega s}{2}\right)\left[-\sin^2\left(\frac{\omega s}{2}\right)\right]^{-2-k} \\
&\times {}_2F_1\left(k, 2+k; 1+2k; \mathrm{cosec}^2\left(\frac{\omega s}{2}\right)\right).
\end{aligned}$$

Quantum thermal expectation values

- The t.e.v. of the SET and of the FC are given by:

$$\langle : T^{\hat{\alpha}\hat{\sigma}} : \rangle_\beta = (E_\beta + P_\beta) u^{\hat{\alpha}} u^{\hat{\sigma}} + P_\beta \eta^{\hat{\alpha}\hat{\sigma}}, E_\beta + P_\beta =$$

$$\times\, {}_2F_1 \left[k, 3+k; 1+2k; -\frac{\cos^2 \omega r}{\sinh^2 \frac{\omega j \beta}{2}} \right],$$

$$P_\beta = -\frac{\omega^4 \Gamma(2+k) (\cos \omega r)^{4+2k}}{\pi^{3/2} 4^{1+k} \Gamma(\frac{1}{2}+k)} \sum_{j=1}^\infty (-1)^j \frac{\cosh \frac{\omega j \beta}{2}}{(\sinh \frac{\omega j \beta}{2})^{4+2k}}$$

$$\times\, {}_2F_1 \left[k, 2+k; 1+2k; -\frac{\cos^2 \omega r}{\sinh^2 \frac{\omega j \beta}{2}} \right],$$

$$\mathrm{FC}_\beta = \langle : \overline{\psi} \psi : \rangle_\beta = -\frac{2\omega^3 \Gamma(2+k) (\cos \omega r)^{4+2k}}{\pi^{3/2} 4^{1+k} \Gamma(\frac{1}{2}+k)} \sum_{j=1}^\infty (-1)^j \frac{\cosh \frac{\omega j \beta}{2}}{(\sinh \frac{\omega j \beta}{2})^{4+2k}}$$

$$\times\, {}_2F_1 \left(1+k, 2+k; 1+2k; -\frac{\cos^2 \omega r}{\sinh^2 \frac{\omega j \beta}{2}} \right).$$