Rigidly-rotating quantum thermal states in bounded systems

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Outline

Motivation

- In QFT, the boundary conditions are imposed at the level of the field operator / quantum modes.
- The boundary conditions must preserve the mathematical consistency of the theory (self-adjointness of the Hamiltonian, time-invariance of the inner product, charge conjugation symmetry).
- For the Klein-Gordon field: Dirichlet, von Neumann, Robin boundary conditions.[1]
- For the Dirac field: spectral,[2] Berry,[3] MIT bag boundary conditions.[4]
- The connection between the boundary conditions and the emergent expectation values is far from trivial.
- This contribution discusses a procedure to prescribe b.c.s in order to achieve given macroscopic variables in the setup of rigidly-rotating thermal states.
- The discussion is limited to the Klein-Gordon field, but can be extended to the Dirac case.[5]

[1] A. Romeo, A. A. Saharian, Phys. Rev. D **63** (2001) 105019.
[2] M. Hortaçsu, K. Rothe, B. Schroer, Nucl. Phys. **B171** (1980) 530.
[3] M. V. Berry, R. J. Mondragon, Proc. R. Soc. Lond. A **412** (1987) 53–74.
[4] A. Chodos, R. Jaffe, K. Johnson, C. Thorn, V. Weisskopf, PRD **9** (1974) 3471.
[5] V. Ambruș, E. Winstanley, Phys. Rev. D **93** (2016) 104014.

Landau frame

- The Landau (energy) frame[6] energy density E and macroscopic velocity u^μ can be obtained by solving the eigenvalue equation:

$$T^\mu{}_\nu u^\nu = -E\, u^\mu.$$

- In the Landau frame, $T^{\mu\nu}$ can be decomposed as follows:

$$T^{\mu\nu} = E u^\mu u^\nu + (P + \varpi)\Delta^{\mu\nu} + \Pi^{\mu\nu},$$

where $\Delta^{\mu\nu} = u^\mu u^\nu + g^{\mu\nu}$ and

$$E = u_\mu T^{\mu\nu} u_\nu, \qquad P + \varpi = \frac{1}{3}\Delta_{\mu\nu} T^{\mu\nu},$$

$$\Pi^{\mu\nu} = T^{<\mu\nu>} = \left[\frac{1}{2}\left(\Delta^\mu{}_\alpha \Delta^\nu{}_\beta + \Delta^\mu{}_\beta \Delta^\nu{}_\alpha\right) - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}\right] T^{\alpha\beta}.$$

- In thermal equilibrium, $T^{\mu\nu} = (E + P)u^\mu u^\nu + P\eta^{\mu\nu}$.
- The dynamic pressure (ϖ) and pressure deviator ($\Pi^{\mu\nu}$) represent non-equilibrium contributions to the SET $T^{\mu\nu}$.

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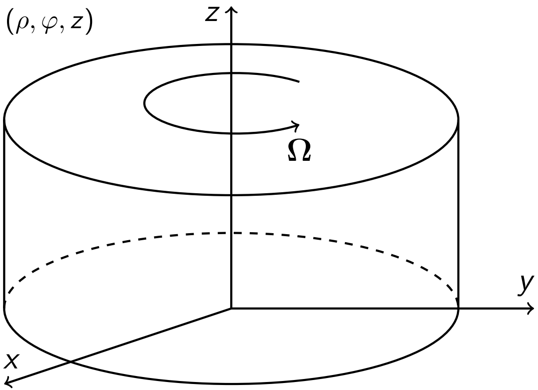
[6] L. D. Landau, E. M. Lifshitz, *Fluid mechanics*, 2nd ed. (1987).

Rigid rotation: Kinetic theory results



- $u^\mu = \gamma(\partial_t + \Omega\partial_\varphi) \Rightarrow v = \rho\Omega$ and $\gamma = (1 - \rho^2\Omega^2)^{-1/2}$.
- The local inverse temperature is $\widetilde{\beta} = \beta\gamma^{-1}$.
- The energy density for massless B-E particles is $E_{\rm B-E} = \dfrac{\pi^2\gamma^4}{30\beta^4}$.[7]

[7] V. E. Ambruș, I. Cotăescu, Phys. Rev. D **94** (2016) 085022.

Klein-Gordon field: mode solutions

- The K-G field operator can be written as:

$$\phi = \sum_j (a_j f_j + a_j^\dagger f_j^*), \qquad f_j = \frac{e^{-i\omega_j t + ik_j z + im_j\varphi}}{\sqrt{8\pi^2\omega_j}} J_{m_j}(q_j\rho),$$

where $q_j = \sqrt{\omega_j^2 - k_j^2 - \mu^2}$ and $[a_j, a_{j'}^\dagger] = \delta(j, j')$.

- f_j are normalised using the inner product:

$$\langle f, h\rangle = i\int d^3x\,\sqrt{-g}(f^*\partial_t h - h\partial_t f^*),$$

such that $\langle f_j, f_{j'}\rangle = \delta(j, j')$ and $\langle f_j, f_{j'}^*\rangle = 0$.

- The stress-energy tensor (SET) operator is:[8]

$$T_{\mu\nu} = \frac{1}{3}\{\nabla_\mu\phi, \nabla_\nu\phi\} - \frac{1}{6}\{\phi, \nabla_\mu\nabla_\nu\phi\} - \frac{1}{6}g_{\mu\nu}[(\nabla\phi)^2 - \mu^2\phi^2].$$

[8] N. D. Birrell, P. C. W. Davies, *Quantum fields in curved space*, Cambridge University Press (1982).

Rigidly-rotating thermal states

- RRTS can be constructed using

$$\langle A \rangle_\beta = Z^{-1}\mathrm{Tr}(WA), \qquad Z = \mathrm{Tr}(W), \qquad W = \exp\left(-\beta H + \boldsymbol{\Omega}\cdot\boldsymbol{J}\right).$$

- The t.e.v. of the one-particle operators is:[9]

$$\langle : a_j^\dagger a_{j'} : \rangle_\beta = \frac{\delta(j,j')}{e^{\beta\widetilde{\omega}_j} - 1}, \qquad \widetilde{\omega} = \omega - \Omega m.$$

- The first problem is that $\langle : a_j^\dagger a_{j'} : \rangle_\beta \rightarrow -\delta(j,j')$ when $\widetilde{\omega}_j < 0$ and $\beta \rightarrow \infty$ (vacuum) $\Rightarrow$ spurious β-independent contributions.
- The second problem can be seen by investigating the t.e.v. of ϕ^2:

$$\langle : \phi^2 : \rangle_\beta = \frac{1}{8\pi^2} \sum_{m=-\infty}^{\infty} \int_\mu^\infty d\omega \int_{-p}^{p} \frac{dk}{e^{\beta\widetilde{\omega}} - 1} J_m^2(q\rho),$$

which $\rightarrow \infty$ when $\rho > 0$ since at sufficiently large m, there are values of ω such that $\widetilde{\omega} = 0$.
- Note: finite quantum corrections can be obtained perturbatively.[10]

[9] A. Vilenkin, Phys. Rev. D **8** (1980) 2260–2269.

[10] F. Becattini, E. Grossi, Phys. Rev. D 92, 045037 (2015).

Bounded K-G field

- Let the system be bounded at $\rho = R$.
- The b.c.s must leave $\langle , \rangle$ time-invariant:

$$i\partial_t \langle f, h\rangle = -\int d\Sigma^i \sqrt{g}(f^* \partial_i h - h\partial_i f^*).$$

- For the cylindrical boundary and $f_{R;j} = N_{R;j} f_j$, this entails:

$$\langle f_{R;j}, f_{R;j'}\rangle = \frac{N_j^* N_{j'}}{2\sqrt{\omega_j \omega_{j'}}} e^{i(\omega_j - \omega_{j'})t} \delta_{m_j, m_{j'}} \delta(k_j - k_{j'}) \times \left\{ J_{m_j}(q_j R)[q_{j'} R J'_{m_{j'}}(q_{j'} R)] - [q_j R J'_{m_j}(q_j R)] J_{m_{j'}}(q_{j'} R) \right\}.$$

- The time-independence is ensured if:
  1. $J_{m_j}(q_j R) = 0$ (Dirichlet),[11]
  2. $J'_{m_j}(q_j R) = 0$ (von Neumann),
  3. $[q_j R J'_{m_j}(q_j R)] + \Psi J_{m_j}(q_j R) = 0$ (Robin).[1]

[11] G. Duffy, A. Ottewill, Phys. Rev. D **67** (2003) 044002.

[1] A. Romeo, A. A. Saharian, Phys. Rev. D **63** (2001) 105019.

Bounded K-G: SET

- Imposing $\langle f_{km\ell}, f_{k'm'\ell'} \rangle = \delta(k-k')\delta_{m,m'}\delta_{\ell,\ell'}$ gives:[1]

$$|N_{km\ell}|^2 = \frac{2q_{m,\ell}^2}{R^2 J_m^2(q_{m,\ell}R)(q_{m,\ell}^2 R^2 + \Psi^2 - m^2)}.$$

- The t.e.v. of the SET is:[12]

$$\langle : T_{\mu\nu} : \rangle_\beta = \sum_{m=-\infty}^{\infty} \sum_{\ell=1}^{\infty} \int_{-\infty}^{\infty} \frac{|N_{km\ell}|^2 dk}{12\pi^2 \omega_{km\ell}(e^{\beta\widetilde{\omega}_{km\ell}} - 1)} F_{\mu\nu},$$

where $\widetilde{\omega} = \omega - \Omega m$ and

$$\begin{aligned}
F_{\hat{0}\hat{0}} &= \left(6\omega^2 + \rho^{-2}m^2 - q^2\right) J_m^2 + q^2 {J'_m}^2, \\
F_{\hat{\varphi}\hat{0}} &= -6\omega\rho^{-1} m J_m^2, \\
F_{\hat{\rho}\hat{\rho}} &= \left(-3\rho^{-2}m^2 + 3q^2\right) J_m^2 + 2q\rho^{-1} J_m J'_m + 3q^2 {J'_m}^2, \\
F_{\hat{\varphi}\hat{\varphi}} &= \left(5\rho^{-2}m^2 + q^2\right) J_m^2 - 2q\rho^{-1} J_m J'_m - q^2 {J'_m}^2, \\
F_{\hat{z}\hat{z}} &= \left(6k^2 - \rho^{-2}m^2 + q^2\right) J_m^2 - q^2 {J'_m}^2,
\end{aligned}$$

where m, ℓ and the explicit dependence on $q\rho$ where dropped.

[1] A. Romeo, A. A. Saharian, Phys. Rev. D **63** (2001) 105019.
[12] V. E. Ambruș, Phys. Lett. B **771** (2017) 151–156.

Macroscopic boundary conditions

- The eigenvalue eq. $T^{\hat{\alpha}}{}_{\hat{\gamma}} u^{\hat{\gamma}} = -E u^{\hat{\alpha}}$ can be solved exactly:

$$v = \frac{T^{\hat{0}\hat{\varphi}}}{E + T^{\hat{\varphi}\hat{\varphi}}}, \qquad E = \frac{1}{2}[T^{\hat{0}\hat{0}} - T^{\hat{\varphi}\hat{\varphi}} + \sqrt{(T^{\hat{0}\hat{0}} + T^{\hat{\varphi}\hat{\varphi}})^2 - 4(T^{\hat{0}\hat{\varphi}})^2}],$$

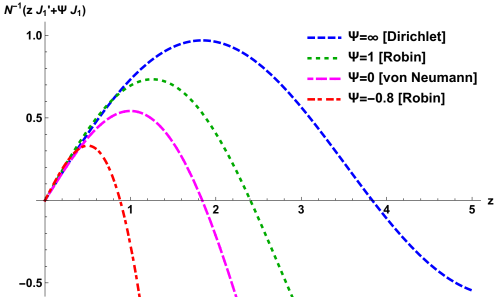
- The following relation holds:

$$T^{\hat{0}\hat{\varphi}} = \frac{v}{1+v^2}(T^{\hat{0}\hat{0}} + T^{\hat{\varphi}\hat{\varphi}}).$$

- In order to impose $v = v_b$ on the cylinder ($\rho = R$), Ψ can be obtained iteratively:

$$\Psi = \frac{3 \sum_{m,\ell} \left(1 + \frac{\Psi^2 - m^2}{q^2 R^2}\right)^{-1} \int \frac{dk}{\omega(e^{\beta\widetilde{\omega}} - 1)} \left[\omega R m - \frac{v_{\rm b}\left(\omega^2 R^2 + m^2\right)}{1 + v_{\rm b}^2}\right]}{\frac{v_{\rm b}}{1+v_{\rm b}^2} \sum_{m,\ell} \left(1 + \frac{\Psi^2 - m^2}{q^2 R^2}\right)^{-1} \int \frac{dk}{\omega(e^{\beta\widetilde{\omega}} - 1)}}$$

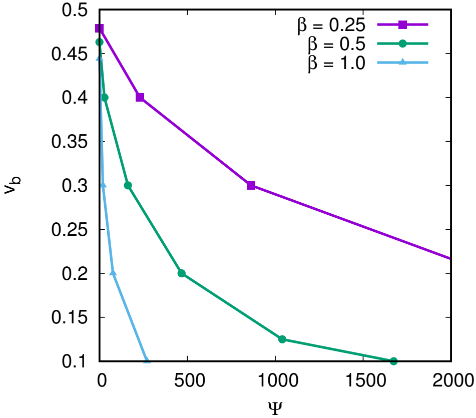
Solving for Ψ



($m = 1$; N provides an arbitrary normalisation)

- The purpose of the boundary is to eliminate the modes with $\widetilde{\omega} < 0$.
- This is guaranteed for $\Psi \geq 0$ ($\xi'_{m,\ell} > m$).
- There are ranges for $\Psi < 0$ where the modes with $\widetilde{\omega} < 0$ reappear.
- There are ranges where they don't (e.g., $0 < \Psi < -0.7$, $-1 \leq \Psi < -1.2$).

Connection between Ψ and $v_{\rm b}$



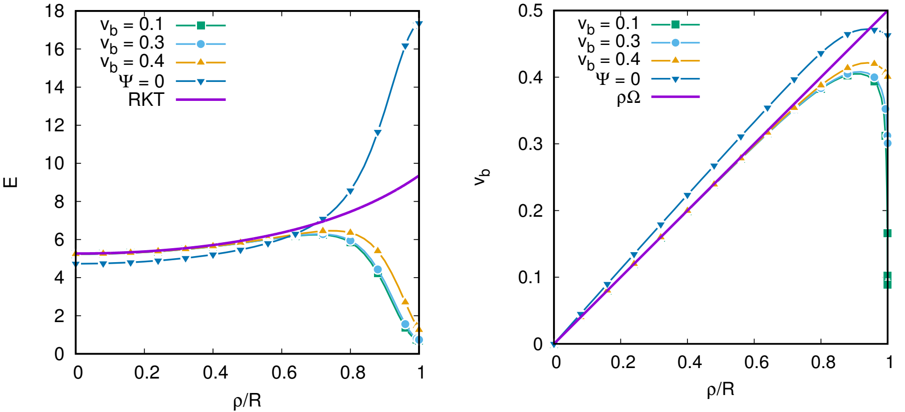
v_b	Ψ
0	∞
0.1	1673.76
0.125	1040.09
0.2	467.099
0.3	162.162
0.4	28.602
0.463053	0
0.472814	-0.7
0.510355	-1

$(\beta = 0.5)$

$$(R = 1;\ \Omega = 0.5;\ \mu = 0)$$

- For fixed $v_{\rm b}$, Ψ decreases as β is increased.
- At fixed β, $v_{\rm b} = \Omega R = 0.5$ is unattainable for $\Psi > 0$.
- Negative Ψ required to reach $v_{\rm b} = 0.5$.

Landau energy E and velocity v



$$(R = 1;\ \Omega = 0.5;\ \beta = 0.5;\ \mu = 0)$$

- Relativistic Kinetic Theory (RKT): $E = \frac{\pi^2\gamma^4}{30\beta^4}$, $v = \rho\Omega$.
- For $\Psi \gg 0$: close to RKT at $\rho \simeq 0$; The b.c.s cause the decrease of v and E as $\rho \to R$.
- For $\Psi = 0$: $E < E_{\rm RKT}$ at $\rho \simeq 0$ and increases wildly as $\rho \to R$.

Non-equilibrium effects

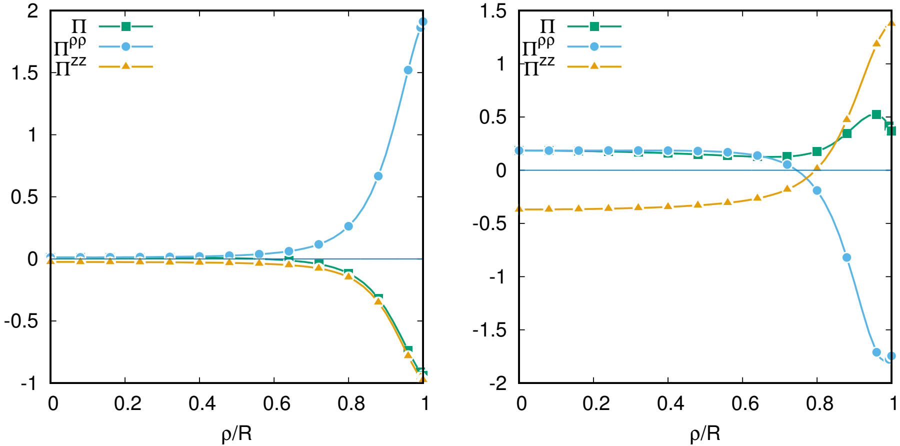
- Rigid rotation is an exact equilibrium solution of the relativistic Boltzmann equation, giving $T^{\hat{\alpha}\hat{\gamma}} = (E + P)u^{\hat{\alpha}}u^{\hat{\gamma}} + P\eta^{\hat{\alpha}\hat{\gamma}}$.
- For the current system, $u^{\hat{\alpha}} = \gamma(1, 0, v, 0)^T$ and

$$T^{\hat{\alpha}\hat{\gamma}} = \begin{pmatrix} T^{\hat{0}\hat{0}} & 0 & T^{\hat{0}\hat{\varphi}} & 0 \\ 0 & T^{\hat{\rho}\hat{\rho}} & 0 & 0 \\ T^{\hat{0}\hat{\varphi}} & 0 & T^{\hat{\varphi}\hat{\varphi}} & 0 \\ 0 & 0 & 0 & T^{\hat{z}\hat{z}} \end{pmatrix}, \quad \Pi^{\hat{\alpha}\hat{\gamma}} = \begin{pmatrix} \beta^2\gamma^2\Pi & 0 & \beta\gamma^2\Pi & 0 \\ 0 & \Pi^{\hat{\rho}\hat{\rho}} & 0 & 0 \\ \beta\gamma^2\Pi & 0 & \gamma^2\Pi & 0 \\ 0 & 0 & 0 & \Pi^{\hat{z}\hat{z}} \end{pmatrix},$$

  where Π is the shear pressure, while $\Pi^{\hat{\rho}\hat{\rho}} = T^{\hat{\rho}\hat{\rho}} - P$ and $\Pi^{\hat{z}\hat{z}} = T^{\hat{z}\hat{z}} - P$ are the longitudinal and transversal pressure deviators.
- The tracelessness of $\Pi^{\hat{\alpha}\hat{\gamma}}$ implies:

$$\Pi + \Pi^{\hat{\rho}\hat{\rho}} + \Pi^{\hat{z}\hat{z}} = 0.$$

Pressure deviators



$(v_\mathrm{b} = 0.4,\ \Psi = 28.602)$ $(\Psi = 0)$

$(R = 1;\ \Omega = 0.5;\ \beta = 0.5;\ \mu = 0)$

- For $\Psi \gg 0$, $\Pi \simeq \Pi^{\hat{z}\hat{z}} \lesssim 0$ and $\Pi^{\hat{\rho}\hat{\rho}} \gtrsim 0$.
- For $\Psi = 0$, $\Pi^{\hat{\rho}\hat{\rho}}$ and $\Pi^{\hat{z}\hat{z}}$ change sign and are both larger in magnitude than Π almost everywhere.

Conclusion

- Boundary conditions are required in order to render the t.e.v. of the rigidly-rotating K-G field finite.
- The most general formulation is given as Robin boundary conditions.
- $\Psi \geq 0$ ensures that the t.e.v.s remain finite.
- For some negative values of Ψ, the t.e.v.s can become divergent.
- A systematic procedure giving Ψ in terms of $v_{\rm b}$ was derived.
- There is a maximum value for $v_{\rm b}$ which can be reached for $\Psi \geq 0$ and it is less than ΩR.
- Since the SET is not in perfect fluid form, thermodynamic equilibrium is not achieved.
- An open challenge is to disentangle quantum corrections from boundary interaction effects.
- Future work: extend the analysis to the Dirac field.

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