Quantum thermal states on anti-de Sitter space

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Abstract

Starting from the closed form analytic expression for the vacuum Feynman Green's function on the anti-de Sitter (adS) spacetime, thermal expectation values are constructed using the (anti-)periodicity properties of the finite temperature Feynman Green's function for scalars (fermions). The results are compared with classical expectation values computed using relativistic kinetic theory, highlighting the regimes where quantum corrections are important. Focussing on the static (non-rotating) case, we find that the structure of the stress-energy tensor (SET) for the K-G field is no longer that of a perfect fluid, with anisotropies developing between the radial and angular pressure components. For simplicity, the analysis is restricted to the case of massless particles. [V. E. Ambruș, C. Kent and E. Winstanley, Int. J. Mod. Phys. D **27** (2018) 1843014]

CadS: coordinates and metric

- Denoting by ω the inverse radius of curvature of adS ($R = -12\omega^2$), the line element can be parametrised as follows:

$$ds^2 = \frac{1}{\cos^2 \omega r}\left[-dt^2 + dr^2 + \left(\frac{\sin \omega r}{\omega}\right)^2 d\Omega^2\right],$$

where the range of $t \in (-\infty, \infty)$ is infinite in the covering space of adS (CadS).

Classical approach: Relativistic kinetic theory

- In global thermal equilibrium and at vanishing chemical potential, massless uncharged boson and charged fermion particles are distributed according to B-E and F-D statistics:

$$f^{(\rm eq)}_{\rm B-E} = \frac{1/(2\pi)^3}{e^{-\widetilde{\beta}(p\cdot u)} - 1}, \qquad f^{(\rm eq)}_{\rm F-D} = \frac{4/(2\pi)^3}{e^{-\widetilde{\beta}(p\cdot u)} + 1},$$

where the inverse of the local temperature ($\widetilde{\beta}$) and the four-velocity of the fluid (u) are given by:

$$\widetilde{\beta} = \frac{\beta}{\cos \omega r}, \qquad u = \cos \omega r\, \partial_t,$$

where β is the inverse temperature when $r = 0$.

- The stress-energy tensor (SET) takes the form:

$$T^\mu{}_\nu = {\rm diag}(-E, P, P, P),$$

where $P = \frac{1}{3}E$ and

$$E_{\rm B-E} = \frac{\pi^2}{30\beta^4}(\cos \omega r)^4, \quad E_{\rm F-D} = \frac{7\pi^2}{60\beta^4}(\cos\omega r)^4.$$

Feynman Green's functions for adS vacuum

- Due to the maximal symmetry of adS, the Feynman Green's functions $G_F(x, x')$ and $S_F(x, x')$ can be written as:[a]

$$G_F \equiv G_F(s), \quad S_F \equiv [\mathcal{A}_F(s) + \mathcal{B}_F(s)\not{n}]\Lambda(x, x'),$$

where s is the geodesic interval between x and x', $n_\mu = \nabla_\mu s$ is the geodesic tangent at x and Λ is the bi-spinor of parallel transport:[b]

$$\Lambda(x,x') = \frac{\sec(\omega s/2)}{\sqrt{\cos\omega r \cos\omega r'}}\Big[\cos\tfrac{\omega\Delta t}{2}\left(\cos\tfrac{\omega r}{2}\cos\tfrac{\omega r'}{2} + \sin\tfrac{\omega r}{2}\sin\tfrac{\omega r'}{2}\tfrac{x\cdot\gamma}{r}\tfrac{x'\cdot\gamma}{r'}\right) + \sin\tfrac{\omega\Delta t}{2}\left(\sin\tfrac{\omega r}{2}\cos\tfrac{\omega r'}{2}\tfrac{x\cdot\gamma}{r}\gamma^{\hat t} + \sin\tfrac{\omega r'}{2}\cos\tfrac{\omega r}{2}\tfrac{x'\cdot\gamma}{r'}\gamma^{\hat t}\right)\Big].$$

[valid for the tetrad in the Cartesian gauge and the Dirac rep. of the γ matrices]

- Solving $(\Box - \xi R)G_F = \delta^4(x - x')/\sqrt{-g}$ yields:

$$G_F^{\xi=0} = \frac{\omega^2}{4\pi^2}\left[\frac{\cos\omega s}{\sin^2\omega s} + \frac{1}{2}\ln\left(-\cot^2\frac{\omega s}{2}\right)\right],$$

$$G_F^{\xi=\frac{1}{6}} = -\frac{\omega^2}{4\pi^2\sin^2\omega s}.$$

- Solving $i\not{D}S_F = \delta^4(x - x')/\sqrt{-g}$ yields:

$$iS_F(x,x') = \frac{\omega^3[\tan^3\left(\frac{\omega s}{2}\right) + i\not{n}]}{16\pi^2\sin^3(\omega s/2)}\Lambda(x,x').$$

[a] W. Mück, J. Phys. A: Math. Gen. **33** (2000) 3021–3026.
[b] V. E. Ambruș, E. Winstanley, Class. Quantum Grav. **34** (2017) 145010.

Green's functions for static thermal states

- The thermal Feynman Green's functions can be obtained from G_F and S_F due to their periodicity/anti-periodicity w.r.t. imaginary time:[a]

$$G_F^\beta(x,x') = \sum_{j=-\infty}^{\infty} G_F(t + ij\beta, x; t', x'),$$

$$S_F^\beta(x,x') = \sum_{j=-\infty}^{\infty} (-1)^j S_F(t + ij\beta, x; t', x').$$

[a] N. D. Birrell, P. C. W. Davies, *Quantum fields in curved space* (CUP, 1982).

Results for the scalar field

- In the case of the scalar field, the SET is given by:

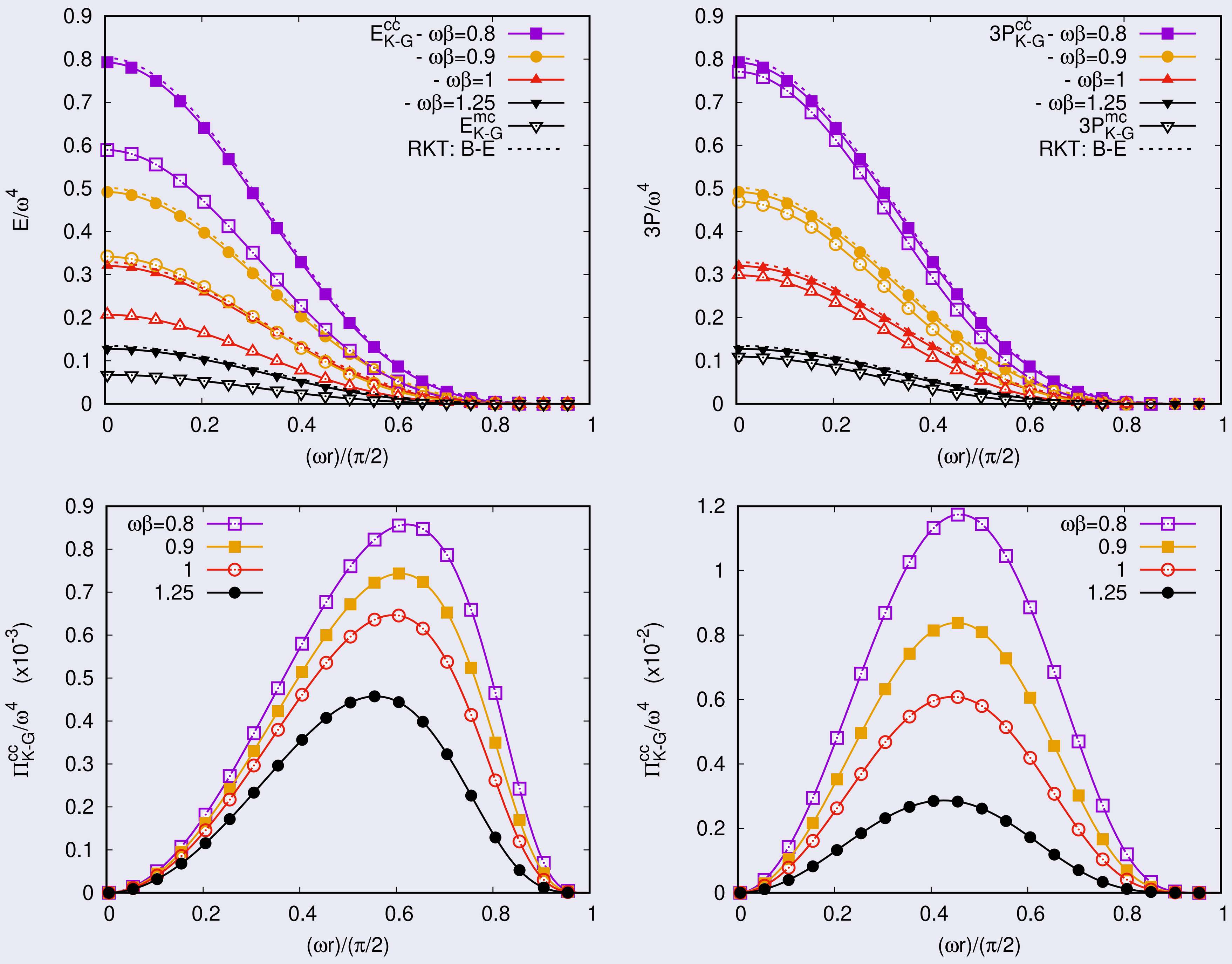
$$\langle : T^\mu{}_\nu : \rangle_\beta = {\rm diag}(-E, P + \Pi, P - \tfrac{1}{2}\Pi, P - \tfrac{1}{2}\Pi),$$

where the pressure deviator Π is a purely quantum effect.

- As a typical example, $E^{\rm mc}_{\rm K-G}$ corresponding to the minimally coupled (mc) K-G field can be computed using:

$$E^{\rm mc}_{\rm K-G} = \frac{3\omega^4}{8\pi^2}(\cos\omega r)^6 \sum_{j=1}^{\infty} \frac{\sinh^{-4}(j\beta\omega/2)}{[\cos(2\omega r) + \cosh(j\beta\omega)]}.$$

- It is worth noting that, while $E = 3P$ in the conformally coupled (cc) case, this relation does not hold in the mc case.

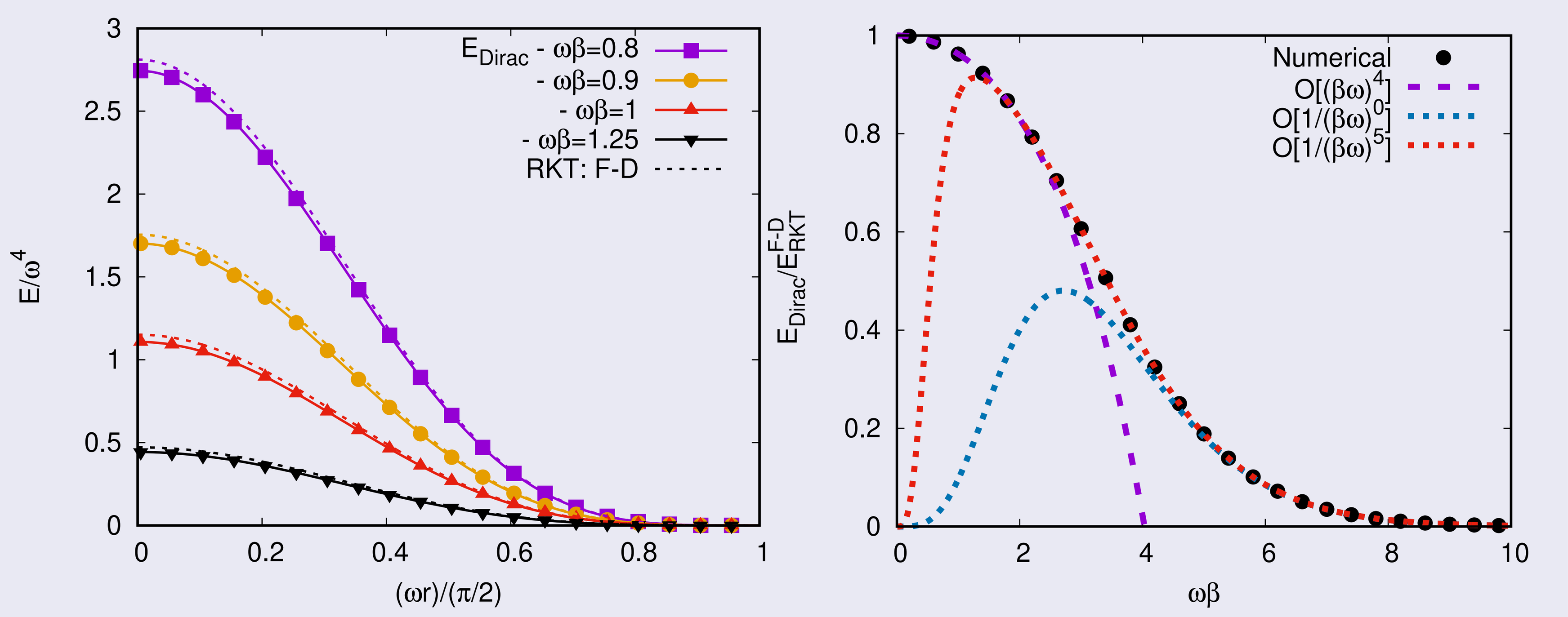


The QFT results for E and P (solid lines) are generally below the RKT results, more markedly for the mc case. In the RKT case, $\Pi = 0$, while $\Pi^{\rm mc}_{\rm K-G}$ seems to exceed the values of $\Pi^{\rm cc}_{\rm K-G}$.

Results for the Dirac field

- In the case of the massless Dirac field, $\Pi = 0$ and $E = 3P$, where

$$E_{\rm D} = \frac{3\omega^4}{4\pi^2}\sum_{j=1}^{\infty}(-1)^{j-1}\frac{\cosh(j\beta\omega/2)}{\sinh^4(j\beta\omega/2)} \simeq E_{\rm F-D}\left(1 - \frac{5\beta^2\omega^2}{14\pi^2} - \frac{17\beta^4\omega^4}{112\pi^4} + \dots\right).$$



Like in the K-G CC case, the QFT result is quite close to the RKT result. On the right, it can be seen that $E_{\rm Dirac}$ decreases w.r.t. the RKT result as $\omega\beta$ is increased.

Conclusion

- On adS, analytic closed form expressions can be obtained for the vacuum two-point functions for the Klein-Gordon and Dirac fields.
- The point-splitting formalism allows these expressions to be employed when constructing thermal states.
- The K-G field exhibits non-vanishing pressure deviator Π, even for conformal coupling $\xi = 1/6$.
- Quantum corrections are stronger for mc compared to cc.
- In the case of the Dirac field, the SET is in perfect fluid form.
- Quantum corrections are stronger at larger $\omega\beta$.
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