Shakhov model for relativistic flows

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Outline

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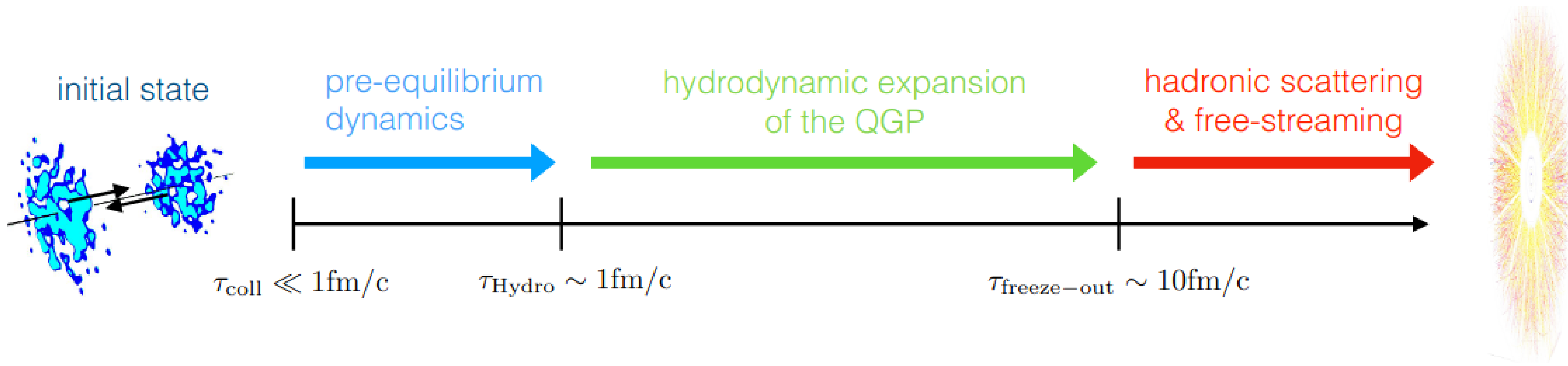
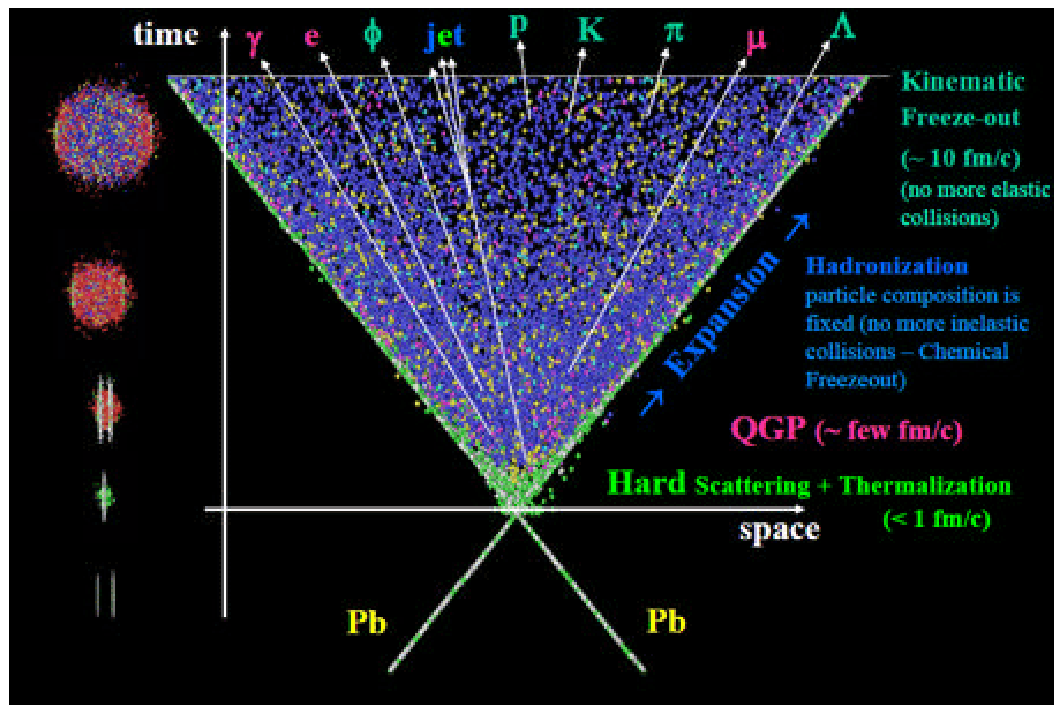
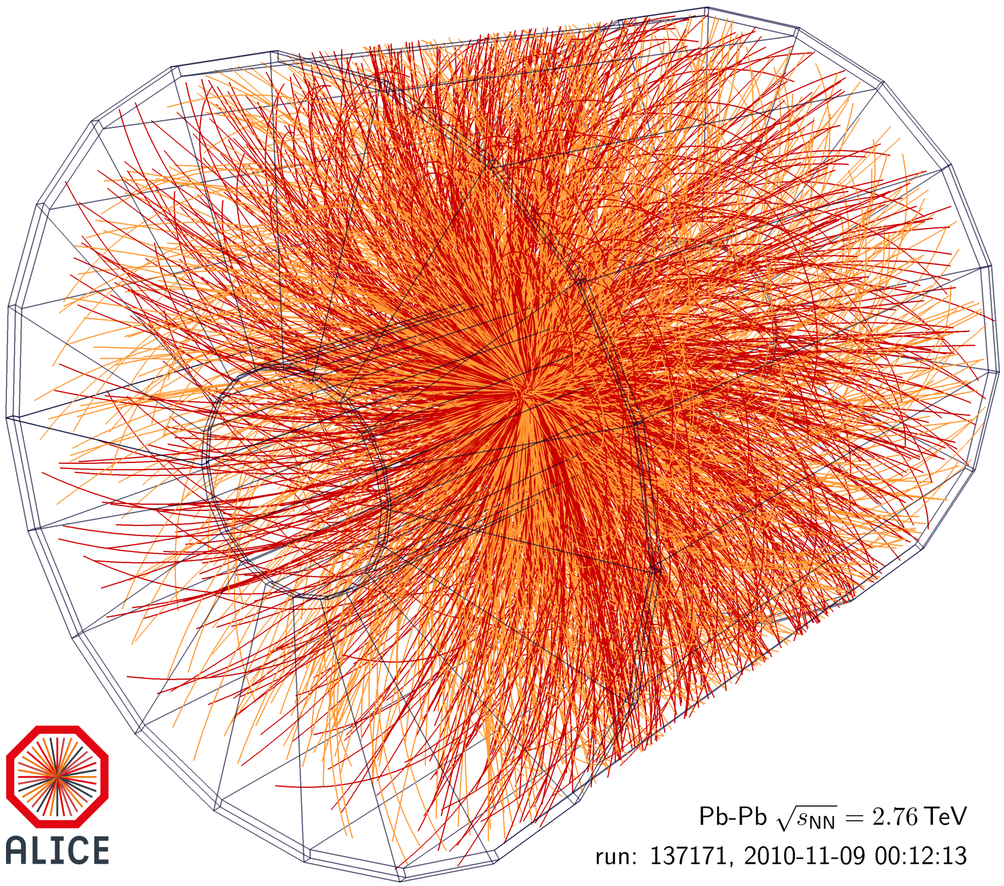


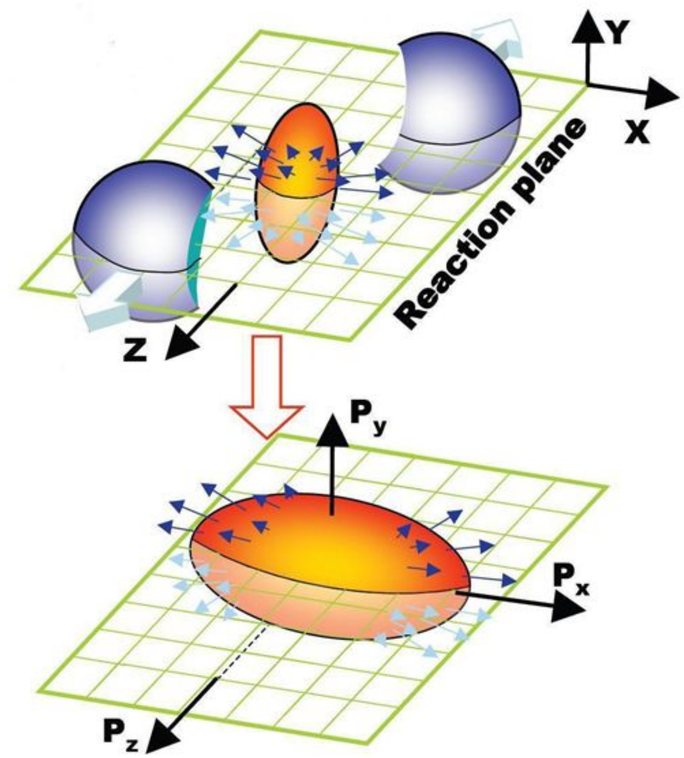
- Shortly after the collision, the system is far-from-equilibrium.
- Pre-eq. dynamics require a non-eq. description.
- Strongly-interacting QGP leaves imprints of thermalization and collectivity in final-state observables.



[Venaruzzo, PhD Thesis, 2011]

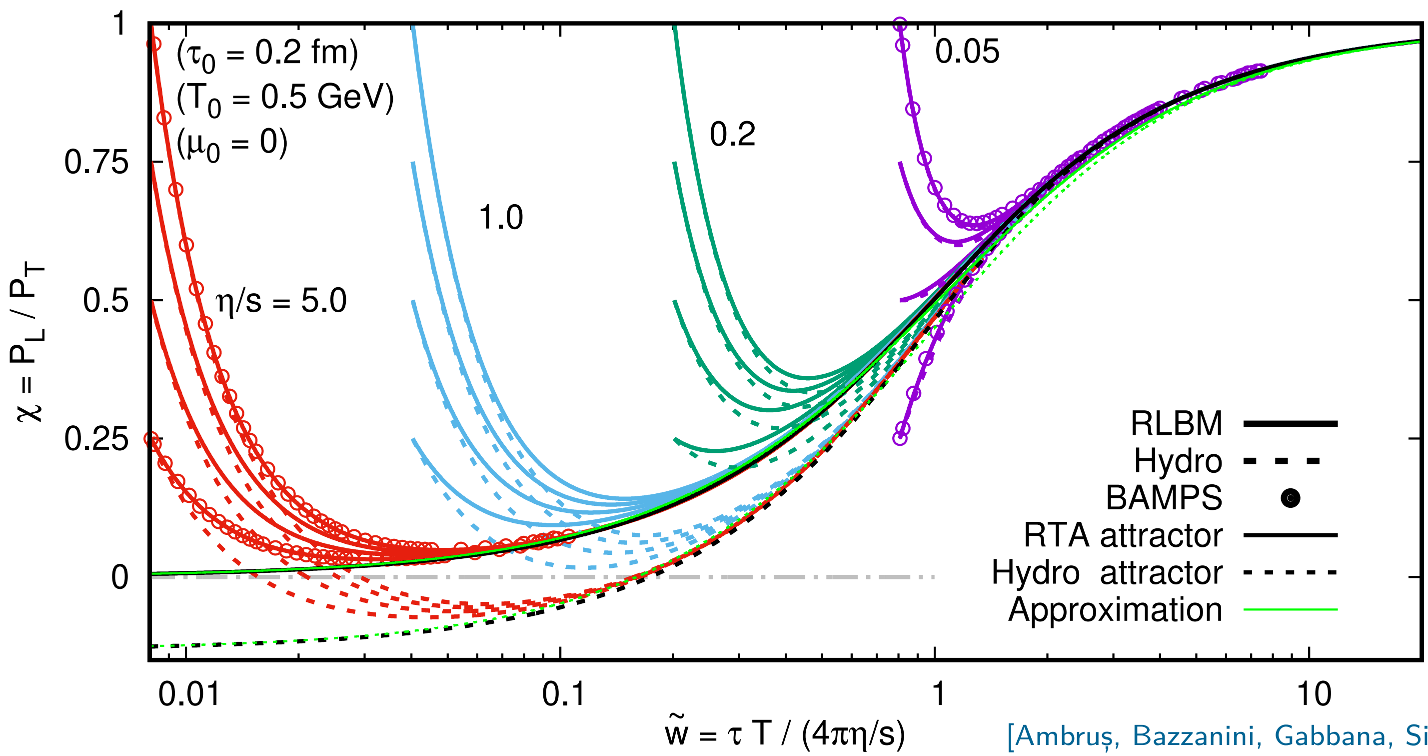
Hadronic Collisions in Experiment





- Relativistic fluid dynamics is indispensible when studying the dynamics of the QGP fireball produced in HICs.
- Realistic models account for the QCD equation of state; realistic transport coefficients; chiral phase transition (hadronization).

Hydro vs Kinetic theory



[Ambruș, Bazzanini, Gabbana, Simeoni, Succi, Nature Comput. Sci. **2** (2022) 641]

- Hydro takes the above into account, but it breaks down far from eq.
- Kinetic theory overcomes this limitation, but realistic simulations are expensive due to $C[f]$.

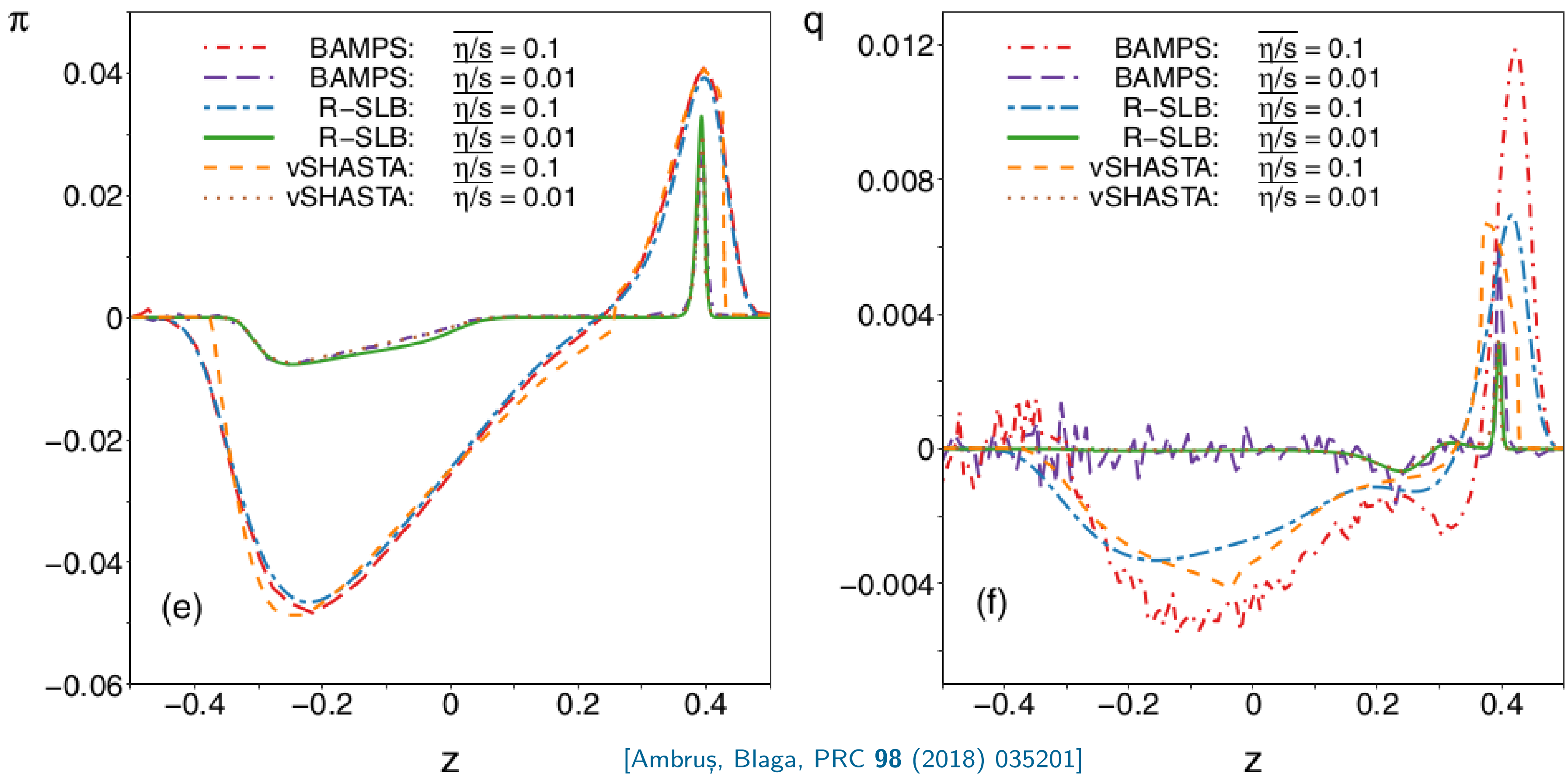
  AMPT: He, Edmonds, Lin, Liu, Molnar, Wang [PLB 753 (2016) 506]
  BAMPS: Greif, Greiner, Schenke, Schlichting, Xu [PRD 96 (2017) 091504]

- RTA: $C[f] \sim -\frac{1}{\tau_R}(f_{\mathbf{k}} - f_{0\mathbf{k}}) \Rightarrow 1-2$ o.m. faster than BAMPS.

  VEA, Busuioc, Fotakis, Gallmeister, Greiner [PRD **104** (2021) 094022]

- τ_R fixes the IR limit of RTA by matching e.g. η to that of $C[f] \Rightarrow$ good agreement with BAMPS.

RTA vs BAMPS



[Ambruș, Blaga, PRC **98** (2018) 035201]

- τ_R governs all dissipative transport $\Rightarrow$ can fix only shear (η) or diffusion (κ), but not both.
- Fixing η via τ_R gives good agreement with BAMPS for $\pi^{\mu\nu}$ but q^μ is not captured correctly.
- **Aim of this work:** Extend RTA with extra parameters allowing multiple transport coefficients to be controlled independently.

BGK model

- In non-relativistic kinetic theory, the RTA was proposed by Bhatnagar, Gross and Krook (BGK): [Bhatnagar, Gross, Krook, Phys. Rev. **94** (1954) 511]

$$C_{\rm BGK}[f] = -\frac{1}{\tau_R}(f_{\mathbf{k}} - f_{0\mathbf{k}}), \quad f_{0\mathbf{k}} = \frac{n e^{-\boldsymbol{\xi}^2/2mk_BT}}{(2\pi m k_B T)^{3/2}}, \tag{1}$$

where $\boldsymbol{\xi} = \boldsymbol{p} - m\boldsymbol{u}$ is the peculiar momentum.

- Applying the Chapman-Enskog expansion gives

$$\delta f_{\mathbf{k}} \equiv f_{\mathbf{k}} - f_{0\mathbf{k}} = -\tau_R \left(\frac{\partial}{\partial t} + \frac{\boldsymbol{k}}{m}\cdot\nabla\right) f_{0\mathbf{k}}. \tag{2}$$

- At first order, $\pi_{ij} = T_{ij} - P\delta_{ij}$ and $\boldsymbol{q}$ are

$$\pi_{ij} = \int d^3k \frac{\xi_i \xi_j}{m} \delta f_{\mathbf{k}} \simeq -2\eta\sigma_{ij}, \tag{3a}$$

$$\boldsymbol{q} = \int d^3k \frac{\boldsymbol{\xi}^2}{2m}\frac{\boldsymbol{\xi}}{m}\delta f_{\mathbf{k}} \simeq -\lambda\nabla T, \tag{3b}$$

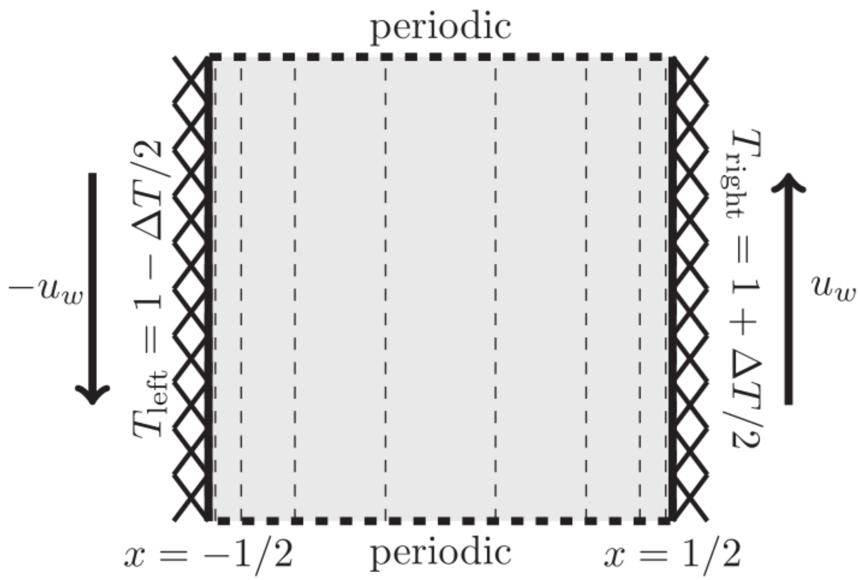
where $\sigma_{ij} = \partial_{(i}u_{j)} - \frac{1}{3}(\nabla\cdot\boldsymbol{u})\delta_{ij}$ is the shear tensor, while

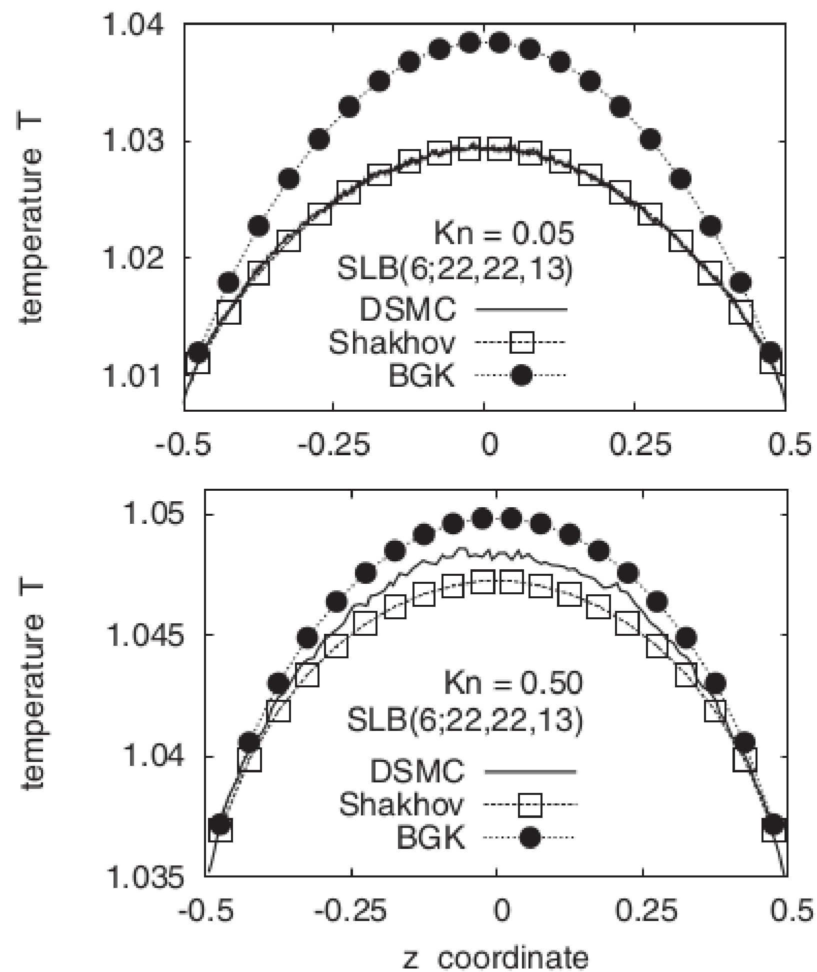
$$\eta = \tau_R P, \quad \lambda = c_p \tau_R P, \tag{4}$$

where $c_p = 5k_B/2m \equiv$ specific heat at constant P of the monatomic ideal gas.

Shakhov model

[Shakhov, Fluid Dyn. **3** (1968) 112]





[Ambruș, Sofonea, PRE **86** (2012) 016708]

- In BGK, $\mathrm{Pr} = \dfrac{c_p \eta}{\lambda} = 1$.
- Hard-sphere ideal gas: $\mathrm{Pr} \simeq \frac{2}{3}$.
- The Shakhov model employs $C_{\mathrm{S}}[f] = -\dfrac{1}{\tau_R}(f_{\mathbf{k}} - f_{\mathrm{S}\mathbf{k}})$ with

$$f_{\mathrm{S}\mathbf{k}} = f_{0\mathbf{k}}(1 + \mathbb{S}_{\mathbf{k}}), \quad \mathbb{S}_{\mathbf{k}} = \frac{1 - \mathrm{Pr}}{P k_B T}\left(\frac{\boldsymbol{\xi}^2}{5 m k_B T} - 1\right) \boldsymbol{q} \cdot \boldsymbol{\xi}. \qquad (5)$$

- Comparison with DSMC validates Shakhov for small Kn.

Relativistic fluids

- In special relativity, space $\mathbf{x}$ & time $t \to$ space-time $x^\mu = (ct, \mathbf{x})$.
- Causality: $|\mathbf{v}| < c$. [Planck units: $c = k_B = \hbar = 1$]
- Energy (E) & momentum $(\mathbf{k}) \to k^\mu = (k^0, \mathbf{k})$, with

$$k^0 = m\gamma = \sqrt{m^2 + \mathbf{k}^2}, \quad \mathbf{k} = m\gamma\mathbf{v}, \quad \gamma = 1/\sqrt{1 - \mathbf{v}^2}. \tag{6}$$

- Main hydro quantities (in equilibrium and in local rest frame):
  - Particle four-flow: $N_0^\mu\rfloor_{\rm LRF} = (n, \mathbf{0})$
  - Energy-momentum tensor: $T_0^{\mu\nu}\rfloor_{\rm LRF} = {\rm diag}(\epsilon, P, P, P)$.
- In an arbitrary frame but in local equilibrium:

$$N_0^\mu = n u^\mu, \quad T_0^{\mu\nu} = \epsilon u^\mu u^\nu - P\Delta^{\mu\nu}, \tag{7}$$

with $u^0 = \sqrt{1 + \mathbf{u}^2}$, $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ and $g^{\mu\nu} = {\rm diag}(1, -1, -1, -1)$.

- Away from equilibrium: $[V^\mu u_\mu = W^\mu u_\mu = \pi^{\mu\nu} u_\nu = 0,\ \pi^\mu{}_\mu = 0]$

$$N^\mu = N_0^\mu + V^\mu, \quad T^{\mu\nu} = T_0^{\mu\nu} - \Pi\Delta^{\mu\nu} + 2W^{(\mu}u^{\nu)} + \pi^{\mu\nu}. \tag{8}$$

- Relativistic NS: constitutive eqs. for Π, V^μ, W^μ, $\pi^{\mu\nu}$ (acausal).
- Relativistic hydro: $\{\Pi, V^\mu, W^\mu, \pi^{\mu\nu}\}$ become dynamical dofs.

Anderson-Witting model

- The Anderson & Witting RTA reads [Anderson, Witting, Physica **74** (1974) 466]

$$k^\mu \partial_\mu f_{\mathbf{k}} = C_{\rm AW}[f], \quad C_{\rm AW}[f] = -\frac{E_{\mathbf{k}}}{\tau_R}(f_{\mathbf{k}} - f_{0\mathbf{k}}), \tag{9}$$

  where $E_{\mathbf{k}} = k^\mu u_\mu \simeq mc^2 + \frac{m}{2}(\mathbf{v} - \mathbf{u})^2 + \dots$ and

$$f_{0\mathbf{k}} = \frac{g e^\alpha}{(2\pi\hbar)^3} e^{-E_{\mathbf{k}}/k_B T} = \frac{nc^3}{8\pi k_B^3 T^3} e^{-E_{\mathbf{k}}/k_B T}, \tag{10}$$

  with g the degeneracy factor.

- The macroscopic quantities N^μ and $T^{\mu\nu}$ are obtained from $f_{\mathbf{k}}$ via

$$N^\mu = \int dK\, k^\mu\, f_{\mathbf{k}}, \quad T^{\mu\nu} = \int dK\, k^\mu k^\nu f_{\mathbf{k}}, \tag{11}$$

  with Lorentz-invariant $dK = g\, d^3k/[k_0(2\pi)^3]$.

- Imposing $\partial_\mu N^\mu = \partial_\nu T^{\mu\nu} = 0$ requires Landau matching:

$$n = n_0, \quad \epsilon = \epsilon_0, \quad T^{\mu\nu} u_\nu = \epsilon u^\mu, \tag{12}$$

- Since $T^{\mu\nu} u_\nu = \epsilon u^\mu + W^\mu$, we have $W^\mu = 0$ in the Landau frame.

Chapman-Enskog expansion

- We are now interested to obtain constitutive relations for the non-equilibrium quantities

$$N^\mu - N_0^\mu = V^\mu, \quad T^{\mu\nu} - T_0^{\mu\nu} = -\Pi\Delta^{\mu\nu} + \pi^{\mu\nu}. \tag{13}$$

- Employing the Chapman-Enskog procedure gives

$$\delta f_{\mathbf{k}} \equiv f_{\mathbf{k}} - f_{0\mathbf{k}} = -\frac{\tau_R}{E_{\mathbf{k}}} k^\mu \partial_\mu f_{0\mathbf{k}}, \tag{14}$$

such that

$$\Pi = -\zeta_R \theta, \quad V^\mu = \kappa_R \nabla^\mu \alpha, \quad \pi^{\mu\nu} = 2\eta_R \sigma^{\mu\nu}. \tag{15}$$

- ζ_R, κ_R and η_R are given by

$$\zeta_R = \frac{m^2}{3}\tau_R \alpha_0^{(0)}, \quad \kappa_R = \tau_R \alpha_0^{(1)}, \quad \eta_R = \tau_R \alpha_0^{(2)}. \tag{16}$$

where $\alpha_0^{(\ell)}$ are τ_R-independent thermodynamic functions.

Shakhov-like extension

[Ambruș, Molnár, under review]

- ζ, η and κ are governed by the same parameter, τ_R.
- We consider a Shakhov-like extension:

$$C_{\rm S}[f] = -\frac{E_{\mathbf{k}}}{\tau_R}(f_{\mathbf{k}} - f_{{\rm S}\mathbf{k}}), \tag{17}$$

where $f_{{\rm S}\mathbf{k}} \to f_{0\mathbf{k}}$ as $\delta f_{\mathbf{k}} = f_{\mathbf{k}} - f_{0\mathbf{k}} \to 0$.

- The cons. eqs. $\partial_\mu N^\mu = \partial_\nu T^{\mu\nu} = 0$ imply:

$$u_\mu N^\mu = u_\mu N^\mu_{\rm S}, \quad u_\nu T^{\mu\nu} = u_\nu T^{\mu\nu}_{\rm S}, \tag{18}$$

which allows for plenty of degrees of freedom (δn, $\delta\epsilon$, W^μ, etc).

- For simplicity, we stick to the Landau matching conditions:

$$\delta n = \delta\epsilon = 0, \qquad T^{\mu\nu} u_\nu = \epsilon u^\mu. \tag{19}$$

Shakohv-like extension

- Employing the Chapman-Enskog procedure gives

$$\delta f_{\mathbf{k}} - \delta f_{\mathrm{S}\mathbf{k}} = -\frac{\tau_R}{E_{\mathbf{k}}} k^\mu \partial_\mu f_{0\mathbf{k}}, \tag{20}$$

leading to

$$\Pi - \Pi_{\mathrm{S}} = -\zeta_R \theta, \quad V^\mu - V^\mu_{\mathrm{S}} = \kappa_R \nabla^\mu \alpha, \quad \pi^{\mu\nu} - \pi^{\mu\nu}_{\mathrm{S}} = 2\eta_R \sigma^{\mu\nu}. \tag{21}$$

- We seek to replace ζ_{R} etc by independent transport coefficients:

$$\begin{aligned} \Pi &\simeq -\zeta_{\mathrm{S}}\theta, & V^\mu &\simeq \kappa_{\mathrm{S}} \nabla^\mu \alpha, & \pi^{\mu\nu} &\simeq 2\eta_{\mathrm{S}} \sigma^{\mu\nu}, \\ \zeta_{\mathrm{S}} &= \frac{\tau_\Pi}{\tau_R}\zeta_R, & \kappa_{\mathrm{S}} &= \frac{\tau_V}{\tau_R}\kappa_R, & \eta_{\mathrm{S}} &= \frac{\tau_\pi}{\tau_R}\eta_R. \end{aligned} \tag{22}$$

- Eq. (22) can be obtained from Eq. (21) when

$$\Pi_{\mathrm{S}} = \Pi\left(1 - \frac{\tau_\Pi}{\tau_R}\right), \quad V^\mu_{\mathrm{S}} = V^\mu\left(1 - \frac{\tau_V}{\tau_R}\right),$$
$$\pi^{\mu\nu}_{\mathrm{S}} = \pi^{\mu\nu}\left(1 - \frac{\tau_\pi}{\tau_R}\right). \tag{23}$$

Minimal $\delta f_{\mathrm{S}\mathbf{k}}$

- Writing $f_{\mathrm{S}\mathbf{k}} = f_{0\mathbf{k}} + \delta f_{\mathrm{S}\mathbf{k}}$, we require:

$$\begin{pmatrix} \rho_{\mathrm{S};0} \\ \rho_{\mathrm{S};1} \\ \rho_{\mathrm{S};2} \end{pmatrix} = \int dK \begin{pmatrix} 1 \\ E_{\mathbf{k}} \\ E_{\mathbf{k}}^2 \end{pmatrix} \delta f_{\mathrm{S}\mathbf{k}} = \begin{pmatrix} -3\Pi_{\mathrm{S}}/m^2 \\ 0 \\ 0 \end{pmatrix},$$
$$\begin{pmatrix} \rho^{\mu}_{\mathrm{S};0} \\ \rho^{\mu}_{\mathrm{S};1} \end{pmatrix} = \int dK \begin{pmatrix} 1 \\ E_{\mathbf{k}} \end{pmatrix} k^{\langle\mu\rangle} \delta f_{\mathrm{S}\mathbf{k}} = \begin{pmatrix} V^{\mu}_{\mathrm{S}} \\ 0 \end{pmatrix},$$
$$\rho^{\mu\nu}_{\mathrm{S};0} = \int dK\, k^{\langle\mu} k^{\nu\rangle} \delta f_{\mathbf{k}} = \pi^{\mu\nu}_{\mathrm{S}}, \qquad (24)$$

with $k^{\langle\mu\rangle} = \Delta^{\mu}_{\alpha} k^{\alpha}$ and $k^{\langle\mu} k^{\nu\rangle} = \Delta^{\mu\nu}_{\alpha\beta} k^{\alpha} k^{\beta}$ *irreducible* tensors.

- Thus, $\delta f_{\mathbb{S}\mathbf{k}} = f_{0\mathbf{k}} \mathbb{S}_{\mathbf{k}}$ can be written as

$$\mathbb{S}_{\mathbf{k}} = -\frac{3\Pi}{m^2}\left(1 - \frac{\tau_R}{\tau_\Pi}\right)\mathcal{H}^{(0)}_{\mathbf{k}0} + k_{\langle\mu\rangle} V^{\mu}\left(1 - \frac{\tau_R}{\tau_V}\right)\mathcal{H}^{(1)}_{\mathbf{k}0} + k_{\langle\mu} k_{\nu\rangle} \pi^{\mu\nu}\left(1 - \frac{\tau_R}{\tau_\pi}\right)\mathcal{H}^{(2)}_{\mathbf{k}0}, \qquad (25)$$

where the functions $\mathcal{H}^{(\ell)}_{\mathbf{k}0}$ are Lorentz scalars depending only on $E_{\mathbf{k}}$.

[DNMR, PRD **85** (2012) 114047]

First-order model

- Specifically, $\mathcal{H}^{(\ell)}_{\mathbf{k}0}$ must satisfy:

$$\int dK\, f_{0\mathbf{k}} \begin{pmatrix} 1 \\ E_\mathbf{k} \\ E_\mathbf{k}^2 \end{pmatrix} \mathcal{H}^{(0)}_{\mathbf{k}0} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$
$$\frac{1}{3}\int dK\, f_{0\mathbf{k}} \begin{pmatrix} 1 \\ E_\mathbf{k} \end{pmatrix} (\Delta^{\alpha\beta}k_\alpha k_\beta)\mathcal{H}^{(1)}_{\mathbf{k}0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$
$$\frac{2}{15}\int dK\, f_{0\mathbf{k}} (\Delta^{\alpha\beta}k_\alpha k_\beta)^2\mathcal{H}^{(2)}_{\mathbf{k}0} = 1. \tag{26}$$

- The lowest-order polynomials satisfying these relations are

$$\mathcal{H}^{(0)}_{\mathbf{k}0} = \frac{G_{33} - G_{23}E_\mathbf{k} + G_{22}E_\mathbf{k}^2}{J_{00}G_{33} - J_{10}G_{23} + J_{20}G_{22}},$$
$$\mathcal{H}^{(1)}_{\mathbf{k}0} = \frac{J_{31}E_\mathbf{k} - J_{41}}{J_{21}J_{41} - J_{31}^2}, \quad \mathcal{H}^{(2)}_{\mathbf{k}0} = \frac{1}{2J_{42}}, \tag{27}$$

where $G_{nm} = J_{n0}J_{m0} - J_{n-1,0}J_{m+1,0}$, while

$$J_{nq} = \frac{(-1)^q}{(2q+1)!!}\int dK\, E_\mathbf{k}^{n-2q}\left(\Delta^{\alpha\beta}k_\alpha k_\beta\right)^q f_{0\mathbf{k}}. \tag{28}$$

Entropy production

- In kinetic theory, the entropy current is given by

$$S^\mu = -\int dK\, k^\mu \left(f_{\mathbf{k}} \ln f_{\mathbf{k}} - f_{\mathbf{k}}\right). \tag{29}$$

- In the Shakhov model, $k^\mu \partial_\mu f = C_{\rm S}[f]$ and

$$\partial_\mu S^\mu = -\int dK\, C_{\rm S}[f] \ln f_{\mathbf{k}} = \frac{1}{\tau_R} \int dK\, E_{\mathbf{k}} (\delta f_{\mathbf{k}} - \delta f_{{\rm S}\mathbf{k}}) \ln f_{\mathbf{k}}. \tag{30}$$

- $\partial_\mu S^\mu$ difficult for generic $f_{\mathbf{k}}$.
- When $\phi_{\mathbf{k}} = \delta f_{\mathbf{k}} / f_{0\mathbf{k}}$ is small, detailed manipulations lead to

$$\partial_\mu S^\mu \simeq \frac{\beta}{\zeta_{\rm S}} \Pi^2 - \frac{1}{\kappa_{\rm S}} V_\mu V^\mu + \frac{\beta}{2\eta_{\rm S}} \pi_{\mu\nu} \pi^{\mu\nu} \geq 0. \tag{31}$$

- Close to eq., the S-model satisfies the $2^{\rm nd}$ law of thermodynamics.

Application: Bjorken flow

- Bjorken model: flow invariant under longitudinal boosts:

$$u^\mu \partial_\mu = \frac{t}{\tau}\partial_t + \frac{z}{\tau}\partial_z, \quad \tau = \sqrt{t^2 - z^2}, \quad \eta_s = \tanh^{-1}(z/t). \tag{32}$$

- In Bjorken coordinates $(\tau, \mathbf{x}_\perp, \eta_s)$,

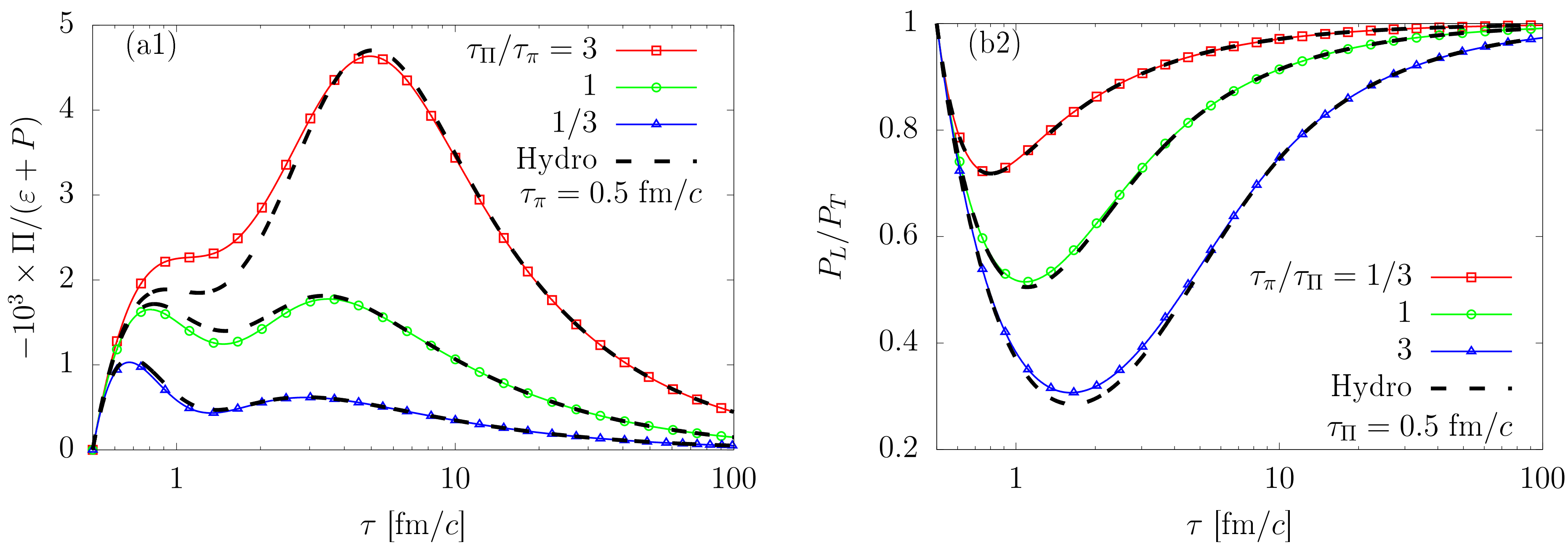
$$T^{\mu\nu} = \mathrm{diag}(e, P_T, P_T, \tau^{-2} P_L),$$
$$P_T = P + \Pi - \frac{\pi_d}{2}, \qquad P_L = P + \Pi + \pi_d. \tag{33}$$

- In 2^{nd}-order hydro, we have: [Denicol, Florkowski, Ryblewski, Strickland, PRC **90** (2014) 044905]

$$\tau\dot{\epsilon} + \epsilon + P_L = 0, \tag{34a}$$
$$\tau\dot{\Pi} + \left(\frac{\delta_{\Pi\Pi}}{\tau_\Pi} + \frac{\tau}{\tau_\Pi}\right)\Pi + \frac{\lambda_{\Pi\pi}}{\tau_\Pi}\pi_d = -\frac{\zeta}{\tau_\Pi},$$
$$\tau\dot{\pi}_d + \left(\frac{\delta_{\pi\pi}}{\tau_\pi} + \frac{\tau_{\pi\pi}}{3\tau_\pi} + \frac{\tau}{\tau_\pi}\right)\pi_d + \frac{2\lambda_{\pi\Pi}}{3\tau_\pi}\Pi = -\frac{4\eta}{3\tau_\pi}. \tag{34b}$$

- We employ the Shakhov model to control ζ independently from η.

Shakhov model: ζ vs. η



- Choosing $\tau_R = \tau_\Pi$, the Shakhov distribution becomes

$$f_{S\mathbf{k}} = f_{0\mathbf{k}} \left[1 + \frac{\beta^2 k_\mu k_\nu \pi^{\mu\nu}}{2(e+P)} \left(1 - \frac{\tau_\Pi}{\tau_\pi}\right)\right]. \tag{35}$$

- Left panel: τ_π is fixed and τ_Π is varied using the Shakhov model.
- Right panel: τ_Π is fixed and τ_π is varied using the Shakhov model.
- $m = 1$ GeV; $\tau_0 = 0.5$ fm; $\beta_0^{-1} = 0.6$ GeV; For $\tau_\pi = 0.5$ fm, $4\pi\eta/s \simeq 3.3$ at $\tau = \tau_0$.

Application: Sound waves

- We now consider an infinitesimal perturbation propagating in an ultrarelativistic fluid at rest.
- Writing $u^\mu \simeq (1, 0, 0, \delta v)$, $\epsilon = \epsilon_0 + \delta\epsilon$ and $n = n_0 + \delta n$, we have

$$
\begin{aligned}
\partial_t \delta n + n_0 \partial_z \delta v + \partial_z \delta V &= 0, \\
\partial_t \delta\epsilon + (\epsilon_0 + P_0)\partial_z \delta v &= 0, \\
(\epsilon_0 + P_0)\partial_t \delta v + \partial_z \delta P + \partial_z \delta\pi &= 0, \\
\tau_V \partial_t \delta V + \delta V + \kappa \partial_z \delta\alpha - \ell_{V\pi}\partial_z \delta\pi &= 0, \\
\tau_\pi \partial_t \delta\pi + \delta\pi + \frac{4\eta}{3}\partial_z \delta v + \ell_{\pi V}\partial_z \delta V = 0,
\end{aligned}
\tag{36}
$$

where $\delta V = V^z$ and $\delta\pi = \pi^{zz}/\gamma^2$.

- In RTA, $\ell_{V\pi} = \ell_{\pi V} = 0$. [Ambruș, Molnár, Rischke, PRD **106** (2022) 076005]
- We track the time evolution of the amplitudes

$$
\widetilde{\delta V} = \frac{2}{L}\int_0^L dz\, \delta V \, \cos(kz), \quad \widetilde{\delta\pi} = \frac{2}{L}\int_0^L dz\, \delta\pi \, \sin(kz). \tag{37}
$$

- We employ the Shakhov model to control κ independently from η.

Sound waves: linear modes

- Inserting $A(t,x) = A_0 + \int_{-\infty}^{\infty} dk \sum_\omega e^{-i(\omega t - kz)} \delta A_\omega(k)$ gives

$$\begin{pmatrix} -3\frac{\omega}{k} & 4P_0 & 0 & 0 & 0 \\ 1 & -\frac{4\omega}{k}P_0 & 1 & 0 & 0 \\ 0 & \frac{4\eta}{3} & -\frac{i}{k} - \frac{\omega}{k}\tau_\pi & 0 & \ell_{\pi V} \\ 0 & n_0 & 0 & -\frac{\omega}{k} & 1 \\ -\frac{3\kappa}{P_0} & 0 & -\ell_{V\pi} & \frac{4\kappa}{n_0} & -\frac{i}{k} - \frac{\omega}{k}\tau_V \end{pmatrix} \begin{pmatrix} \delta P_\omega(k) \\ \delta v_\omega(k) \\ \delta \pi_\omega(k) \\ \delta n_\omega(k) \\ \delta V_\omega(k) \end{pmatrix} = 0.$$

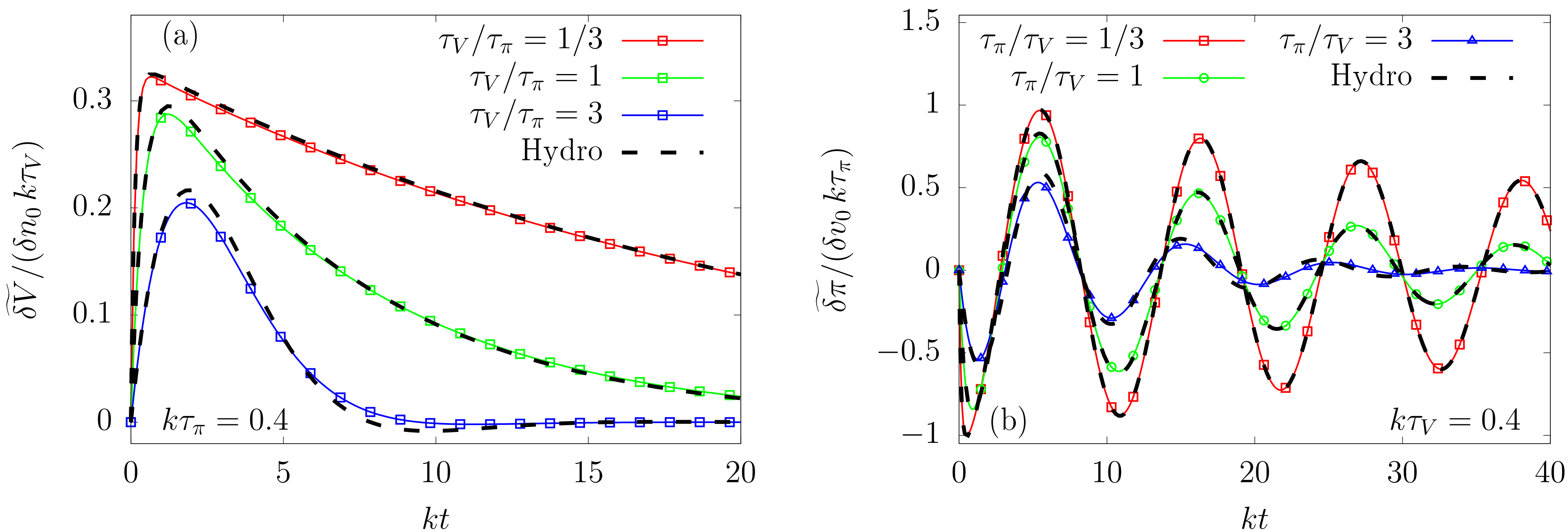
- Thanks to $\ell_{V\pi} = \ell_{\pi V} = 0$, the shear and diffusion sectors decouple:

$$(k^2 - 3\omega^2)(1 - i\omega\tau_\pi) - \frac{ik^2\omega}{P_0}\eta = 0, \quad \omega(1 - i\omega\tau_V) + \frac{4ik^2}{n_0}\kappa = 0.$$

- The shear and diffusion modes are:

$$\begin{aligned} \omega_a^\pm &= \pm|k|c_{s;a} - i\xi_a, \qquad \omega_\eta = -i\xi_\eta; \qquad \omega_\kappa^\pm = -i\xi_\kappa^\pm, \\ c_{s;a} &\simeq \frac{1}{\sqrt{3}}, \quad \xi_a \simeq \frac{k^2\eta}{6P_0}, \quad \xi_\eta \simeq \frac{1}{\tau_\pi} - \frac{k^2\eta}{3P_0}, \\ \xi_\kappa^- &\simeq \frac{4k^2\kappa}{n_0}, \qquad \xi_\kappa^+ \simeq \frac{1}{\tau_V} - \frac{4k^2\kappa}{n_0}. \end{aligned} \tag{38}$$

Shakhov model: κ vs. η



- Setting $\tau_R = \tau_\pi$ for definiteness, the Shakhov distribution becomes

$$f_{S\mathbf{k}} = f_{0\mathbf{k}} \left[1 + \frac{k_\mu V^\mu}{P}(\beta E_{\mathbf{k}} - 5)\left(1 - \frac{\tau_\pi}{\tau_V}\right)\right]. \qquad (39)$$

- At initial time, $n(0, z) = n_0 + \delta n_0 \cos(kz)$ and $v(0, z) = \delta v_0 \sin(kz)$.
- The approximate solution is

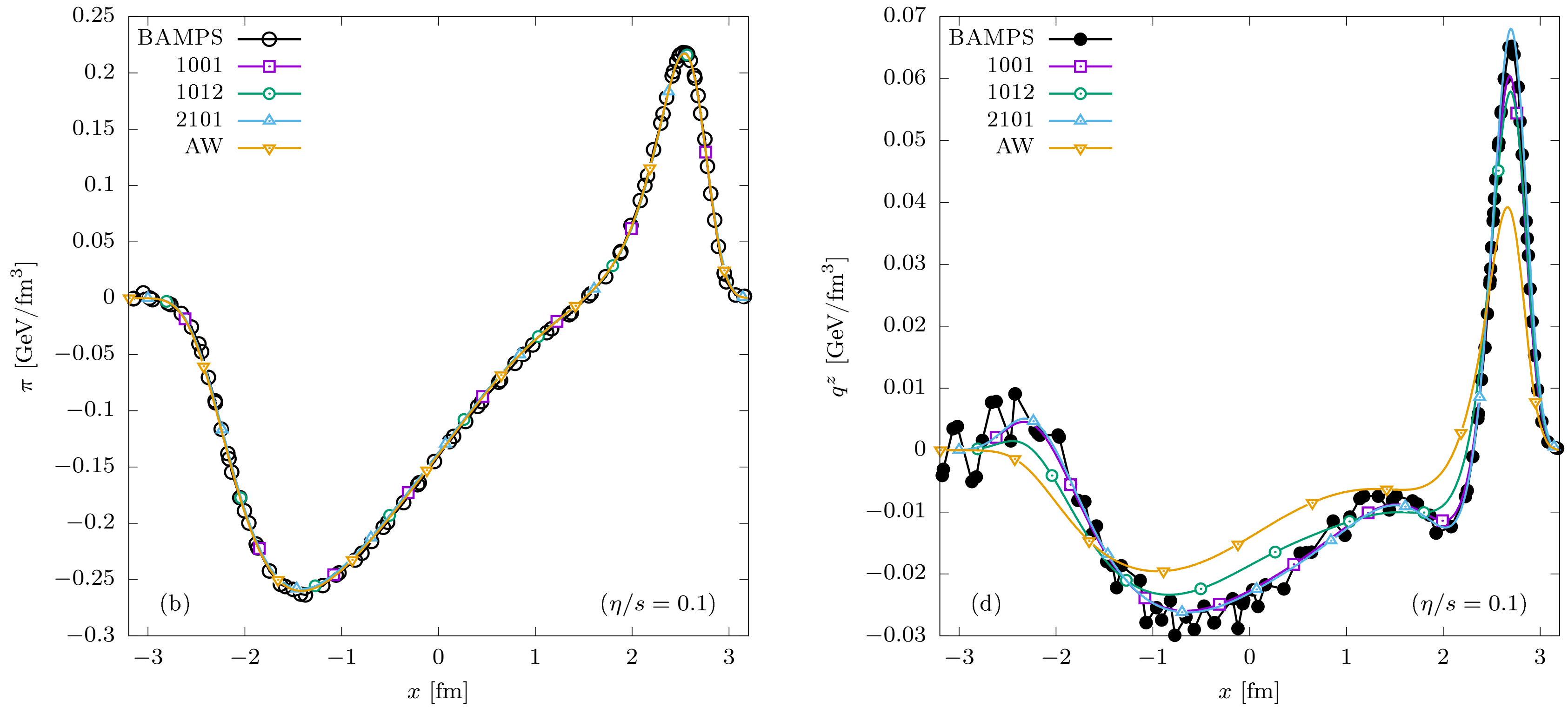
[Ambruș, PRC **97** (2018) 024914.]

$$\widetilde{\delta V} \simeq \frac{4k\kappa\delta n_0}{\tau_V n_0} \frac{e^{-\xi_\kappa^+ t} - e^{-\xi_\kappa^- t}}{\xi_\kappa^+ - \xi_\kappa^-},$$

$$\widetilde{\delta\pi} \simeq -\frac{4\eta}{3}\delta v_0 \left\{e^{-\xi_a t}\left[\cos(kc_s t) - \frac{\xi_a}{kc_s}\sin(kc_s t)\right] - e^{-t/\tau_\pi}\right\}. \qquad (40)$$

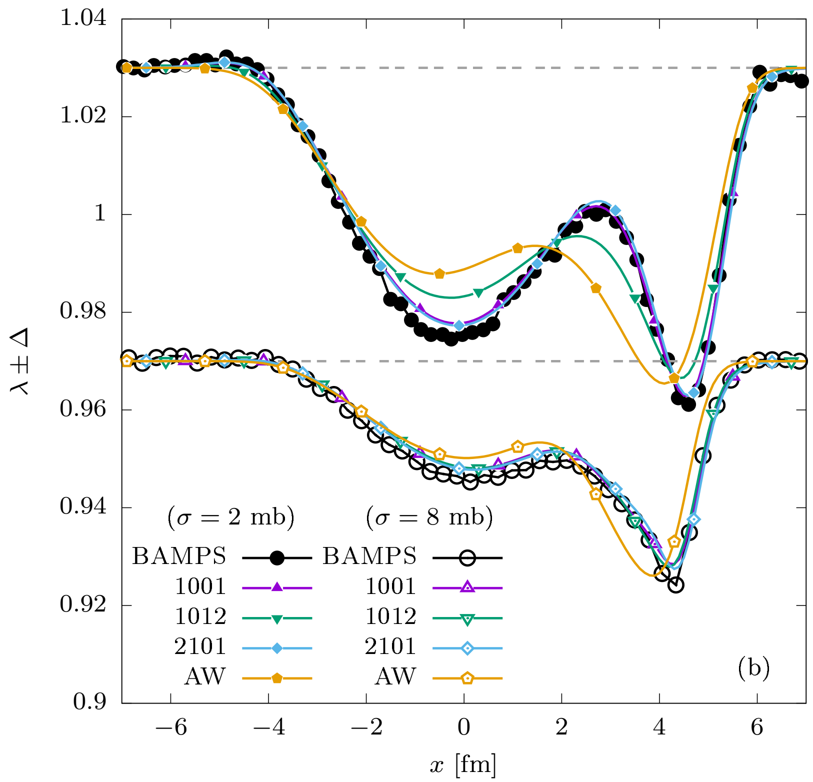
Sod shock tube: Comparison to BAMPS

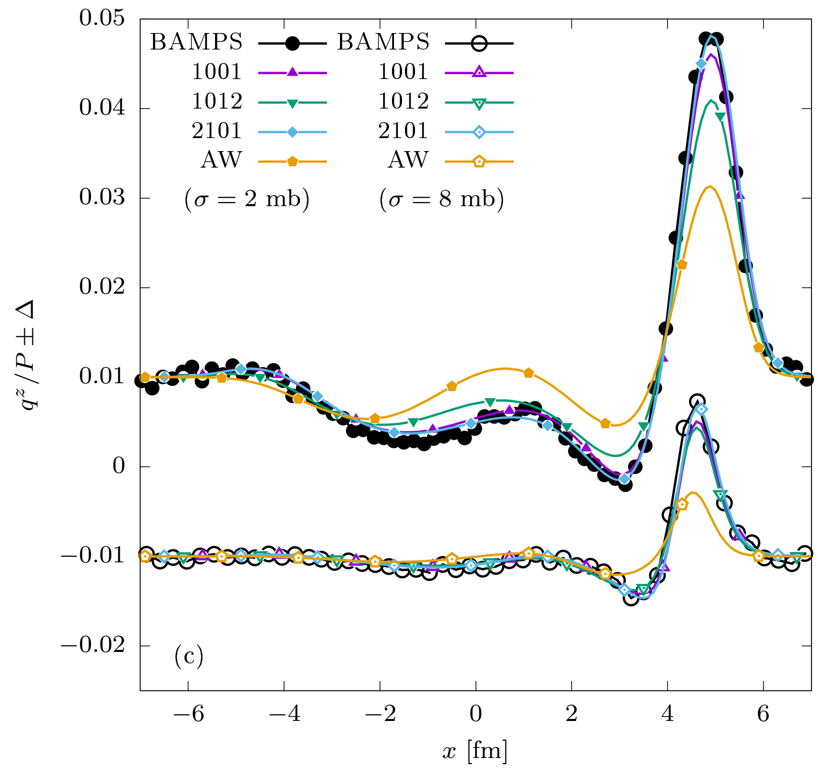
[Bouras *et al*, PRC **82** (2010) 024910]



- In the frame of the Sod shock tube, we considered a comparison to BAMPS for hard-sphere interactions.
- Using τ_R to tune η, shear comes out well with AW and Shakhov.
- For diffusion: $1001 \equiv$ first-order Shakhov underestimates peak.
- Higher-order (2101) Shakhov required to tune $2^{\rm nd}$ order t. coeffs.

Sod shock tube: Comparison to BAMPS [DNBMXRG, PRD **89** (2014) 074005]





- In the heat-flow problem (const. initial λ, pressure jump), again higher-order 2101 Shahkov required.

Conclusions

- Shakhov model generalized for the relativistic Anderson-Witting RTA, allowing ζ, κ and η to be controlled independently.
- Numerical simulations of the Bjorken flow and of sound waves damping confirmed that the model is robust.
- Extending the Shakhov model allows $2^{\rm nd}$-order t. coeffs. to be controlled $\Rightarrow$ agreement with BAMPS in Sod shock tube.
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Appendix

Arbitrary Shakhov matrix

- The model can be extended to control $2^{\rm nd}$-order transport coeffs..
- Systematic extensions can be obtained by writing in general

$$\mathbb{S}_{\mathbf{k}} = \sum_{\ell=0}^{\infty} \sum_{n=-s_\ell}^{N_\ell} \rho_{\mathrm{S};n}^{\mu_1\cdots\mu_\ell} E_{\mathbf{k}}^{-s_\ell} k_{\langle\mu_1} \cdots k_{\mu_\ell\rangle} \widetilde{\mathcal{H}}_{\mathbf{k},n+s_\ell}^{(\ell)}, \tag{41}$$

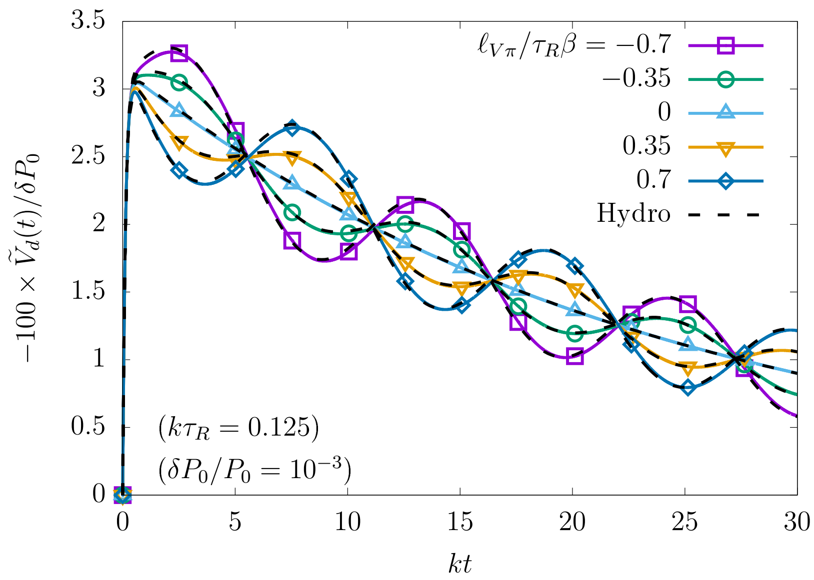
where $N_\ell \equiv$ expansion order and $s_\ell \equiv$ basis-shift allowing to access negative-order moments.
- The Shakhov irreducible moments are taken as

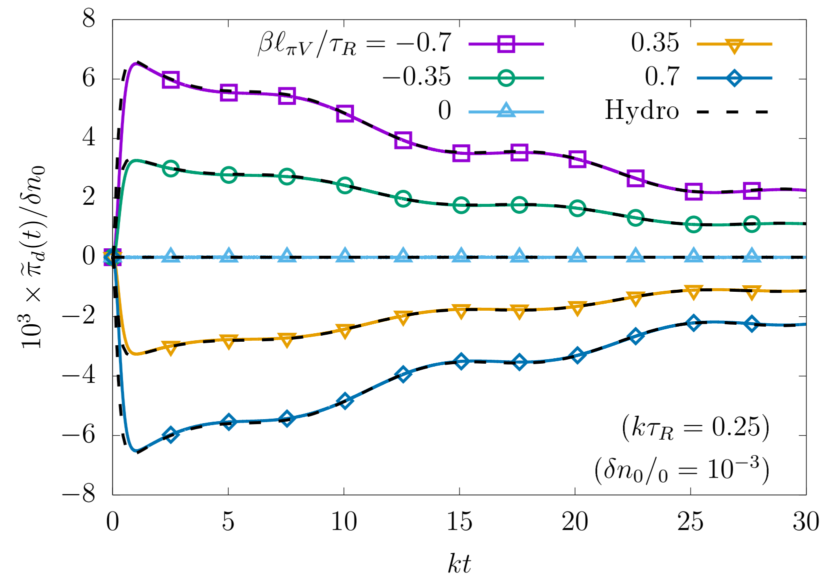
$$\rho_{\mathrm{S};r}^{\mu_1\cdots\mu_\ell} = \sum_{n=-s_\ell}^{N_\ell} \left(\delta_{rn} - \tau_R \mathcal{A}_{\mathrm{S};rn}^{(\ell)}\right) \rho_n^{\mu_1\cdots\mu_\ell}. \tag{42}$$

with arbitrary entries $\mathcal{A}_{\mathrm{S};rn}^{(\ell)}$ defined for $-s_\ell \le r, n \le N_\ell$.
- The irreducible moments $C_{\mathrm{S};r-1}^{\mu_1\cdots\mu_\ell}$ of the collision term can be written as

$$C_{\mathrm{S};r-1}^{\mu_1\cdots\mu_\ell} = -\sum_n \mathcal{A}_{rn}^{(\ell)} \rho_n^{\mu_1\cdots\mu_\ell}, \quad \mathcal{A}_{rn}^{(\ell)} = \begin{pmatrix} \frac{1}{\tau_R}\delta_{rn} & \mathcal{A}_{<;rn}^{(\ell)} & 0 \\ 0 & \mathcal{A}_{\mathrm{S};rn}^{(\ell)} & 0 \\ 0 & \mathcal{A}_{>;rn}^{(\ell)} & \frac{1}{\tau_R}\delta_{rn} \end{pmatrix}. \tag{43}$$

$(N_1, N_2, s_1, s_2) = (1, 0, 0, 1)$ model





- We consider a simple extension of the tenso matrix to cover the $r = -1$ row.
- Setting $\mathcal{A}_S^{(1)} = 1/\tau_V$ and

$$\mathcal{A}_S^{(2)} = \frac{1}{\tau_\pi H(H + L_{V\pi}L_{\pi V})} \begin{pmatrix} H - L_{\pi V} & \frac{\beta}{4}(HL_{V\pi} + L_{\pi V}) \\ -\frac{4}{\beta}L_{\pi V} & H + L_{\pi V} \end{pmatrix}, \tag{44}$$

allows $\ell_{V\pi}$ and $\ell_{\pi V}$ to be controlled independently via

$$L_{V\pi} = \frac{4}{\beta\tau_V}\ell_{V\pi}, \qquad L_{\pi V} = \frac{5\beta}{8\tau_\pi}\ell_{\pi V}, \qquad H = \frac{5\eta}{4\tau_\pi P}, \tag{45}$$