Quantum corrections in rigidly-rotating thermal states on anti-de Sitter space

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Outline

Motivation

- Why QFT analysis of rigidly rotating states?
  - Frame dragging for Kerr black holes.[1]
  - Anomalous transport in QGP formed at RHIC.[2]
  - Tractable analytically.[3]
- Why adS?
  - Relevant to QGP through adS/CFT correspondence.[4]
  - On Minkowski, a boundary is necessary to prevent superluminal rotation.[5]
  - AdS has timelike boundary → no SOLS for "mild" Ω.[6]
  - Tractable analytically due to the maximal symmetry.[7]

[1] M. Casals *et al.*, Phys. Rev. D **87** (2013) 064027.
[2] STAR Collaboration, Nature **548** (2017) 62–65.
[3] V. E. Ambruș, E. Winstanley, Phys. Lett. B **734** (2014) 296–301.
[4] O. Aharony *et al.*, Phys. Rept. **323** (2000) 183–386.
[5] V. E. Ambruș, E. Winstanley, Phys. Rev. D **93** (2016) 104014.
[6] R. Panerai, Phys. Rev. D **93** (2016) 104021.
[7] V. E. Ambruș, E. Winstanley, Class. Quant. Grav. **34** (2017) 145010.

Anti-de Sitter space

- In the $5D$ embedding space, adS satisfies

$$z_0^2 - z_1^2 - z_2^2 - z_3^2 + z_4^2 = \frac{1}{\omega^2},$$

  where $\omega \equiv$ inverse radius of curvature.

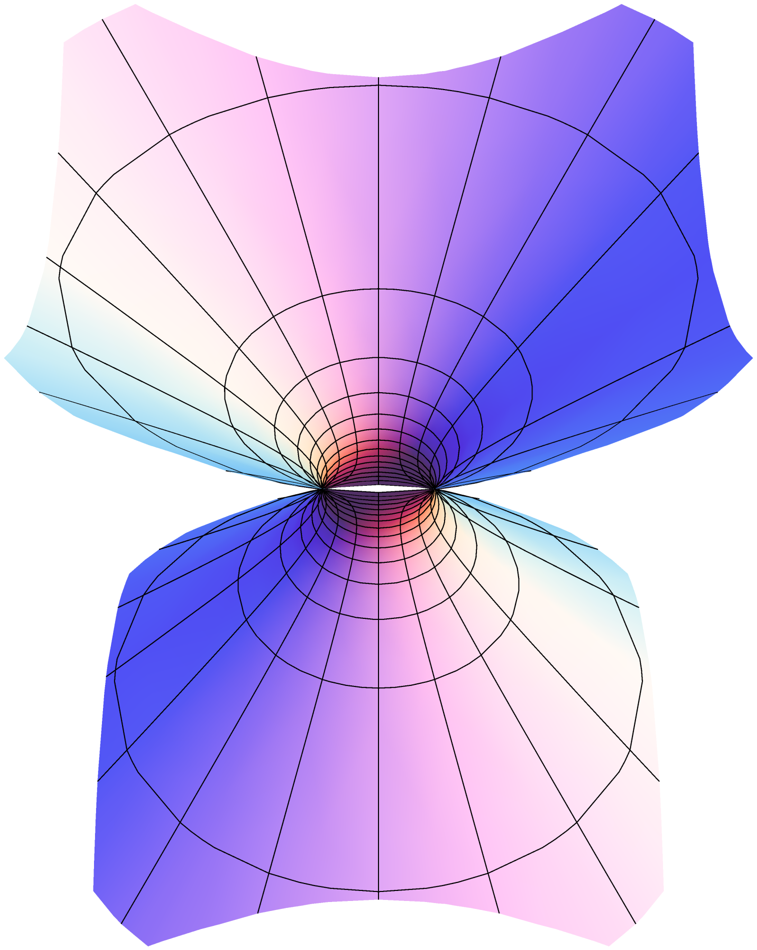
- Coordinates for static chart:

$$z^0 = \frac{1}{\omega}\frac{\cos\omega t}{\cos\omega r}, \qquad z^5 = \frac{1}{\omega}\frac{\sin\omega t}{\cos\omega r},$$
$$z^i = \frac{\tan\omega r}{\omega r} x^i,$$

  where $t \in (-\infty, \infty)$ and $0 \le \omega r < \frac{\pi}{2}$.

- Line element of adS:

$$ds^2 = \frac{1}{\cos^2\omega r}\left[-dt^2 + dr^2 + \frac{\sin^2\omega r}{\omega^2}\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right].$$



Rigidly-rotating distribution

- In global thermal equilibrium, the Fermi-Dirac distribution for fermions (q) and anti-fermions ($\overline{q}$) is:[8]

$$f_{q/\overline{q}} = \frac{g_s}{(2\pi)^3} \frac{1}{e^{\beta(u\cdot p \mp \mu)} + 1}, \qquad (\beta u_\mu)_{;\nu} + (\beta u_\nu)_{;\mu} = 0, \qquad \beta\mu = \text{const},$$

- Since the Killing for rotation is $\beta u \sim \partial_\varphi$, rigid rotation corresponds to:

$$\beta = \frac{\beta_0}{\Gamma \cos\omega r}, \qquad u = \Gamma \cos\omega r (\partial_t + \Omega \partial_\varphi), \qquad \mu = \Gamma \cos\omega r \mu_0,$$

  where β_0 and μ_0 are the inverse temperature and chemical potential at the origin, while the Lorentz factor is

$$\Gamma = \frac{1}{\sqrt{1 - v^2}}, \qquad v = \frac{\rho\Omega}{\omega r} \sin\omega r = \frac{\Omega}{\omega} \sin\theta \sin\omega r,$$

  where $\rho = r\sin\theta$ is the distance to the rotation axis.
- β^{-1} and μ blow up on the SOLS, when $\Gamma \to \infty$.
- The SOL can form only when $\Omega > \omega$, having the equation:

$$\theta = \sin^{-1}\left(\frac{\omega}{\Omega \sin\omega r}\right).$$

[8] C. Cercignani, G. Kremer, *The relativistic Boltzmann equation*, Birkhäuser Verlag (2002).

Macroscopic quantities

- The CC and SET are:

$$\begin{pmatrix} J^\mu \\ T^{\mu\nu} \end{pmatrix} = \int \frac{d^3p}{-p_t} \begin{pmatrix} (f_q - f_{\overline{q}})p^\mu \\ (f_q + f_{\overline{q}})p^\mu p^\nu \end{pmatrix}.$$

- In global equilibrium, $J^\mu = Q u^\mu$ and $T^{\mu\nu} = (E+P)u^\mu u^\nu + P g^{\mu\nu}$.
- For $M = 0$ ($E = 3P$):

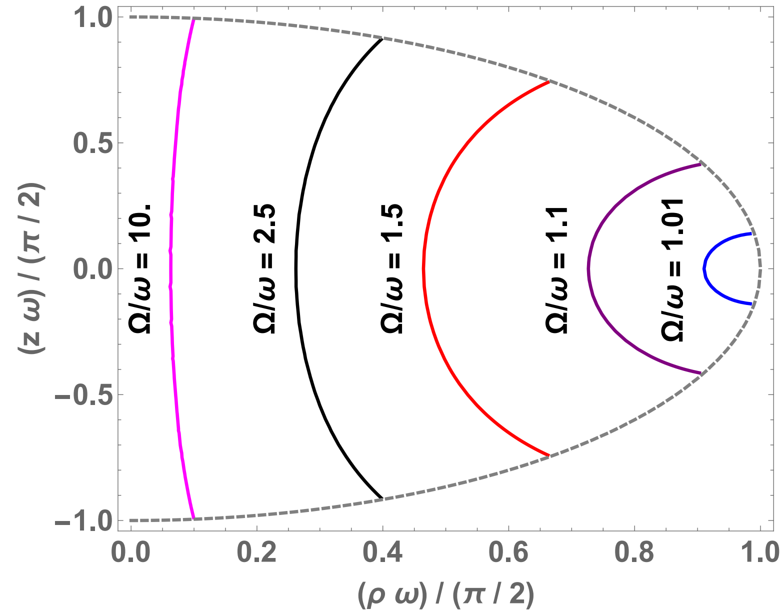
$$Q_{\rm RKT} = \frac{g_s \mu}{6}\left(\frac{1}{\beta^2} + \frac{\mu^2}{\pi^2}\right),$$
$$E_{\rm RKT} = \frac{7\pi^2 g_s}{120\beta^4} + \frac{g_s\mu^2}{4\beta^2} + \frac{g_s\mu^4}{8\pi^2}.$$

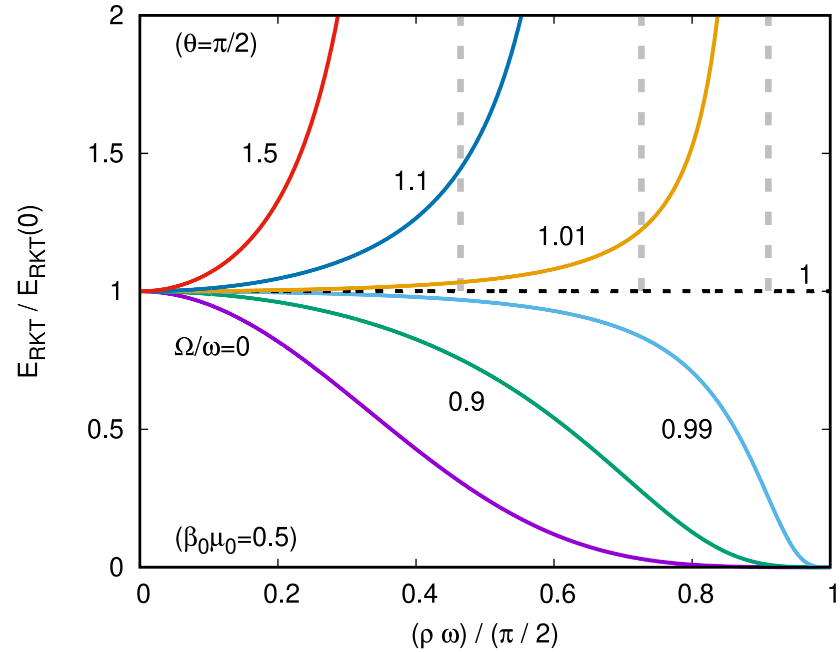
- For $\sin\theta = \pi/2$, we have:

$\Omega < \omega$: $\lim\limits_{\sin\omega r \to 1} \Gamma\cos\omega r = 0.$

$\Omega > \Omega$: $\lim\limits_{\sin\omega r \to \frac{\omega}{\Omega}} \Gamma\cos\omega r \to \infty.$

$\Omega = \omega$: $\Gamma\cos\omega r = 1$ $(\forall r)$.





The quantum approach: Point-splitting

- Consider the Feynman two-point function:

$$iS_F(x,x') = \langle \theta(t-t')\psi(x)\overline{\psi}(x') - \theta(t'-t)\overline{\psi}(x')\psi(x)\rangle \,.$$

- The expectation values of $\overline{\psi}\psi$, J^μ and $T_{\mu\nu}$ can be obtained as:[9]

$$\begin{aligned}
\langle\overline{\psi}\psi\rangle &= -\lim_{x'\to x} \mathrm{tr}[iS_F(x,x')\Lambda(x',x)],\\
\langle J^\mu\rangle &= -\lim_{x'\to x} \mathrm{tr}[\gamma^\mu iS_F(x,x')\Lambda(x',x)],\\
\langle T_{\hat{\alpha}\hat{\sigma}}\rangle &= -\frac{i}{2}\lim_{x'\to x} \mathrm{tr}\left\{\left[\gamma_{(\mu}D_{\nu)} iS_F - g_\mu{}^{\mu'} g_\nu{}^{\nu'} iS_F \overleftarrow{D}_{(\mu'}\gamma_{\nu')}\right]\Lambda(x',x)\right\},
\end{aligned}$$

where $\{\gamma^\mu,\gamma^\nu\} = -2g^{\mu\nu}$ and $D_\mu = \partial_\mu + \Gamma_\mu$.

- The spin connection $\Gamma_\mu = \omega^{\hat{\alpha}}_\mu \Gamma_{\hat{\alpha}}$ is defined with respect to the tetrad $\omega^{\hat{\alpha}}$ and $e_{\hat{\alpha}}$:

$$\Gamma_{\hat{\alpha}} = -\frac{i}{2}\Gamma_{\hat{\beta}\hat{\gamma}\hat{\alpha}} S^{\hat{\beta}\hat{\gamma}}, \quad \Gamma_{\hat{\beta}\hat{\gamma}\hat{\alpha}} = \frac{1}{2}(c_{\hat{\beta}\hat{\gamma}\hat{\alpha}} + c_{\hat{\beta}\hat{\gamma}\hat{\alpha}} - c_{\hat{\beta}\hat{\gamma}\hat{\alpha}}), \quad c_{\hat{\beta}\hat{\gamma}}{}^{\hat{\alpha}} = \omega^{\hat{\alpha}}_\mu [e_{\hat{\beta}}, e_{\hat{\gamma}}]^\mu.$$

[9] P. B. Groves, P. R. Anderson, E. D. Carlson, Phys. Rev. D **66** (2002) 124017.

The Feynman two-point function for thermal states

- The t.e.v. of an operator $\hat{A}$ is $\langle \hat{A} \rangle_{\beta_0} = Z^{-1} \text{tr}(\hat{\rho}\hat{A})$, where

$$\hat{\rho} = e^{-\beta_0(\hat{H} - \Omega \hat{M}^{\hat{z}} - \mu_0 \hat{Q})}.$$

- Noting that:

$$\hat{\rho}\hat{\Psi}(t,\varphi)\hat{\rho}^{-1} = e^{-\beta_0\mu_0} e^{\beta_0 \Omega S^{\hat{z}}} \hat{\Psi}(t + i\beta_0, \varphi + i\beta_0\Omega),$$

where $e^{\beta_0 \Omega S^{\hat{z}}} = \cosh \frac{\beta_0 \Omega}{2} - 2S^{\hat{z}} \sinh \frac{\beta_0 \Omega}{2}$, the following expression can be obtained:[10]

$$S^F_\beta(t,\varphi;t',\varphi') = \sum_{j=-\infty}^{\infty} (-1)^j e^{-j\beta_0\mu_0} \left(\cosh \frac{j\beta_0\Omega}{2} - 2S^{\hat{z}} \sinh \frac{j\beta_0\Omega}{2}\right) \times S^F(t + ij\beta_0, \varphi + ij\beta_0\Omega; t', \varphi').$$

- $j = 0$ is the (regularised) vacuum contribution.[11]

[10] N. D. Birrell, P. C. W. Davies, *Quantum fields in curved space* (CUP, 1982).
[11] V. E. Ambruș, E. Winstanley, Phys. Lett. B **749** (2015) 597–602.

The vacuum Hadamard two-point function

- For the maximally symmetric vacuum state, S_F can be written as:

$$iS^F_{\rm vac}(x,x') = [\mathcal{A}(s) + \mathcal{B}(s)\not{n}]\Lambda(x,x').$$

- The geodesic interval s can be given through:

$$\cos\omega s = \frac{\cos\omega\Delta t}{\cos\omega r \cos\omega r'} - \cos\gamma \tan\omega r \tan\omega r',$$

  where $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\Delta\varphi$.
- $n_\mu = \nabla_\mu s(x,x')$ is the normalised tangent to the geodesic at x.
- $\mathcal{A}$ and $\mathcal{B}$ depend only on s and satisfy:

$$i\frac{d}{ds}\begin{pmatrix}\mathcal{A}\\ \mathcal{B}\end{pmatrix} + \frac{3i\omega}{2}\begin{pmatrix}-\mathcal{A}\tan(\omega s/2)\\ \mathcal{B}\cot(\omega s/2)\end{pmatrix} - M\begin{pmatrix}\mathcal{B}\\ \mathcal{A}\end{pmatrix} = \begin{pmatrix}0\\ i(-g)^{-1/2}\delta(x,x')\end{pmatrix}$$

- The equations can be solved exactly. When $M=0$, we have:

$$\mathcal{A}\rfloor_{M=0} = \frac{\omega^3}{16\pi^2}\left(\cos\frac{\omega s}{2}\right)^{-3}, \qquad \mathcal{B}\rfloor_{M=0} = \frac{i\omega^3}{16\pi^2}\left(\sin\frac{\omega s}{2}\right)^{-3}.$$

- The bi-spinor of parallel transport satisfies:

$$D_\mu \Lambda(x,x') = -i\omega S_{\mu\nu} n^\nu \Lambda(x,x') \tan\left(\frac{\omega s}{2}\right).$$

- Employing the Cartesian gauge for the tetrad:[12]

$$e_{\hat{t}} = \cos\omega r\,\partial_t, \qquad e_{\hat{\imath}} = \cos\omega r\left[\frac{\omega r}{\sin\omega r}\left(\delta_{ij} - \frac{x^i x^j}{r^2}\right) + \frac{x^i x^j}{r^2}\right]\partial_j,$$
$$\omega^{\hat{t}} = \frac{dt}{\cos\omega r}, \qquad \omega^{\hat{\imath}} = \frac{1}{\cos\omega r}\left[\frac{\sin\omega r}{\omega r}\left(\delta_{ij} - \frac{x^i x^j}{r^2}\right) + \frac{x^i x^j}{r^2}\right] dx^j.$$

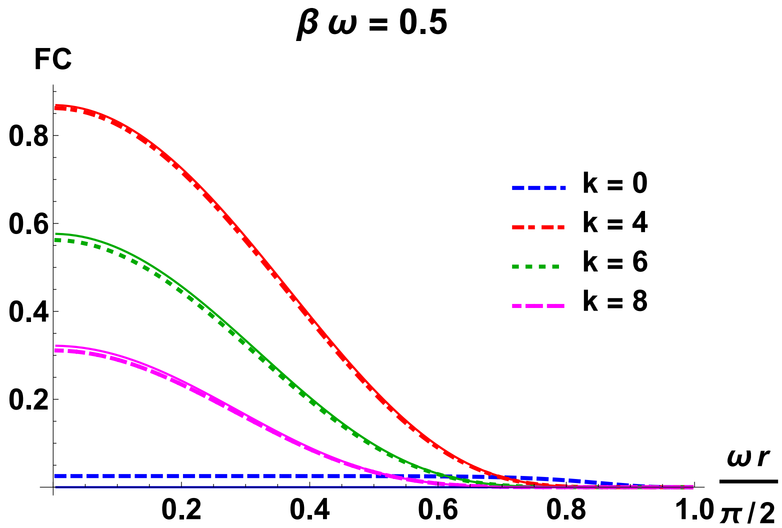
allows $\Lambda(x,x')$ to be expressed as:[13]

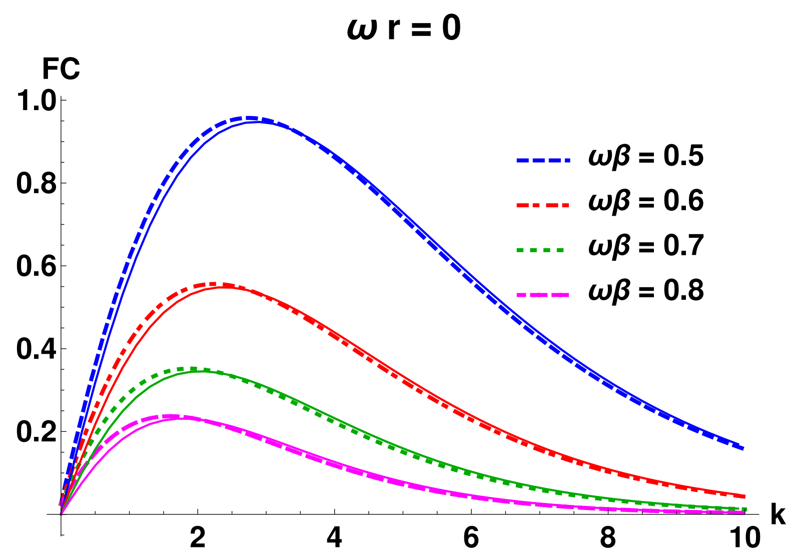
$$\Lambda(x,x') = \frac{\sec(\omega s/2)}{\sqrt{\cos\omega r \cos\omega r'}}\Bigg[\cos\frac{\omega\Delta t}{2}\left(\cos\frac{\omega r}{2}\cos\frac{\omega r'}{2} + \sin\frac{\omega r}{2}\sin\frac{\omega r'}{2}\frac{\mathbf{x}\cdot\boldsymbol{\gamma}}{r}\frac{\mathbf{x}'\cdot\boldsymbol{\gamma}}{r'}\right) + \sin\frac{\omega\Delta t}{2}\left(\sin\frac{\omega r}{2}\cos\frac{\omega r'}{2}\frac{\mathbf{x}\cdot\boldsymbol{\gamma}}{r}\gamma^{\hat{t}} + \sin\frac{\omega r'}{2}\cos\frac{\omega r}{2}\frac{\mathbf{x}'\cdot\boldsymbol{\gamma}}{r'}\gamma^{\hat{t}}\right)\Bigg].$$

[12] I. I. Cotăescu, Rom. J. Phys. **52** (2007) 895.

[13] V. E. Ambruș, E. Winstanley, Class. Quant. Grav. **34** (2017) 145010.

Fermion condensate ($\Omega = 0$)





- When $\Omega = 0$, the FC can be computed using

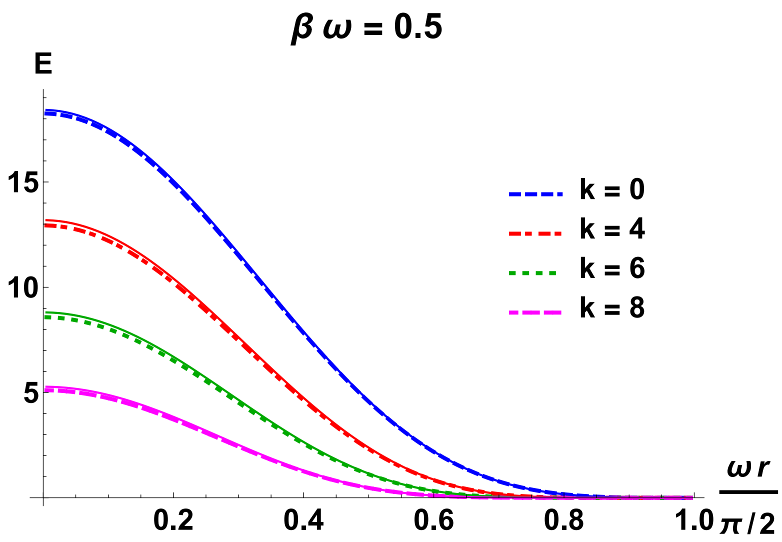
$$\langle : \hat{\overline{\Psi}} \Psi : \rangle_{\beta_0} = -\frac{2\omega^3 \Gamma(2+k)(\cos \omega r)^{4+2k}}{\pi^{3/2} 4^{1+k} \Gamma(\frac{1}{2} + k)} \sum_{j=1}^{\infty} (-1)^j \frac{\cosh(\omega j \beta_0/2)}{\sinh(\omega j \beta_0/2)^{4+2k}} \times {}_2F_1\left(1+k; 2+k; 1+2k; -\frac{\cos^2 \omega r}{\sinh^2 \frac{\omega j \beta_0}{2}}\right).$$

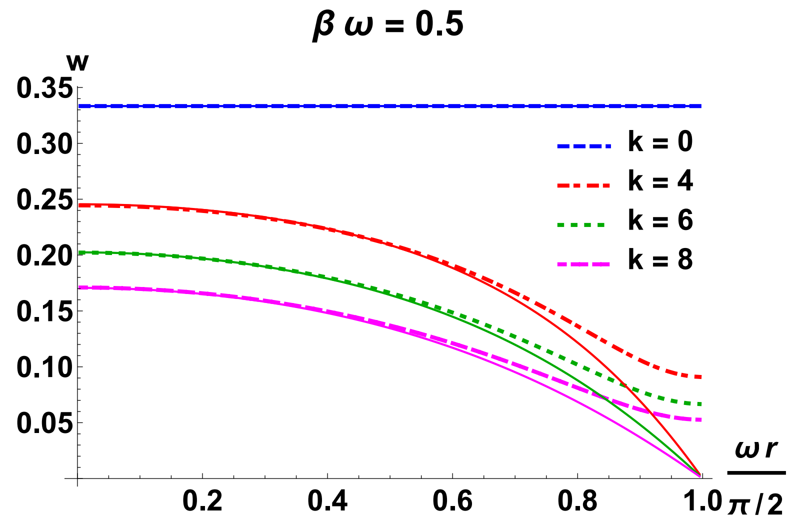
- At vanishing mass, $M = \omega k = 0$, we have:

$$\langle : \hat{\overline{\Psi}} \Psi : \rangle_{\beta_0} = -\frac{\omega^3}{2\pi^2} (\cos \omega r)^4 \sum_{j=1}^{\infty} (-1)^j \frac{\cosh(\omega j \beta_0/2)}{[\sinh(\omega j \beta_0/2)^2 + \cos^2 \omega r]^2}.$$

- $\hat{\overline{\Psi}} \hat{\Psi} = \hat{T}^{\mu}{}_{\mu}/M$ vanishes in RKT when $M \to 0$.

Stress-energy tensor ($\Omega = 0$)





- When $\Omega = 0$, $\langle : T_{\hat{\alpha}\hat{\sigma}} : \rangle_{\beta_0} = \text{diag}(E, P, P, P)$, with

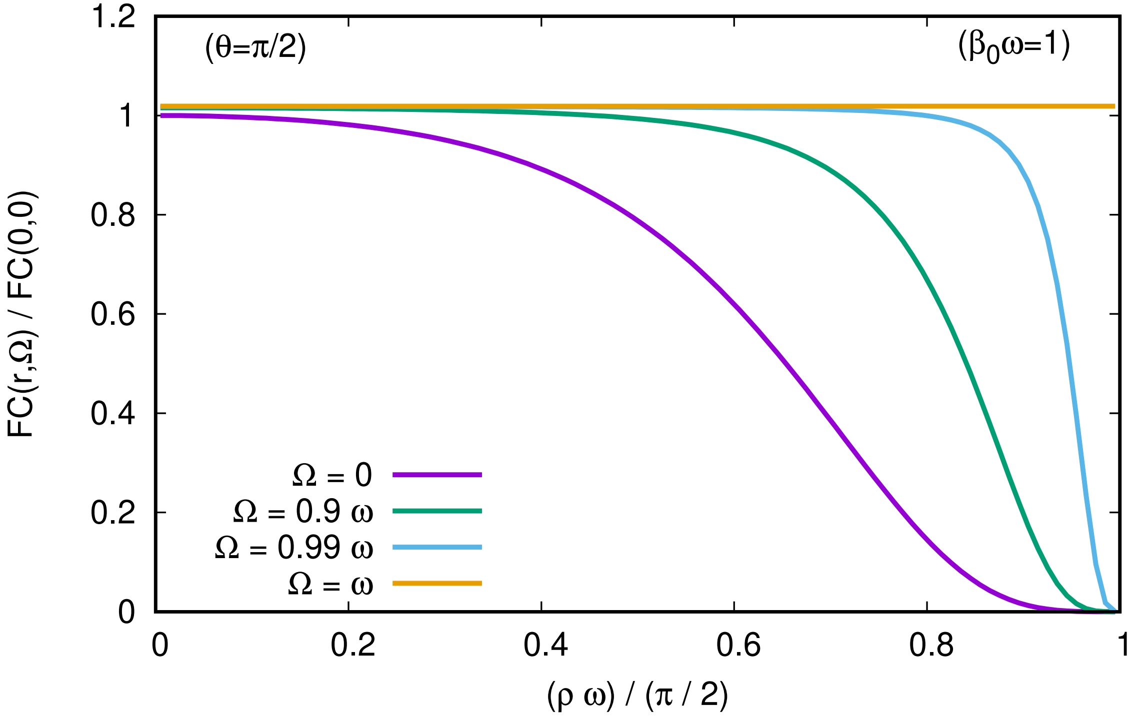
$$\begin{pmatrix} E + P \\ P \end{pmatrix} = -\frac{\omega^4 \Gamma(2+k)(\cos \omega r)^{4+2k}}{\pi^{3/2} 4^{1+k} \Gamma(\frac{1}{2} + k)} \sum_{j=1}^{\infty} (-1)^j \frac{\cosh \frac{\omega j \beta_0}{2}}{\sinh(\omega j \beta_0 / 2)^{4+2k}} \times \begin{pmatrix} 2(2+k)\, {}_2F_1\left(k; 3+k; 1+2k; -\cos^2 \omega r / \sinh^2 \frac{\omega j \beta_0}{2}\right) \\ {}_2F_1\left(k; 2+k; 1+2k; -\cos^2 \omega r / \sinh^2 \frac{\omega j \beta_0}{2}\right) \end{pmatrix}.$$

- At vanishing mass, $w = P/E$ takes a finite value on the boundary:

$$\lim_{\omega r \to \frac{\pi}{2}} w = \frac{1}{3 + 2k}, \qquad (1)$$

while $w_{\rm RKT} = P_{\rm RKT}/E_{\rm RKT} = 0$ for all $k > 0$ ($w_{\rm RKT} = 1/3$ when $k = 0$).

Fermion condensate ($\Omega > 0$)



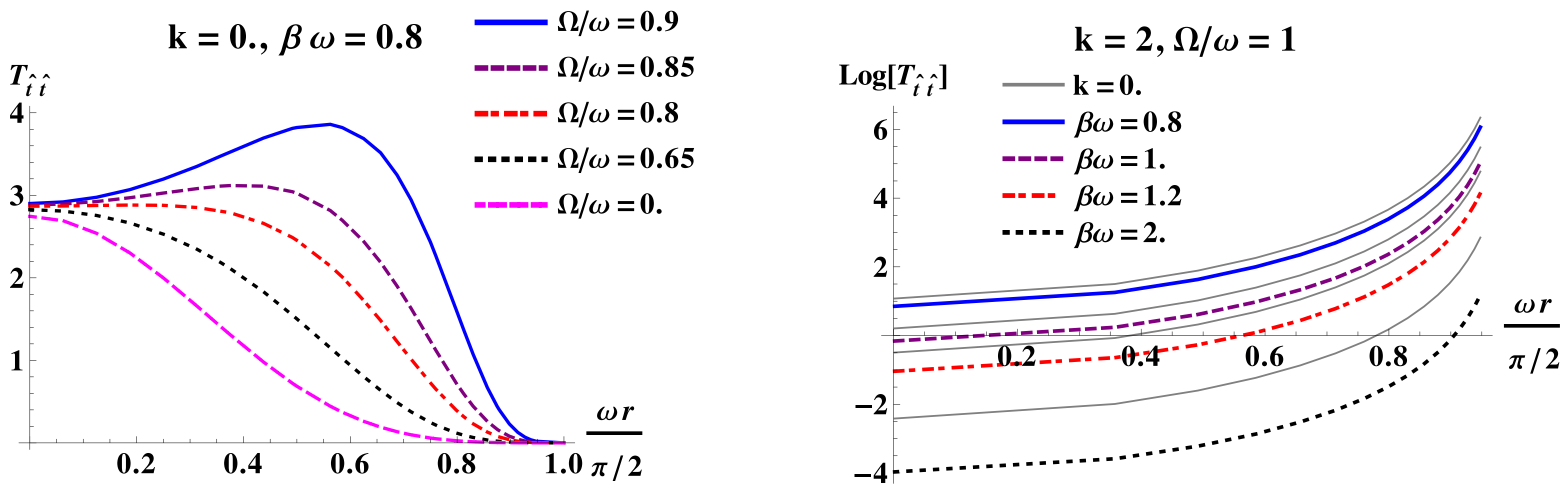
- The QFT expression at $M = 0$ can be obtained as:

$$\langle : \hat{\bar{\Psi}} \hat{\Psi} : \rangle_{\beta_0} = -\frac{\omega^3}{2\pi^2} \cos^4 \omega r \sum_{j=1}^{\infty} \frac{(-1)^j \cosh(j\beta_0\mu_0) \cosh \frac{\omega j \beta_0}{2} \cosh \frac{\Omega j \beta_0}{2}}{(\cos^2 \omega r + \sinh^2 \frac{\omega j \beta_0}{2} - \sin^2 \omega r \sin^2 \theta \sinh^2 \frac{\Omega j \beta_0}{2})^2}.$$

- When $\theta = \pi/2$ and $\Omega = \omega$, $\langle : \hat{\bar{\Psi}} \hat{\Psi} : \rangle_{\beta_0}$ is independent of r:

$$\langle : \hat{\bar{\Psi}} \hat{\Psi} : \rangle_{\beta_0} = -\frac{\omega^3}{2\pi^2} \sum_{j=1}^{\infty} (-1)^j \frac{\cosh(j\beta_0\mu_0)}{\cosh^2(\omega j \beta_0/2)}.$$

Stress-energy tensor ($\Omega > 0$)



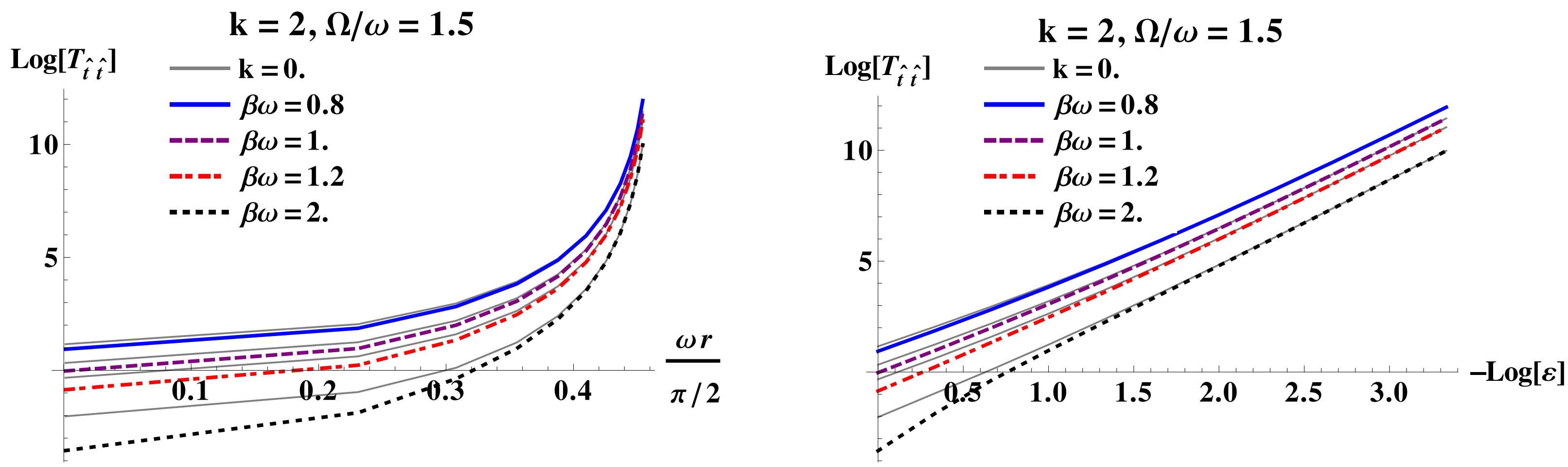
$$\langle : T_{\hat{t}\hat{t}} : \rangle_{\beta_0} = -\frac{\omega^4}{4\pi^2} \cos^4 \omega r \sum_{j=1}^{\infty} \frac{(-1)^j \cosh(j\beta_0\mu_0) \cosh \frac{\omega j \beta_0}{2} \cosh \frac{\Omega j \beta_0}{2}}{[\sinh^2(\omega j \beta_0/2) - \sin^2 \omega r \sin^2 \theta \sinh^2(\Omega j \beta_0/2)]^2} \times \left[\frac{4\sinh^2(\omega j\beta_0/2)}{\sinh^2(\omega j \beta_0/2) - \sin^2 \omega r \sin^2 \theta \sinh^2(\Omega j \beta_0/2)} - 1\right].$$

- When $\omega = \Omega$, the r dependence is through $\Gamma = (1 - \sin^2 \omega r \sin^2 \theta)^{-1/2}$:

$$\langle : T_{\hat{t}\hat{t}} : \rangle_{\beta_0} = -\frac{\omega^4}{4\pi^2} \cos^4 \omega r \Gamma^4 (4\Gamma^2 - 1) \sum_{j=1}^{\infty} \frac{(-1)^j \cosh(j\beta_0\mu_0) \cosh^2 \frac{\omega j \beta_0}{2}}{\sinh^4 \frac{\omega j \beta_0}{2}}$$

$$\simeq T^{\rm RKT}_{\hat{t}\hat{t}} + \frac{\omega^2}{36} (\Gamma \cos \omega r)^4 (4\Gamma^2 - 1) \left(\frac{1}{\beta_0^2} + \frac{3\mu_0^2}{\pi^2}\right) + O(\omega^4).$$

Stress-energy tensor ($\Omega > \omega$)



- When $\Omega > \omega$, the rotating vacuum is no longer maximum symmetric.
- The bi-spinor of p.t. approach is no longer applicable.
- Instead, one has to rely on mode sums:

$$\hat{\Psi} = \sum_j (U_j b_j + V_j d_j^\dagger), \quad \hat{\rho} \hat{b}_j^\dagger \hat{\rho}^{-1} = e^{-\beta_0(\widetilde{E}_j - \mu_0)} b_j^\dagger, \quad \langle b_j^\dagger b_{j'} \rangle_{\beta_0} = \frac{\delta(j, j')}{e^{\beta_0(\widetilde{E}_j - \mu_0)} + 1},$$

  where $\widetilde{E}_j = E_j - \Omega m_j$.
- Since $\Omega > \omega$, the SOL appears and the SET diverges.

Conclusion

- Using an analytic expression for $S_F(x, x')$ written in terms of $\Lambda(x, x')$, the t.e.v.s of the quantum Dirac field were investigated.
- For $\Omega = 0$:
  - The quantum SET describes a perfect fluid.
  - The FC and E decrease like $\cos^4 \omega r$.
  - $w = P/E = (3 + 2k)^{-1}$ on the boundary, while $w_{\rm RKT} \to 0$ when $k = M\omega > 0$.
  - When $k = M/\omega = 0$, the FC is finite while $T^\mu{}_\mu/M = 0$ in RKT.
- For $\Omega = \omega$:
  - RKT predicts that Q, E and P are constant in the equatorial plane.
  - The conclusion is supported by preliminary QFT results.
- This work is supported by a Grant from the Romanian National Authority for Scientific Research and Innovation, CNCS-UEFISCDI, project number PN-III-P1-1.1-PD-2016-1423.