Point splitting in quantum field theory on curved spaces

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Outline

Quantum field theory on curved spaces

Classical background geometry $g_{\mu\nu}$ (from Einstein's field equations)

Klein-Gordon / Dirac / Maxwell-Proca equations

Construct mode solutions

Second quantisation - particle/anti-particle interpretation of modes

Time-ordered product

Inhomogeneous Dirac equation

Obtain the Feynman propagator $S_F(x, x')$

Calculate vacuum expectation value of $T_{\mu\nu}$

Motivation

- Point splitting uses two-point functions to calculate expectation values of position-dependent operators, e.g. of the SET:

$$\langle T_{\mu\nu}\rangle = \frac{i}{2}\lim_{x'\to x} \mathrm{tr}\left[\gamma_{(\mu}D_{\nu)}S_F(x,x') - g_\mu^{\mu'}g_\nu^{\nu'}S_F(x,x')\overleftarrow{D}_{(\mu'}\gamma_{\nu')}\right]\Lambda^{-1}(x,x').$$

- Useful for regularisation of quantum states on curved spaces:

$$S_F(x,x') \to S_F^{\mathrm{reg}}(x,x') = S_F(x,x') - S_F^{\mathrm{div}}(x,x').$$

- Useful for the construction of quantum expectation values at finite temperature:

$$S_F(x,x') \to \Delta S_F^{\beta_0}(x,x') = \sum_{j\neq 0} S_F(\boldsymbol{x}, t + ij\beta_0; \boldsymbol{x'}, t').$$

- Especially useful when the propagator can be expressed in closed form, i.e. on maximally symmetric spaces, when:

$$S_F(x,x') = [A(s) + B(s)\not{n}]\Lambda(x,x').$$

Synge's world function

- Let $z^\mu(x, x'; \lambda)$ describe the unique geodesic passing through x and x' (i.e. x is in the normal convex neighbourhood of x'), with:

$$z^\mu(x, x'; \lambda_0) = x^\mu, \qquad z^\mu(x, x'; \lambda_1) = x'^\mu, \qquad t^\nu \nabla_\nu t^\mu = 0,$$

  where $t^\mu = dz^\mu / d\lambda$ is the tangent to the geodesic at $z^\mu(\lambda)$.
- Synge's world function is defined as[1]:

$$\sigma(x, x') = -\frac{1}{2}(\lambda_1 - \lambda_0) \int_{\lambda_0}^{\lambda_1} d\lambda \, g_{\mu\nu} t^\mu t^\nu,$$

  being linked to the geodetic interval $s \equiv s(x, x')$ through $\sigma = -\frac{1}{2} s^2$.
- The normalised tangents n_μ and $n_{\mu'}$ to the geodesic at x and x' are given by:

$$n_\mu = \nabla_\mu s, \qquad n_{\mu'} = \nabla_{\mu'} s.$$

[1] E. Poisson, *The Motion of Point Particles in Curved Spacetime*, Living Rev. Relativity 7, (2004).

Bi-vector and bi-spinor of parallel transport

- Tensors can be parallel-transported along z^μ using the bi-vector of parallel transport ${g^\mu}_{\mu'} \equiv {g^\mu}_{\mu'}(x,x')$:

$$T_{||}^{\mu\nu\dots}(x) = {g^\mu}_{\mu'}{g^\nu}_{\nu'}\dots T^{\mu'\nu'\dots}(x'),$$

where ${g^\mu}_{\mu'}$ satisfies the parallel transport equation:

$$n^\lambda \nabla_\lambda {g^\mu}_{\mu'} = 0.$$

- Spinors can be parallel transported using the bi-spinor of parallel transport $\Lambda(x,x')$:

$$\psi_{|}|(x) = \Lambda(x,x')\psi(x'),$$

where
$$n^\lambda D_\lambda \Lambda(x,x') = 0, \quad \Lambda(x,x) = I, \quad \Lambda(x,x')^{-1} = \Lambda(x',x).$$

The spinor covariant derivative $D_\lambda = \nabla_\lambda - \Gamma_\lambda$ is defined using the spin connection Γ_λ:

$$\Gamma_\lambda = \frac{1}{2}\omega_\mu^{\hat\alpha}\Gamma_{\hat\beta\hat\gamma\hat\alpha}\Sigma^{\hat\beta\hat\gamma}.$$

Expectation values from two-point functions

- Consider the Feynman two-point function:

$$S^{(1)}(x,x') = \langle \theta(t-t')\psi(x)\overline{\psi}(x') - \theta(t'-t)\overline{\psi}(x')\psi(x)\rangle \,,$$

  where $\langle \cdot \rangle$ denotes an expectation value with respect to some quantum state.
- The expectation values of $\overline{\psi}\psi$, J^μ and $T_{\mu\nu}$ can be obtained as[2]:

$$\langle \overline{\psi}\psi\rangle = -\lim_{x'\to x} \mathrm{tr} S_F(x,x')\Lambda(x',x),$$

$$\langle J^\mu\rangle = -\lim_{x'\to x} \mathrm{tr}\gamma^\mu S_F(x,x')\Lambda(x',x),$$

$$\langle T_{\mu\nu}\rangle = \frac{i}{2}\lim_{x'\to x} \mathrm{tr}\left[\gamma_{(\mu}D_{\nu)}S_F(x,x') - g_\mu{}^{\mu'} g_\nu{}^{\nu'} S_F(x,x')\overleftarrow{\overline{D}}_{(\mu'}\gamma_{\nu')}\right]\Lambda(x',x).$$

[2] C. Dappiaggi, T.-P. Hack, and N. Pinamonti, Rev. Math. Phys. **21** (2009) 1241

Maximally symmetric space-times

- On maximally symmetric space-times, the following equations hold[3]:

$$n_{\mu;\nu} = -A(g_{\mu\nu} + n_\mu n_\nu), \qquad n_{\mu;\nu'} = -C(g_{\mu\nu'} - n_\mu n_{\nu'}),$$
$$D_\mu \Lambda(x,x') = -(A+C)\Sigma_{\mu\nu} n^\nu \Lambda(x,x').$$

- Also, the Feynman two-point function can only depend on geometric quantities:

$$S_F(x,x') = [A(s) + B(s)\not{n}]\Lambda(x,x'),$$

where the Dirac equation constrains $A(s)$ and $B(s)$ through[4]:

$$iA' + \frac{3i}{2}(A+C)A - \mu B = 0,$$
$$iB' + \frac{3i}{2}(A-C)B - \mu A = \frac{1}{\sqrt{-g}}\delta(x,x').$$

[3] B. Allen, T. Jacobson, Commun. Math. Phys **103** (1986) 669
[4] W. Mück, J. Phys. A: Math. Gen. **33** (2000) 3021

Anti-de Sitter space

- In the $5D$ embedding space, adS is a hyperboloid:

$$z_0^2 - z_1^2 - z_2^2 - z_3^2 + z_4^2 = \frac{1}{\omega^2},$$

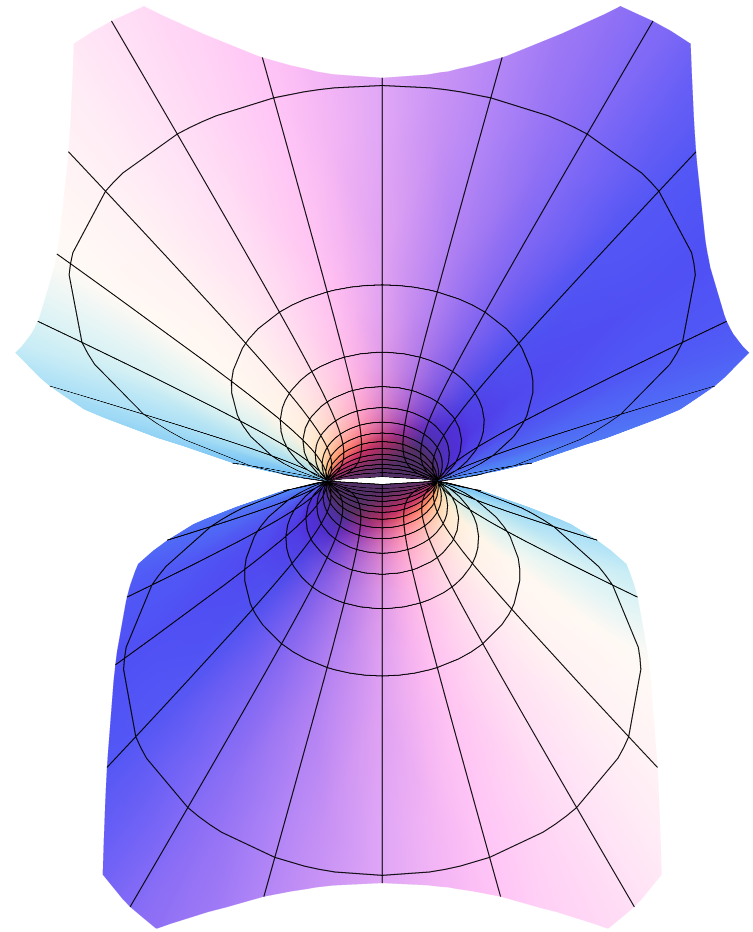
where ω is the inverse radius of curvature.

- AdS is a maximally symmetric vacuum solution of the Einstein equations with negative cosmological constant $\Lambda = -3\omega^2$.
- Coordinates for static chart:

$$z^0 = \frac{1}{\omega}\frac{\cos\omega t}{\cos\omega r}, \qquad z^5 = \frac{1}{\omega}\frac{\sin\omega t}{\cos\omega r},$$

$$z^i = \frac{\tan\omega r}{\omega r}x^i,$$

where $t \in (-\infty, \infty)$ and $0 \le \omega r < \frac{\pi}{2}$.



Geometry of adS

- Line element of adS:

$$ds^2 = \frac{1}{\cos^2 \omega r}\left[-dt^2 + dr^2 + \frac{\sin^2 \omega r}{\omega^2}\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right],$$

- Geodetic interval $s(x, x')$ is given through:

$$\cos\omega s = \frac{\cos\omega\Delta t}{\cos\omega r \cos\omega r'} - \cos\gamma \tan\omega r \tan\omega r',$$

where $\Delta t = t - t'$ and $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\Delta\varphi$.

- Bi-spinor of parallel transport has the form[5]:

$$\Lambda(x, x') = \frac{\left(\cos\frac{\omega s}{2}\right)^{-1}}{\sqrt{\cos\omega r \cos\omega r'}}\Bigg\{ \cos\frac{\omega\Delta t}{2}\left(\cos\frac{\omega r}{2}\cos\frac{\omega r'}{2} + \frac{\mathbf{x}\cdot\gamma}{r}\frac{\mathbf{x}'\cdot\gamma}{r'}\sin\frac{\omega r}{2}\sin\frac{\omega r'}{2}\right) - \gamma^0 \sin\frac{\omega\Delta t}{2}\left(\frac{\mathbf{x}\cdot\gamma}{r}\sin\frac{\omega r}{2}\cos\frac{\omega r'}{2} - \frac{\mathbf{x}'\cdot\gamma}{r'}\cos\frac{\omega r}{2}\sin\frac{\omega r'}{2}\right)\Bigg\}.$$

[5] V. E. Ambruș, E. Winstanley, AIP Conference Proceedings **1634** (2014) 40

Hadamard regularisation of vacuum expectation values

- The Feynman propagator $S_F = (A + B\not{n})\Lambda(x,x')$ diverges in the coincidence limit $s \to 0$:

$$A = -\frac{\mu}{4\pi^2 s^2} + O(\ln \omega s), \qquad B = \frac{i}{2\pi^2 s^3} + O(s^{-1}).$$

- Hadamard regularisation uses the auxiliary function $\mathcal{G}_F$:

$$S_F(x,x') = (i\gamma^\lambda D_\lambda + \mu)\mathcal{G}_F(x,x').$$

- The Hadamard theorem states that[6]:

$$\mathcal{G}_F(x,x') = \frac{1}{8\pi^2}\Bigg(\underbrace{\frac{u}{\sigma} + v \ln|\nu^2 \sigma|}_{\text{Hadamard form } 8\pi^2 \mathcal{G}_H} + w\Bigg)\Lambda(x,x'),$$

where u, v and w are finite as $s \to 0$ and ν is a renormalisation mass scale.

- u and v can be determined from the Dirac equation:

$$u = \left(\frac{\omega s}{\sin \omega s}\right)^{\frac{3}{2}}, \qquad v = \frac{\mu^2 - \omega^2}{2} \cos\tfrac{\omega s}{2}\, {}_2F_1\left(2-k, 2+k; 2; \sin^2\tfrac{\omega s}{2}\right),$$

where $k = \mu/\omega$.

[6] A.-H. Najmi, A. C. Ottewill, Phys. Rev. D **30** (1984) 2573

Renormalised vacuum stress

- The difference $S_F^{\rm reg}(x,x') \equiv S_F(x,x') - S_H(x,x')$ can be used to calculate $\langle 0|T_{\mu\nu}^{\rm ren}|0\rangle = \frac{1}{4}T_{\rm ren}g_{\mu\nu}$[7]:

$$T_{\rm ren} = -\frac{\omega^4}{4\pi^2}\left\{\frac{11}{60} + k - \frac{7k^2}{6} - k^3 + \frac{3k^4}{2} + 2k^2(k^2-1)\left[\ln\frac{\nu\sqrt{2}}{e^\gamma\omega} - \psi(k)\right]\right\},$$

  where ν is an arbitrary renormalisation mass and γ is Euler's constant.
- Conformal anomaly: $T_{\rm ren}\rfloor_{k=0} \neq 0$.
- For $\nu = \mu$, $T_{\rm ren} = 0$ when $k = 0.37$ and $k = 0.68$.

[7] V. E. Ambruș, E. Winstanley, Phys. Lett. B **749** (2015) 597

Feynman propagator for a thermal state

- The Feynman propagator at finite inverse temperature β_0 is antiperiodic with respect to the imaginary time[8]:

$$S_F^{\beta_0}(\Delta t; \boldsymbol{x}, \boldsymbol{x}') = \sum_{j=-\infty}^{\infty} (-1)^j S_F(\Delta t + ij\beta_0; \boldsymbol{x}, \boldsymbol{x}').$$

- The closed form expression of $S_F(x, x')$ can be used to deduce that the Dirac field behaves as a perfect fluid at rest:

$$\langle : T^\mu{}_\nu : \rangle_{\beta_0} = \mathrm{diag}(-\rho, p, p, p),$$

where ρ is the energy density and p is the pressure.

[8] N. D. Birrell, P. C. W. Davies, *Quantum fields in curved space*, Cambridge University Press, 1982

Thermal distribution of massless particles

- The vacuum $S_F(x,x')$ reduces to:

$$S_F(x,x')\rfloor_{\mu=0} = \frac{i\omega^3}{16\pi^2}\left(\sin\frac{\omega s}{2}\right)^{-3} \not{s}\Lambda(x,x').$$

- The field is conformal so $p = \rho/3$, and[5]:

$$\begin{aligned}\rho\rfloor_{\mu=0} =& -\frac{3\omega^4}{4\pi^2}(\cos\omega r)^4 \sum_{j=1}^{\infty}(-1)^j \frac{\cosh(j\omega\beta_0/2)}{[\sinh(j\omega\beta_0/2)]^4} \\ =&(\cos\omega r)^4\left[\frac{7\pi^2}{60\beta_0^4} - \frac{\omega^2}{24\beta_0^2} + O(\omega^2)\right].\end{aligned}$$

[5] V. E. Ambruș, E. Winstanley, AIP Conference Proceedings **1634** (2014) 40

Thermal states using mode sums

- Mode sums can also be used to construct thermal states:

$$\psi(x) = \sum_k [U_k b_k + V_k d_k^\dagger],$$

where the thermal expectation values of the one-particle operators are:

$$\langle b_k^\dagger b_{k'} \rangle_{\beta_0} = \langle d_k^\dagger d_{k'} \rangle_{\beta_0} = \frac{\delta_{kk'}}{e^{\beta_0 E} + 1}.$$

- The resulting expressions for $\langle : T^\mu{}_\nu : \rangle_{\beta_0}$ are considerably more complicated:

$$\langle : T^t{}_t : \rangle_{\beta_0} = -\frac{\omega^2}{\pi} \frac{(\cos \omega r)^4}{\sin^2 \omega r} \sum_{n_+, j} (j + \tfrac{1}{2}) \sum_{\sigma = \pm 1} \frac{E_\sigma \left[(f_\sigma^+)^2 + (f_\sigma^-)^2 \right]}{e^{\beta_0 E_\sigma} + 1},$$

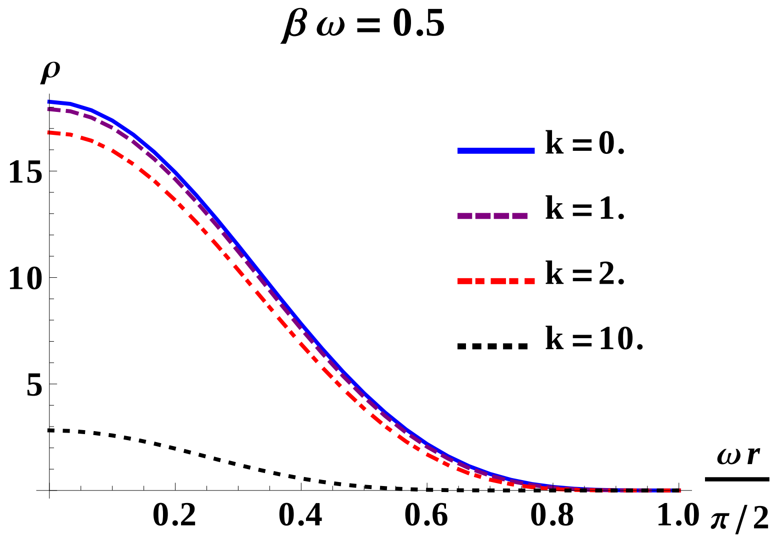
$$\langle : T^r{}_r : \rangle_{\beta_0} = \frac{\omega^3}{\pi} \frac{(\cos \omega r)^4}{\sin^2 \omega r} \sum_{n_+, j} (j + \tfrac{1}{2}) \sum_{\sigma = \pm 1} \frac{W_{\omega r}(f_\sigma^-, f_\sigma^+)}{e^{\beta_0 E_\sigma} + 1},$$

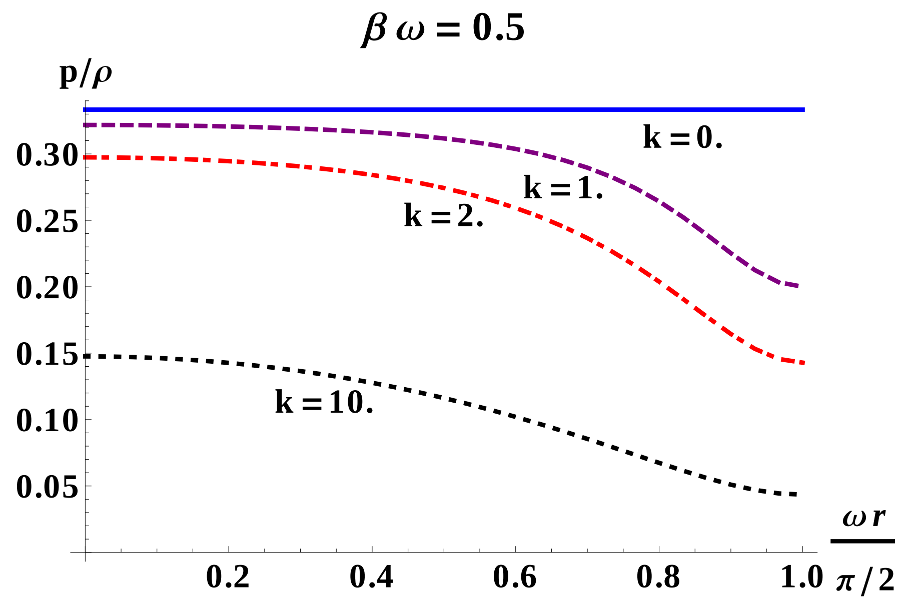
$$\langle : T^\theta{}_\theta : \rangle_{\beta_0} = \frac{\omega^3}{\pi} \frac{(\cos \omega r)^4}{\sin^3 \omega r} \sum_{n_+, j} (j + \tfrac{1}{2})^2 \sum_{\sigma = \pm 1} \sigma \frac{f_\sigma^+ f_\sigma^-}{e^{\beta_0 E_\sigma} + 1},$$

$$\langle : T^\varphi{}_\varphi : \rangle_{\beta_0} = \langle : T^\theta{}_\theta : \rangle_{\beta_0}.$$

Statistical average of $T_{\mu\nu}$

The Dirac field behaves like a perfect fluid: $\langle : T^{\mu}{}_{\nu} : \rangle_{\beta_0} = \mathrm{diag}(-\rho, p, p, p)$.





- $\langle : T^{\mu}{}_{\nu} : \rangle_{\beta_0}$ goes to 0 at the boundary ($\omega r = \frac{\pi}{2}$).
- p/ρ is decreasing with r down to a finite value on the boundary.

Relativistic Boltzmann equation

- The Boltzmann equation in general relativity can be written as[9]:

$$p^\mu \frac{\partial f}{\partial x^\mu} - \Gamma^i{}_{\mu\nu} p^\mu p^\nu \frac{\partial f}{\partial p^i} = J[f],$$

where $J[f]$ is the collision term that governs the relaxation towards thermodynamic equilibrium $f = f^{(\mathrm{eq})}$, where:

$$\begin{aligned}
\text{Maxwell-Jüttner:} \quad & f^{(\mathrm{eq})}_{\mathrm{M-J}} = Z \exp(\beta p^\mu u_\mu), \\
\text{Fermi-Dirac:} \quad & f^{(\mathrm{eq})}_{\mathrm{F-D}} = \frac{Z}{\exp(-\beta p^\mu u_\mu) + 1}, \\
\text{Bose-Einstein:} \quad & f^{(\mathrm{eq})}_{\mathrm{B-E}} = \frac{Z}{\exp(-\beta p^\mu u_\mu) - 1}.
\end{aligned}$$

- The fluid is in global thermodynamic equilibrium if βu^μ is a Killing vector field, i.e.:

$$(\beta u_\mu)_{;\nu} + (\beta u_\nu)_{;\mu} = 0.$$

[9] C. Cercignani, G. M. Kremer, *The relativistic Boltzmann equation: theory and applications*, Birkhäuser Verlag, Basel, Switzerland (2002)

Equilibrium states on adS

- Setting $u^\mu = (\cos\omega r, 0, 0, 0)^T$ and $\beta = \beta_0 / \cos\omega r$ gives:

$$\rho\rfloor_{\mu=0} = (\cos\omega r)^4 \frac{12}{\pi^2 \beta_0^4} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j^4}$$
$$= (\cos\omega r)^4 \frac{7\pi^2}{60\beta_0^4}.$$

- The QFT result shows evidence of quantum effects in the form of corrections:

$$\rho\rfloor_{\mu=0} = (\cos\omega r)^4 \frac{3\omega^4}{4\pi^2} \sum_{j=1}^{\infty} (-1)^{j+1} \frac{\cosh(j\omega\beta_0/2)}{[\sinh(j\omega\beta_0/2)]^4}$$
$$= (\cos\omega r)^4 \frac{7\pi^2}{60\beta_0^4} \left[1 - \frac{5\omega^2\beta_0^2}{14\pi^2} + O(\omega^2)\right].$$

Conclusion

- The point splitting technique in quantum field theory on curved spaces is useful when calculating the expectation values of local operators ($\overline{\psi}\psi$, J^μ, $T_{\mu\nu}$, etc.).
- On maximally symmetric spaces, the technique is extremely easy to use because the propagators are geometric and can be expressed analytically.
- AdS examples: Hadamard renormalisation and thermal states.
- Relativistic Boltzmann results can be recovered as a limit of the results obtained using the Dirac theory at finite temperature.
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