Rest frame vacua of massive scalar fields on spatially flat FLRW spacetimes

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Introduction

The main difficulty in defining scalar modes on curved manifolds is to find the criterion of separating the frequencies defining the particle and antiparticle modes and, implicitly, the current vacuum at a given time.

The principal method used so far is to focus mainly on the asymptotic states whose behavior is similar to the usual Minkowskian particle and antiparticle mode functions.

Another challenge is to solve the ambiguity related to the normalization of these Minkowskian states, which can be done either with respect to the scalar product of the curved manifold or by using the Minkowskian scalar product.

This difficulty can be avoided since the state spaces of different geometries are separable Hilbert spaces which are isometric among themselves.

Here we discuss a new type of vacuum called the rest frame vacuum (r.f.v.).

Scalar states on curved spacetimes

On a given curved manifold (M, g) the quantum states are **prepared or measured** by a **global apparatus** formed by the algebra of the quantum observables, i.e. Hermitian operators defined globally as vector fields on the whole manifold.

The operators proportional with the Killing vector fields are **conserved**, commuting with the operator of the field equation.

The global apparatus prepares quantum modes whose mode functions are common eigenfunctions of a system of commuting conserved operators (s.c.c.o.) which includes the operator of the field equation.

In addition, these mode functions are supposed to be normalized with respect to a specific relativistic scalar product on (M, g).

We consider the $(1+3)$-dimensional local- Minkowskian manifold (M,g) equipped with a local chart $\{x\}$ of coordinates x^μ (labeled by natural indices $\alpha,...\mu,...=0,1,2,3$) with $x^0=t$ and arbitrary space coordinates.

The scalar field, $\Phi: M \to \mathbb{C}$, of mass m, minimally coupled to the gravity of (M,g), satisfies the Klein Gordon equation

$$\frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}\,g^{\mu\nu}\partial_\nu\Phi\right)+m^2\Phi=0\,,\quad g=|\mathrm{det}g_{\mu\nu}|\,, \tag{1}$$

and the other eigenvalue problems determining partially or completely (when the s.c.c.o. is complete) the scalar quantum mode functions $f_{\vec{p}}$ giving the expansion

$$\Phi(x)=\int d^3p\,\left[f_{\vec{p}}(x)\mathfrak{a}(\vec{p})+f^*_{\vec{p}}(x)\mathfrak{a}^c(\vec{p})^\dagger\right]\,, \tag{2}$$

in terms of particle, $\mathfrak{a},\mathfrak{a}^\dagger$, and antiparticle, $\mathfrak{a}^c,\mathfrak{a}^{c\,\dagger}$, field operators which satisfy the canonical commutation relations.

In general, the mode functions, $f \in \mathcal{K}$, behave as tempered distributions or square integrable functions with respect to the indefinite Hermitian form

$$\langle f, f' \rangle_M = i \int_\Sigma d\sigma^\mu \sqrt{g}\, f^* \overleftrightarrow{\partial}_\mu f' = i \int_{\mathbb{R}^3} d^3x\, g^{00} \sqrt{g}\, f^* \overleftrightarrow{\partial}_t f' \in \mathbb{C}\,, \quad (3)$$

written with the notation $f \overleftrightarrow{\partial} f' = f\partial f' - f'\partial f$. The square integrable functions $f \in \mathcal{H} \subset \mathcal{K}$ may have 'squared norms' $\langle f, f \rangle_m$ of any sign splitting thus the space $\mathcal{K}$ as

$$f \in \begin{cases} \mathcal{H}_+ \subset \mathcal{K}_+ & \text{if } \langle f, f \rangle_M > 0\,, \\ \mathcal{H}_0 \subset \mathcal{K}_0 & \text{if } \langle f, f \rangle_M = 0\,, \\ \mathcal{H}_- \subset \mathcal{K}_- & \text{if } \langle f, f \rangle_M < 0\,. \end{cases} \quad (4)$$

From the physical point of view the mode functions of $\mathcal{K}_\pm$ are of positive/negative frequencies while those of $\mathcal{K}_0$ do not have a physical meaning.

For any $f \in \mathcal{K}_+$ we have $f^* \in \mathcal{K}_-$ so that $\langle f^*, f^* \rangle_M = -\langle f, f \rangle_M$ but whether $f^* = f$ then $f \in \mathcal{F}_0$, since $\langle f, f \rangle_M = 0$.

In fact, $\mathcal{H}$ is a Krein space while $\mathcal{K}_\pm$ are the spaces of tempered distributions of the Hilbertian triads associated to the Hilbert spaces $\mathcal{H}_\pm$ equipped with the scalar products $\pm\langle\ ,\ \rangle_M$.

In general, the s.c.c.o. determining the quantum modes is not complete such that the mode functions are determined up to some integration constants which depend on the separation of the positive and negative frequencies defining thus the vacuum state of the Fock space.

A complete system of orthonormal mode functions, $\{f_\alpha\}_{\alpha\in I} \subset \mathcal{K}_+$ forms a (generalized) basis of positive frequencies in $\mathcal{K}_+$ related to the negative frequencies one, $\{f^*_\alpha\}_{\alpha\in I} \subset \mathcal{K}_-$.

On says that different bases define different vacuum states. These bases are related among themselves through Bogolyubov transformations that may mixe the positive and negative frequency modes.

Measuring states from another geometry

Another manner of setting the integration constants is by defining the modes on (M, g) in which one measures the parameters corresponding to another geometry (M', g'), according to the method we present here.

We consider another manifold $(\hat{M}, \hat{g})$ whose local chart $\{\hat{x}\}$ is defined on the **same domain** of the flat model as the chart $\{x\}$ of (M, g) such that there exists the coordinate transformation $\hat{x} = \chi(x)$.

We can relate the set $\mathcal{K}$ to the set $\hat{\mathcal{K}}$ of the scalar mode functions on $(\hat{M}, \hat{g})$ equipped with the Hermitian form $\langle\ ,\ \rangle_{\hat{M}}$, defined as in Eq. (3). Since the physical parts of the sets $\hat{\mathcal{K}}$ and $\mathcal{K}$ are separable Hilbert spaces we can define the isometry $\mu : \mathcal{H}_+ \to \hat{\mathcal{H}}_+$ which satisfies

$$\langle \mu(f), \mu(f') \rangle_{\hat{M}} = \langle f, f' \rangle_M \,. \qquad (5)$$

Then for any normalized mode functions $f_\alpha \in \mathcal{H}_+$ and $\hat{f}_\beta \in \hat{\mathcal{H}}_+$ which satisfy

$$\langle f_\alpha, f_\alpha \rangle_M = \langle \hat{f}_\beta, \hat{f}_\beta \rangle_{\hat{M}} = 1\,. \tag{6}$$

we can construct the amplitude

$$\langle \alpha | \beta \rangle_t = \langle \mu(f_\alpha), \hat{f}_\beta \rangle_{\hat{M}} \Big|_t = \langle f_\alpha, \mu^{-1}(\hat{f}_\beta) \rangle_M \Big|_t \,, \tag{7}$$

which, in general, depends on time.

This gives the quantity $|\langle \alpha | \beta \rangle_t|^2$ which can be interpreted as the probability of measuring at the time t the parameters β in the state α prepared on (M, g) or reversely as the probability of measuring the parameters α in the state β prepared on $(\hat{M}, \hat{g})$.

We say that $\mu(f) \in \hat{\mathcal{K}}$ is the projection of $f \in \mathcal{K}$.

The isometry μ involves the coordinate transformation $\hat{x} = \chi(x)$ but which can be eliminated by choosing the same coordinates for the both manifolds under consideration by taking $\chi = id \to \hat{x} = x$.

Note that this is possible since we assumed that the local charts of (M,g) and $(\hat{M},\hat{g})$ are in the same domain of the flat model.

With this choice the isometry takes the simple form

$$\mu(f)=\left(\frac{g^{00}\sqrt{g}}{\hat{g}^{00}\sqrt{\hat{g}}}\right)^{\frac{1}{2}} f\,, \tag{8}$$

that can be used in applications.

Minkowskian states

When $(\hat{M},\hat{g})$ is just the Minkowski spacetime then we can set at any time $\chi=id$ defining in (M,g) states in which one measures exclusively Minkowskian parameters at a given time t_0.

For any normalized mode function $\hat{f}\in\hat{\mathcal{K}}$ on the Minkowski spacetime we may define the corresponding Mikowskian state on (M,g) whose normalized mode function $f\in\mathcal{K}$ is defined such that the functions,

$$\mu(f)=\left(g^{00}\sqrt{g}\right)^{\frac{1}{2}} f\,, \tag{9}$$

and $\hat{f}$ have a **contact** of order k at the time t_0, satisfying the system of $k+1$ algebraic equations,

$$\begin{aligned} \mu(f)(t_0) &= \hat{f}(t_0)\,, \\ \frac{d\mu(f)}{dt}(t_0) &= \frac{d\hat{f}}{dt}(t_0)\,, \\ &\vdots \\ \frac{d^k\mu(f)}{dt^k}(t_0) &= \frac{d^k\hat{f}}{dt^k}(t_0)\,, \end{aligned} \tag{10}$$

able to give all the integration constants of f in terms of the Minkowskian parameters of the function $\hat{f}$ we chose.

The number $k+1$ of equations depends on the number of the undetermined integration constants or other parameters we need to find out.

With this method we can apply the definitions of Minkowskian particles or antiparticles to any manifold (M,g) but only at a given time since, in general, these states are evolving in time.

FLRW spacetimes

We consider the family of $(1+3)$-dimensional spatially flat FLRW spacetimes, M, for which we use the **same coordinates** of the FLRW chart, $\{t, \vec{x}\}$, i. e. the proper (or cosmic) time $t \in D_t$ and the Cartesian space coordinates $\vec{x} = (x^1, x^2, x^3) \in \mathbb{R}^3$.

The line element on M depends on the scale factor $a(t)$ which is a smooth function on D_t giving the conformal time

$$t_c = \int \frac{dt}{a(t)} \in D_{t_c}\,, \tag{11}$$

of the conformal chart $\{t_c, \vec{x}\}$. The line elements of these charts are

$$ds^2 = dt^2 - a(t)^2 d\vec{x}\cdot d\vec{x} = a(t_c)^2\left(dt^2 - d\vec{x}\cdot d\vec{x}\right)\,, \tag{12}$$

where we denoted $a(t_c) = a[t(t_c)]$. The Minkowski spacetime, denoted from now simply as $\hat{M}$, is the particular case when $a(t) = 1$ and $t_c = t$.

In the chart $\{t, \vec{x}\}$ the massive scalar field $\Phi : M \to \mathbb{C}$ of mass m satisfies the Klein-Gordon equation

$$\left(\partial_t^2 + \frac{3\dot{a}(t)}{a(t)}\,\partial_t - \frac{1}{a(t)^2}\,\Delta + m^2\right)\Phi(t, \vec{x}) = 0\,, \tag{13}$$

which allows a system of plane wave solutions, i. e. eigenfunctions of the momentum operators $P_i = -i\partial_i$ corresponding to the eigenvalues (p_1, p_2, p_3) representing the components of the conserved momentum $\vec{p}$.

These mode functions can be written as

$$f_{\vec{p}}(t, \vec{x}) = \frac{e^{i\vec{x}\cdot\vec{p}}}{[2\pi a(t)]^{\frac{3}{2}}}\mathcal{F}_p(t)\,, \tag{14}$$

in terms of the time modulation functions $\mathcal{F}_p : D_t \to \mathbb{C}$ which depend on $p = |\vec{p}|$ satisfying the oscillator-type equation

$$\frac{d^2\mathcal{F}_p(t)}{dt^2} + \Omega_p(t)^2\mathcal{F}_p(t) = 0\,, \tag{15}$$

whose time dependent frequency reads

$$\Omega_p(t) = \left[\frac{p^2}{a(t)^2} + m^2 - \frac{3}{2}\frac{\ddot{a}(t)}{a(t)} - \frac{3}{4}\frac{\dot{a}(t)^2}{a(t)^2}\right]^{\frac{1}{2}} . \tag{16}$$

The fundamental solutions (14) form an orthonormal basis with respect to the scalar product (3) that now reads

$$\begin{aligned} \langle f, f' \rangle_M &= i \int_{\mathbb{R}^3} d^3x\, a(t)^3 \,(f^* \overleftrightarrow{\partial}_t\, f') \\ &= i \int_{\mathbb{R}^3} d^3x\, a(t_c)^2 \,(f^* \overleftrightarrow{\partial}_{t_c}\, f')\,, \end{aligned} \tag{17}$$

allowing us to impose the normalization condition

$$\delta^3(\vec{p} - \vec{p}') = \langle f_{\vec{p}}, f_{\vec{p}'} \rangle_M = \delta^3(\vec{p} - \vec{p}')i\, \mathcal{F}_p^*(t) \overleftrightarrow{\partial}_t\, \mathcal{F}_p(t)\,, \tag{18}$$

requiring the time modulation functions to satisfy

$$\Big(\mathcal{F}_p, \mathcal{F}_p\Big) = i\, \mathcal{F}_p^*(t) \overleftrightarrow{\partial}_t\, \mathcal{F}_p(t) = 1\,. \tag{19}$$

The most general form of the time modulation function is

$$\mathcal{F}_p(t) = c_1\phi_p(t) + c_2\phi_p^*(t)\,, \tag{20}$$

where ϕ_p is a particular solution satisfying

$$\Big(\phi_p, \phi_p\Big) = 1 \quad \rightarrow \quad \Big(\phi_p^*, \phi_p^*\Big) = -1\,. \tag{21}$$

such that a normalized solution of positive frequency, $f_{\vec{p}} \in \mathcal{K}_+$, must have a time modulation function which satisfies

$$\Big(\mathcal{F}_p, \mathcal{F}_p\Big) = 1 \quad \rightarrow \quad |c_1|^2 - |c_2|^2 = 1\,. \tag{22}$$

Then $\mathcal{F}_p^*$ which satisfies $\Big(\mathcal{F}_p^*, \mathcal{F}_p^*\Big) = -1$ and $\Big(\mathcal{F}_p^*, \mathcal{F}_p\Big) = 0$ is the time modulation function of $f_{\vec{p}}^* \in \mathcal{K}_-$.

Note that the quantity (f, f) defined by Eq. (19) is independent on time only when f is a solution of Eq. (15).

Rest frame vacuum

In the particular case of the Minkowski spacetime $\hat{M}$ the mode functions of positive frequencies of a scalar field of mass $\hat{m}$

$$\hat{f}_{\vec{p}}(t,\vec{x}) = \frac{e^{i\vec{x}\cdot\vec{p}}}{[2\pi]^{\frac{3}{2}}}\,\hat{\mathcal{F}}_p(t)\,, \quad \hat{\mathcal{F}}_p(t) = \frac{1}{\sqrt{2E}}\,e^{-iEt}\,, \tag{23}$$

are eigenfunctions of the energy operator $H_0 = i\partial_t$ depending on the conserved energy $E = \sqrt{p^2+\hat{m}^2}$ and satisfying the orthonormalization condition with respect to the scalar product

$$\langle \hat{f}, \hat{f}' \rangle_{\hat{M}} = i\int_{\mathbb{R}^3} d^3x\, \hat{f}^* \overset{\leftrightarrow}{\partial_t}\, \hat{f}'\,. \tag{24}$$

We have shown that in any FLRW spacetime there exists an **energy operator** that in the FLRW chart, $\{t,\vec{x}\}$, has the form [38, 39]

$$H = i\partial_t + \frac{\dot{a}(t)}{a(t)}\,\vec{x}\cdot\vec{P}\,. \tag{25}$$

In general, this operator does not commute with the momentum operator $\vec{P}$ but in the rest frames (where $\vec{p}=0$) this coincides with the Minkowski one, $H|_{\vec{p}=0} = H_0 = i\partial_t$.

Thus we may separate the frequencies by using the Minkowskian rest states on M defined in the previous section but with another rest energy, $\hat{m} \neq m$, we call here the **dynamical mass**.

Therefore, we may consider the system (10) with $k=2$ giving the following equations

$$\lim_{p\to 0}\left[\mathcal{F}_p(t)-\hat{\mathcal{F}}_p(t)\right]\Big|_{t=t_0} = 0\,, \tag{26}$$

$$\lim_{p\to 0}\frac{d}{dt}\left[\mathcal{F}_p(t)-\hat{\mathcal{F}}_p(t)\right]\Bigg|_{t=t_0} = 0\,, \tag{27}$$

$$\lim_{p\to 0}\frac{d^2}{dt^2}\left[\mathcal{F}_p(t)-\hat{\mathcal{F}}_p(t)\right]\Bigg|_{t=t_0} = 0\,. \tag{28}$$

which are enough for separating the frequencies in the rest frame and finding the dynamical mass $\hat{m}(t_0)$.

We start with the function (20) for $p = 0$ denoting simply $\phi = \phi_p|_{p=0}$ for which we obtain the system

$$c_1\phi(t_0) + c_2\phi^*(t_0) = \frac{1}{\sqrt{2\hat{m}}}e^{-i\hat{m}t_0}\,, \tag{29}$$

$$c_1\dot{\phi}(t_0) + c_2\dot{\phi}^*(t_0) = -i\hat{m}\frac{1}{\sqrt{2\hat{m}}}e^{-i\hat{m}t_0}\,, \tag{30}$$

$$c_1\ddot{\phi}(t_0) + c_2\ddot{\phi}^*(t_0) = -\hat{m}^2\frac{1}{\sqrt{2\hat{m}}}e^{-i\hat{m}t_0}\,. \tag{31}$$

The first two equations give the normalized integration constants corresponding to the r.f.v. while the third one gives us the associated dynamical mass in the rest frame as,

$$\hat{m} = \hat{m}(t_0) = \lim_{p\to 0}\Omega_p(t_0) \equiv \Omega(t_0)\,, \tag{32}$$

since $\ddot{\phi} = -\Omega^2\phi$ as it results from Eq. (15). Thus we obtain the final result

$$c_1 \;\to\; c_1(t_0) = \frac{e^{-i\Omega(t_0)t_0}}{\sqrt{2\Omega(t_0)}}\left(\Omega(t_0)\phi^*(t_0) - i\dot{\phi}^*(t_0)\right)\;, \tag{33}$$

$$c_2 \;\to\; c_2(t_0) = \frac{e^{-i\Omega(t_0)t_0}}{\sqrt{2\Omega(t_0)}}\left(-\Omega(t_0)\phi(t_0) + i\dot{\phi}(t_0)\right)\;, \tag{34}$$

which complies with the normalization condition

$$|c_1(t_0)|^2 - |c_2(t_0)|^2 = \begin{cases} 1 \;\text{if}\; \Omega(t_0)^2 > 0 \\ 0 \;\text{if}\; \Omega(t_0)^2 < 0 \end{cases}\;. \tag{35}$$

A particle prepared in r.f.v. at the time t_0 has the mode function

$$f_{\vec{p},t_0}(t,\vec{x}) = \frac{e^{i\vec{x}\cdot\vec{p}}}{[2\pi a(t)]^{\frac{3}{2}}}\mathcal{F}_p(t_0,t)\,, \tag{36}$$

whose time modulation function reads

$$\mathcal{F}_p(t_0,t) = c_1(t_0)\phi_p(t) + c_2(t_0)\phi_p^*(t)\,. \tag{37}$$

The set $\{f_{\vec{p},t_0}|\vec{p}\in\mathbb{R}^3\}$ forms a basis in $\mathcal{K}_+$ while the set $\{f^*_{\vec{p},t_0}|\vec{p}\in\mathbb{R}^3\}$ is the corresponding basis of $\mathcal{K}_-$ in the r.f.v. prepared at $t=t_0$.

In general, the r.f.v. is dynamic, being associated with a time-dependent dynamical mass $\hat{m}(t)=\Omega(t)\in\mathbb{R}$.

The time domain $D_t = D_t^+\cup D_t^-$ is split in the tardyonic part $D_t^+ = \{t|\Omega(t)^2>0\}$ and the tachyonic one $D_t^- = \{t|\Omega(t)^2<0\}$.

All the tachyonic states with $\Omega(t) = i|\Omega(t)|$ are eliminated as having null norms. Thus in r.f.v. the scalar field survives only on D_t^+.

This vacuum becomes stable on the FLRW manifolds where the energy operator is conserved satisfying

$$[H_0,\Omega]=i\partial_t\Omega=0\,, \tag{38}$$

Only the Minkowski and de Sitter spacetimes comply with this condition having stable r.f.v..

Bogolyubov transformations

According to the previous results, the scalar field

$$\Phi(x) = \int d^3p \left[f_{\vec{p},t_0}(x)\mathfrak{a}(\vec{p},t_0) + f^*_{\vec{p},t_0}(x)\mathfrak{a}^c(\vec{p},t_0)^\dagger \right] , \tag{39}$$

may be expanded in terms of the mode functions (36), with the time modulation functions (37).

The field operators depend on t_0 defining the vacuum state as

$$\mathfrak{a}(\vec{p},t_0)\,|t_0\rangle = 0\,, \quad \langle t_0|\,\mathfrak{a}^\dagger(\vec{p},t_0) = 0\,. \tag{40}$$

The particles created at t_0' can be measured at any moment $t_0 > t_0'$ in the new vacuum state $|t_0\rangle$. These vacua correspond to the different bases, $\{f_{\vec{p},t_0}\}$ and $\{f_{\vec{p},t_0'}\}$, giving the Bogolyubov transformations:

$$\mathfrak{a}(\vec{p},t_0) = \int d^3p' \left[\langle f_{\vec{p},t_0}, f_{\vec{p}',t_0'}\rangle \mathfrak{a}(\vec{p}', t_0') + \langle f_{\vec{p},t_0}, f^*_{\vec{p}',t_0'}\rangle \mathfrak{a}^c(\vec{p}', t_0')^\dagger\right]$$

$$= \alpha(t_0,t_0')\mathfrak{a}(\vec{p},t_0') + \beta(t_0,t_0')\mathfrak{a}^c(\vec{p},t_0')^\dagger \,, \tag{41}$$

$$\mathfrak{a}^c(\vec{p},t_0) = \int d^3p' \left[\langle f_{\vec{p},t_0}, f_{\vec{p}',t_0'}\rangle \mathfrak{a}^c(\vec{p}', t_0') + \langle f_{\vec{p},t_0}, f^*_{\vec{p}',t_0'}\rangle \mathfrak{a}(\vec{p}', t_0')^\dagger\right]$$

$$= \alpha(t_0,t_0')\mathfrak{a}^c(\vec{p},t_0') + \beta(t_0,t_0')\mathfrak{a}(\vec{p},t_0')^\dagger \,, \tag{42}$$

where

$$\langle f_{\vec{p},t_0}, f_{\vec{p}',t}\rangle = \delta^3(\vec{p}-\vec{p}')\alpha(t_0,t_0') \,, \tag{43}$$

$$\langle f_{\vec{p},t_0}, f^*_{\vec{p}',t}\rangle = \delta^3(\vec{p}-\vec{p}')\beta(t_0,t_0'), \tag{44}$$

with

$$\alpha(t_0,t_0') = \left[c_1^*(t_0)c_1(t_0') - c_2^*(t_0)c_2(t_0')\right] \,, \tag{45}$$

$$\beta(t_0,t_0') = \left[c_2^*(t_0)c_1^*(t_0') - c_1^*(t_0)c_2^*(t_0')\right] \,, \tag{46}$$

The constants $c_{1,2}(t_0)$ are given by Eqs. (33) and (34). Moreover, we must have

$$\left|\alpha(t_0, t_0')\right|^2 - \left|\beta(t_0, t_0')\right|^2 = 1\,. \tag{47}$$

According to the standard interpretation, the density of the new particles or antiparticles created between t_0' and t_0 is proportional to,

$$n(t_0, t) \propto \left|\beta(t_0, t_0')\right|^2 = \left|c_2(t_0)c_1(t_0') - c_1(t_0)c_2(t_0')\right|^2\,. \tag{48}$$

In addition, we observe that the rate of c.p.c. can also be estimated as

$$R(t_0, t) \propto \frac{d\,n(t_0, t)}{dt}\,. \tag{49}$$

In our opinion, this is a rough interpretation that could be reafined by studying the unitary evolution operator

$$\mathfrak{U}(t_0, t_0') = |t_0\rangle\langle t_0'|\,. \tag{50}$$

Example I: de Sitter expanding universe

The first example is of a stable r.f.v. on the expanding portion of the de Sitter spacetime, M. The scale factor $a(t) = e^{2\omega t}$ gives the conformal time t_c and the function $a(t_c)$ as

$$t_c = -\frac{1}{\omega} e^{-\omega t} \in (-\infty, 0] \,, \quad a(t_c) = -\frac{1}{\omega t_c} \,. \tag{51}$$

In the conformal chart the mode functions of the momentum basis have the form (14) with the time modulation functions

$$\mathcal{F}_p(t_c) = \; c_1 \phi_p(t) + c_2 \phi_p^*(t) \,, \quad \phi_p(t) = \frac{1}{\sqrt{\pi\omega}} K_\nu(ipt_c) \,, \tag{52}$$

where

$$\nu = \begin{cases} \sqrt{\frac{9}{4} - \mu^2} & \text{for } \; \mu < \frac{3}{2} \\ i\kappa \,, \quad \kappa = \sqrt{\mu^2 - \frac{9}{4}} & \text{for } \; \mu > \frac{3}{2} \end{cases} \,, \quad \mu = \frac{m}{\omega} \,. \tag{53}$$

By using Eq. (A.4) we find that the normalization condition (19) is fulfilled only if we take

$$|c_1|^2 - |c_2|^2 = 1\,, \tag{54}$$

such that we remain with an undetermined integration constant which has to be fixed by setting the r.f.v. according to Eqs. (26), (27) and (28).

We assume first that $m > \frac{3}{2}\omega$ solving this system in the conformal chart $\{t_c, \vec{x}\}$ where the de Sitter time modulation function has the form (52) with $\nu = i\kappa$ while the Minkowski one (23) takes the form

$$\hat{\mathcal{F}}[t(t_c)] = \frac{(-\omega t_c)^{\frac{iE}{\omega}}}{\sqrt{2E}}\,. \tag{55}$$

Moreover, since in this case the limit to $p \to 0$ is sensitive, we solve first this system for $p \neq 0$ and then we evaluate this limit. From the first two equations we obtain the integration constants

$$\begin{aligned} c_1(p) &= \frac{(-\omega t_c)^{\frac{iE}{\omega}}}{\sqrt{2\pi\omega E}}\left[\omega p t_c K_{i\kappa+1}(-ipt_c)\right. \\ &\qquad \left. +(E-\kappa\omega)K_{i\kappa}(-ipt_c)\right] , \end{aligned} \tag{56}$$

$$\begin{aligned} c_2(p) &= -\frac{(-\omega t_c)^{\frac{iE}{\omega}}}{\sqrt{2\pi\omega E}}\left[\omega p t_c K_{i\kappa+1}(ipt_c)\right. \\ &\qquad \left. +(E-\kappa\omega)K_{i\kappa}(ipt_c)\right] , \end{aligned} \tag{57}$$

that substituted in Eq. (28) give the equation

$$\lim_{p\to 0}\left[E^2-\kappa^2\omega^2-\omega^2p^2t_c^2\right] = (\hat{m}^2-\omega^2\kappa^2) = 0\,, \tag{58}$$

determining the expected dynamical mass

$$\hat{m} = \omega\kappa = \sqrt{m^2-\frac{9}{4}\,\omega^2}\,, \tag{59}$$

related to the well-known rest energy [39].

Then for $p \to 0$ we obtain the constants $c_1 = \lim_{p\to 0} c_1(p)$ and $c_2 = \lim_{p\to 0} c_2(p)$ which have the absolute values

$$|c_1| = \frac{e^{\pi\kappa}}{\sqrt{e^{2\pi\kappa}-1}}\,, \tag{60}$$

$$|c_2| = \frac{1}{\sqrt{e^{2\pi\kappa}-1}}\,, \tag{61}$$

resulted from Eqs. (A.1) and (A.6). Finally, by substituting these values in Eq. (52), we obtain the definitive result

$$\mathcal{F}_p(t_c) = \sqrt{\frac{\pi}{\omega}}\left(\frac{p}{2\omega}\right)^{-i\kappa}\frac{I_{i\kappa}(ipt_c)}{\sqrt{e^{2\pi\kappa}-1}}\,, \tag{62}$$

where the general phase factor was introduced for assuring the correct limit for $p \to 0$ as given by Eq. (A.6). These functions are correctly normalized since the integration constants (60) and (61) satisfy the condition (54). Note that these results can be rewritten in terms of the cosmic time t according to Eq. (51).

Furthermore, we consider the case of $m < \frac{3}{2}\omega$ applying the same method for fixing the r.f.v.. We solve first Eqs. (26) and (27) for $p \neq 0$ and an arbitrary time t_c obtaining

$$\begin{aligned} c_1(p,t_c) &= \frac{(-\omega t_c)^{\frac{iE}{\omega}}}{\sqrt{2\pi\omega E}}\left[(E+i\nu\omega)K_\nu(-ipt_c)\right. \\ &\qquad \left. -\omega p t_c K_{\nu+1}(-ipt_c)\right] \,, \end{aligned} \tag{63}$$

$$\begin{aligned} c_2(p,t_c) &= \frac{(-\omega t_c)^{\frac{iE}{\omega}}}{\sqrt{2\pi\omega E}}\left[(E+i\nu\omega)K_\nu(ipt_c)\right. \\ &\qquad \left. +\omega p t_c K_{\nu+1}(ipt_c)\right] \,. \end{aligned} \tag{64}$$

Substituting these results in Eq. (28) we find the condition

$$\lim_{p\to 0}\left[E^2+\nu^2\omega^2-\omega^2p^2t_c^2\right] = (\hat{m}^2+\omega^2\nu^2) = 0\,, \tag{65}$$

giving the tachyonic dynamical mass $\hat{m} = \pm i\nu\omega$. Moreover, we have the surprise to find that in the rest frame we have

$$\lim_{p\to 0} c_1(p,t_c) = \lim_{p\to 0} c_2(p,t_c) = 0\,, \tag{66}$$

which means that if we set the r.f.v. then the particles with $m < \frac{3}{2}\omega$ cannot survive on the de Sitter expanding portion.

The above results can be now gathered in the synthetic form of the mode functions of positive frequency in the conformal chart,

$$f_{\vec{p}}(t_c, \vec{x}) = \left(\frac{-\omega t_c}{2\pi}\right)^{\frac{3}{2}} \sqrt{\frac{\pi}{\omega}} \left(\frac{p}{2\omega}\right)^{-\nu} \frac{I_\nu(ipt_c)\, e^{i\vec{p}\cdot\vec{x}}}{\sqrt{e^{-2i\pi\nu} - 1}}\,, \tag{67}$$

since whether ν, given by Eq. (53), takes real values then the squared norm of $f_{\vec{p}}$ vanishes as $I_\nu(-ipt_c) \stackrel{\leftrightarrow}{\partial_{t_c}} I_\nu(ipt_c) \propto I_\nu(ipt_c) \stackrel{\leftrightarrow}{\partial_{t_c}} I_\nu(ipt_c) = 0$.

Thus we have shown that the scalar r.f.v. on the de Sitter expanding universe is stable corresponding to a time-independent dynamical mass (59) which does make sense only when $m > \frac{3}{2}\omega$.

Otherwise we have either to eliminate the scalar fields with $m < \frac{3}{2}\omega$ or to resort to another vacuum as the adiabatic Bunch-Davies one [37] which can be set for particles of any mass by taking $c_1 = 1$ and $c_2 = 0$.

Example II: Milne-type universe

An example of manifold M without an adiabatic vacuum is the $(1+3)$-dimensional spatially flat FLRW manifold with the scale factor $a(t) = \omega t$ determining the conformal time as

$$t_c = \int \frac{dt}{a(t)} = \frac{1}{\omega}\ln(\omega t) \in (-\infty, \infty) \;\to\; a(t_c) = e^{\omega t_c}\,. \tag{68}$$

The constant ω is a free parameter.

In the case of the genuine Milne's universe (of negative space curvature) this must be fixed to $\omega = 1$ for eliminating the gravitational sources [37].

This spacetime is produced by isotropic gravitational sources, i. e. the density ρ and pressure p, evolving in time as

$$\rho = \frac{3}{8\pi G}\frac{1}{t^2}\,, \quad p = -\frac{1}{8\pi G}\frac{1}{t^2}\,. \tag{69}$$

These sources govern the expansion of M that can be better observed in the chart $\{t,\vec{\hat{x}}\}$, of 'physical' space coordinates $\hat{x}^i=\omega t x^i$, where the line element

$$ds^2=\left(1-\frac{1}{t^2}\vec{\hat{x}}\cdot\vec{\hat{x}}\right)dt^2+2\vec{\hat{x}}\cdot d\vec{\hat{x}}\,\frac{dt}{t}-d\vec{\hat{x}}\cdot d\vec{\hat{x}}\,, \tag{70}$$

lays out an expanding horizon at $|\vec{\hat{x}}|=t$ and tends to the Minkowski spacetime when $t\to\infty$ and the gravitational sources vanish.

In the FLRW chart $\{t,\vec{x}\}$ of this spacetime the Klein-Gordon equation is analytically solvable giving time modulation functions as

$$\mathcal{F}_p(t)=c_1\phi_p(t)+c_2\phi^*_p(t)\,,\quad \phi_p(t)=\sqrt{\frac{t}{\pi}}\,K_\nu(imt) \tag{71}$$

where

$$\nu=\sqrt{1-\frac{p^2}{\omega^2}}\,, \tag{72}$$

can take real or pure imaginary values for $p>\omega$. The integration constants which must satisfy the normalization condition $|c_1|^2-|c_2|^2=1$, have to be determined by fixing the vacuum. here are no adiabatic vacua since the functions (71) are singular in $t=0$.

Therefore, we must focus only on the r.f.v. solving first the Eqs. (26) and (27) for $p=0$ and proper time $t>0$ when we separate the frequencies in the rest frame. We obtain the time-dependent integration constants

$$c_1(t) = \frac{e^{-it\hat{m}}}{2\sqrt{2t\hat{m}}}\left[2tmK_0(-imt) + (i+2t\hat{m}))K_1(-imt)\right] , \quad (73)$$

$$c_2(t) = \frac{e^{-it\hat{m}}}{2\sqrt{2t\hat{m}}}\left[2tmK_0(imt) - (i+2t\hat{m}))K_1(imt)\right] , \quad (74)$$

which have to be substituted in Eq. (28) for obtaining a simple equation giving the time-dependent dynamical mass

$$\hat{m}(t) = \sqrt{m^2 - \frac{3}{4\,t^2}}\,. \quad (75)$$

The functions (73) and (74) are singular in $t=0$ and $t=t_m \equiv \frac{\sqrt{3}}{2m}$ when $\hat{m}(t)$ vanishes (as in Fig. 1). From Eq. (75) we see that a particle of mass

m has a tachyonic behavior in the domain $D_t^- = (0, t_m)$ and a tardyonic one only if $t \in D_t^+ = (t_m, \infty)$. As in the general case we can verify that

$$|c_1(t)|^2 - |c_2(t)|^2 = \begin{cases} 0 \text{ if } 0 < t < t_m \\ 1 \text{ if } t > t_m \end{cases} \tag{76}$$

showing that on the tachyonic domain the wave function is of null norm having thus no physical meaning.

This means that the scalar particles can be prepared only in the tardyonic domain $t > t_m$ where $\hat{m}(t)$ increases with t such that for $t \to \infty$, when M becomes just the Minkowski spacetime, this tends to m.

In this limit we recover the usual Minkowski scalar modes since the functions K behave as in Eq. (A.5) such that

$$\lim_{t\to\infty} |c_1(t)| = 1 \,, \quad \lim_{t\to\infty} |c_2(t)| = 0 \,. \tag{77}$$

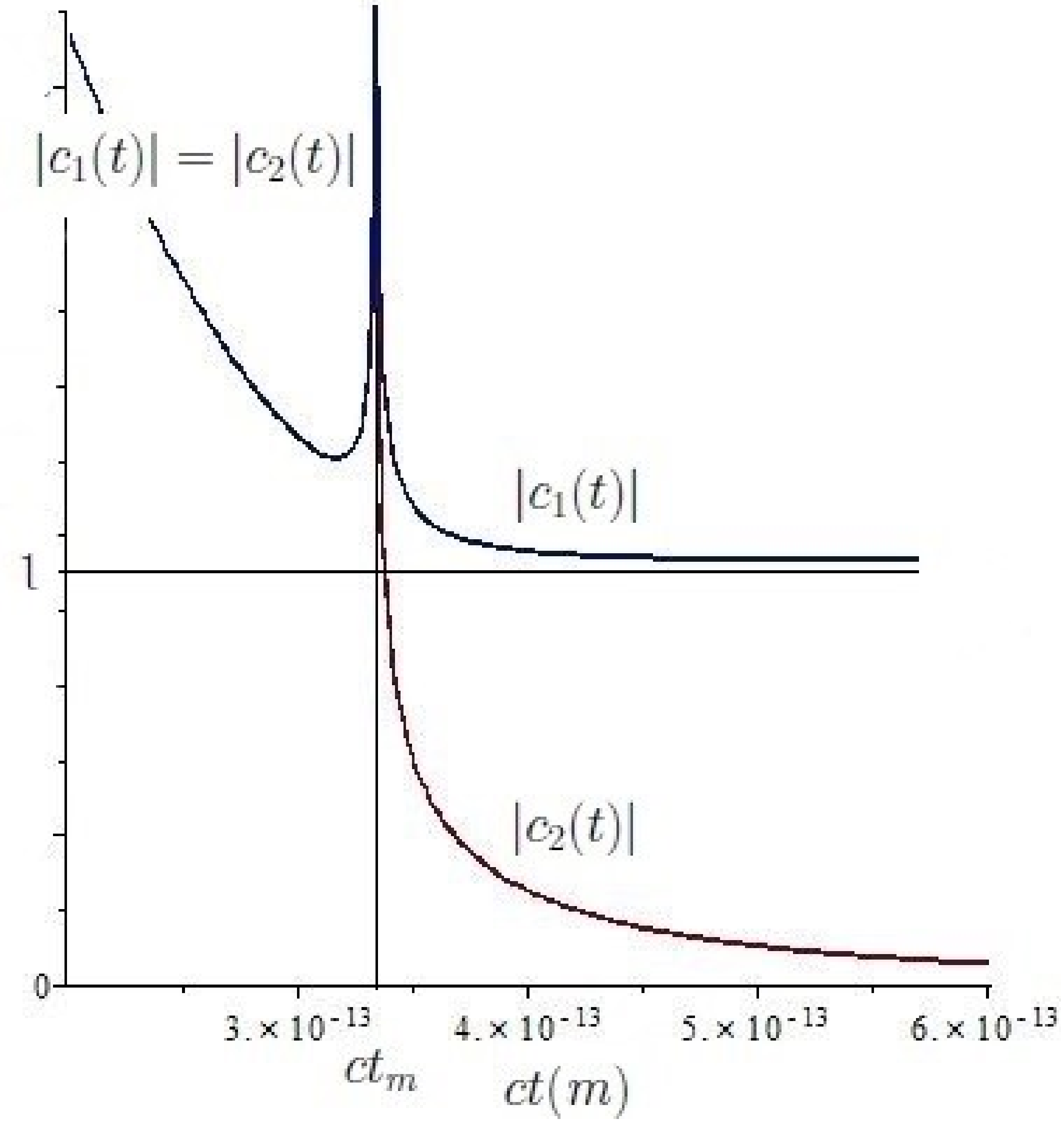

Figure 1: The functions $|c_1(t)|$ and $|c_2(t)|$ versus ct for a light particle having the electron mass, $m = m_e$, for which $t_{m_e} \sim 1.1\,10^{-21}s$. The plotting domain is $0.6\,t_{m_e} < t < 1.8\,t_{m_e}$.

The instability of r.f.v. on this expanding manifold give rise to c.p.c. that can be analyzed thanks to our previous results that hold for any $t > t_0$.

We can study how the particles created at t_0 can be measured at any moment $t > t_0$ by using the Bogolyubov coefficients between the bases $f_{\vec{p}}(t_0)$ and $f_{\vec{p}}(t)$ derived before.

Thus we see that the effects of the dynamic r.f.v. tend to stability when the time is increasing since then

$$\lim_{t\to\infty} n(t_0,t) \sim |c_2(t_0)|^2 \,, \quad \lim_{t\to\infty} R(t_0,t) = 0 \,, \tag{78}$$

as we deduce from Eqs. (77).

Therefore, the dynamical effect is visible only for the very old particles, prepared at $t_0 < 5\,t_m$, since the function $c_2(t_0)$ decreases rapidly to zero when t_0 increases and $\hat{m}(t_0) \to m$.

For the younger particles, prepared at $t_0 > 5 - 10\, t_m$, this is inhibited remaining with an apparently stable r.f.v. of the Bunch-Davies type (with $c_1 = 1$ and $c_2 = 0$) in which the mode functions can be approximated as

$$f_{\vec{p}}(t, \vec{x}) \sim \frac{e^{i\vec{p}\cdot\vec{x}}}{(2\pi\omega t)^{\frac{3}{2}}} \sqrt{\frac{t}{\pi}}\, K_\nu(imt)\,, \tag{79}$$

independent on the moment t_0 when the particle was prepared.

The dynamic effect is very fast, during an extremely short period of time, even at quantum scale, since by definition $t_m = \frac{\sqrt{3}}{2m}$ (or $\frac{\sqrt{3}}{2}\frac{\hbar}{mc^2}$ in SI units) is very small.

For example, if we take $m = m_e$ (the electron mass) then $t_{m_e} \sim 1.1\, 10^{-21} s$ such that for the particles born at cosmic times $t_0 > 10^{-20} s$ the r.f.v. is apparently stable. Only the particles prepared at $t_0 < 10^{-20} s$ lay out this effect as in Figs. 2 and 3 where we plot the functions (48) and (49) versus ct instead of t for avoiding too small numbers.

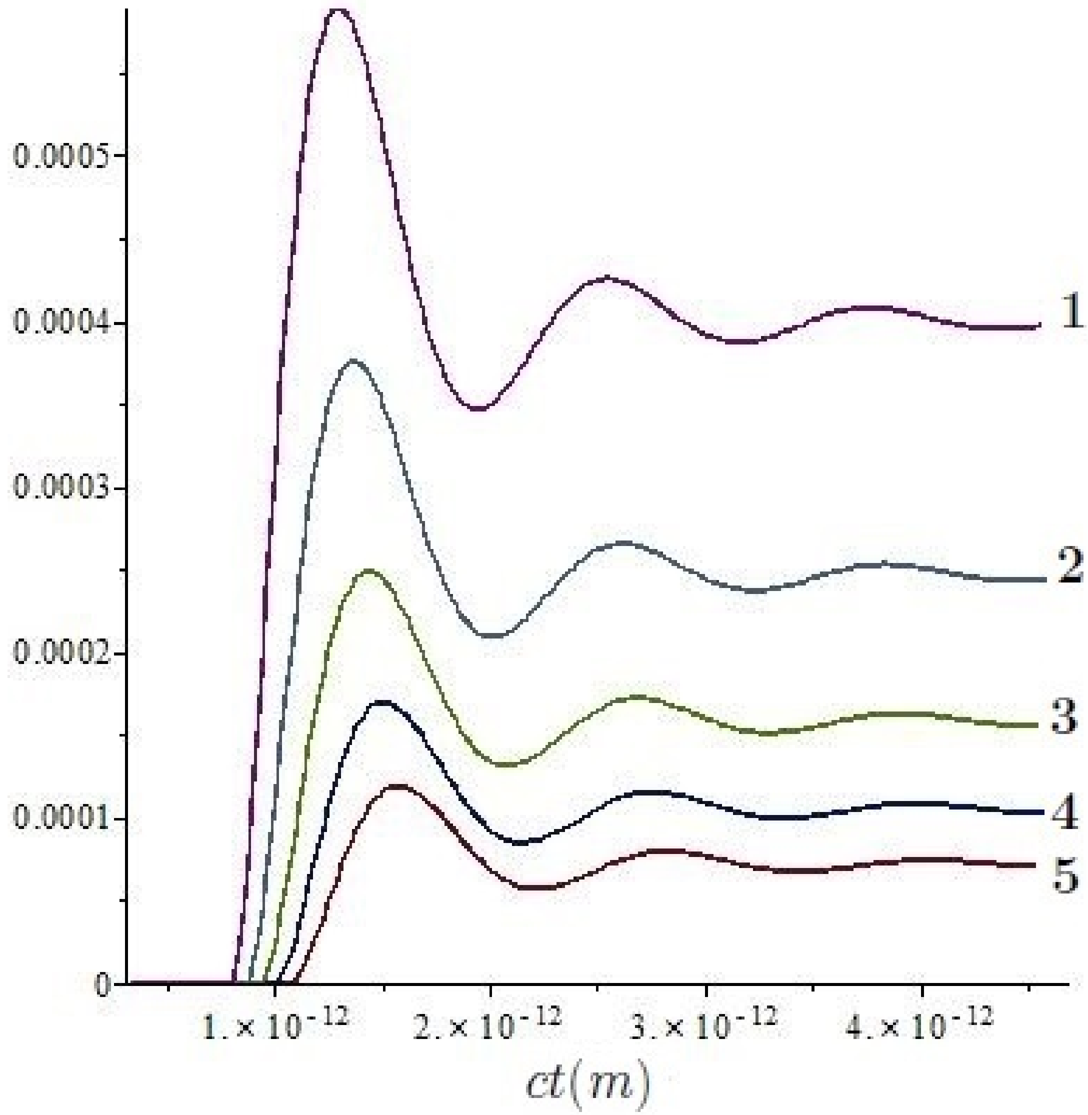

Figure 2: The function $n(t_0, t)$ versus ct in the domain $t_m < t < 14\,t_m$ for $m = m_e$ and: $t_0 = 2.4\,t_m$ (1), $t_0 = 2.6\,t_m$ (2), $t_0 = 2.8\,t_m$ (3), $t_0 = 3\,t_m$ (4), $t_0 = 3.2\,t_m$ (5).

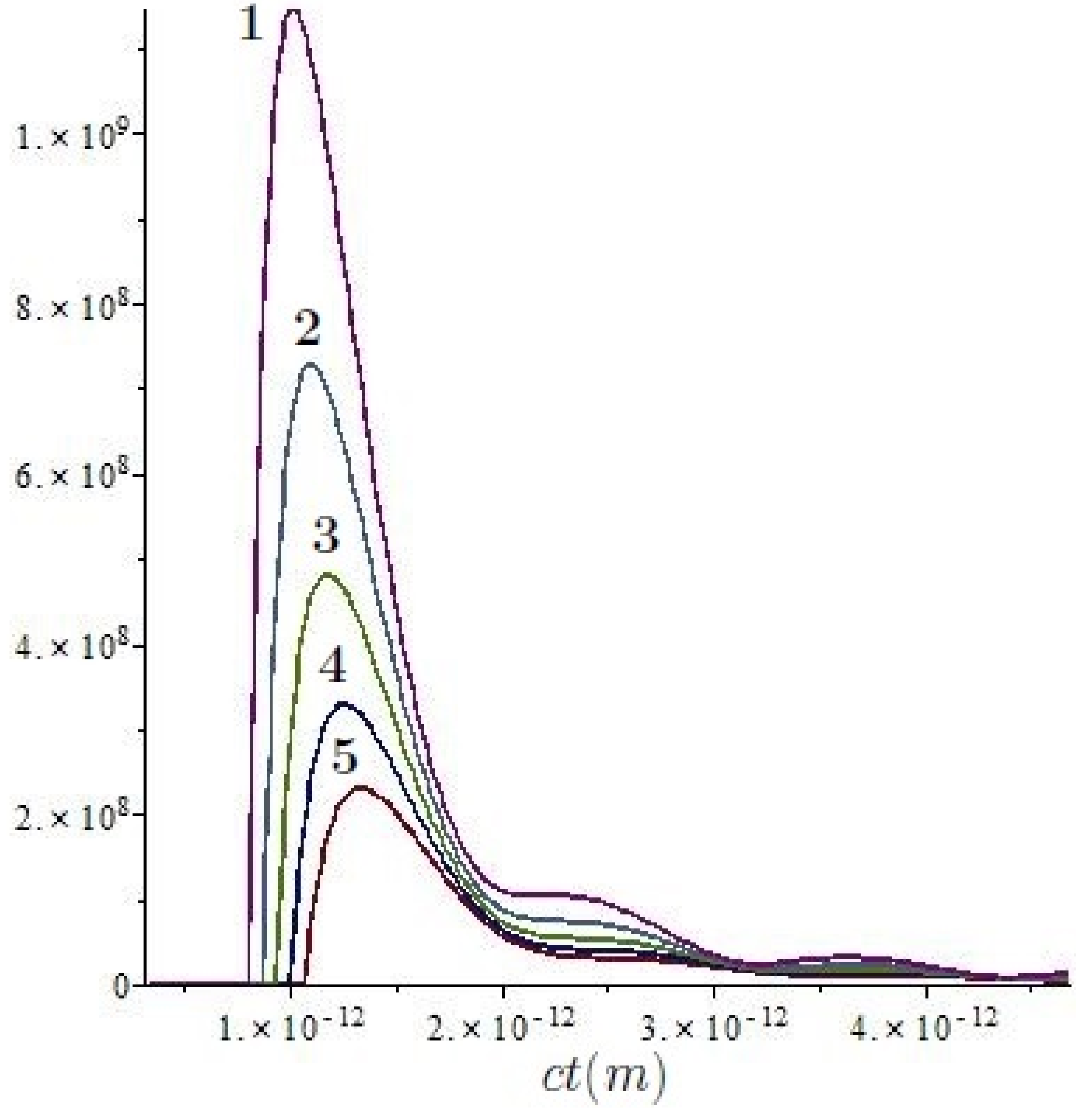

Figure 3: The function $R(t_0, t)$ versus ct in the domain $t_m < t < 14\,t_m$ for $m = m_e$ and: $t_0 = 2.4\,t_m$ (1), $t_0 = 2.6\,t_m$ (2), $t_0 = 2.8\,t_m$ (3), $t_0 = 3\,t_m$ (4), $t_0 = 3.2\,t_m$ (5).

Conclusions

By setting the r.f.v. we obtain a new world where;

1. The cosmological particle creation is forbidden on the de Sitter expanding universe where the r.f.v. is stable. Consequently all the vacuum effects are inhibed on this manifold.

2. This can arise only on the spatially flat FLRW manifolds where the energy is not conserved such that the dynamical mass as well as the vacuum state depend on time. The only example we have so far is the above Milne-type spacetime.

3. The dynamical effect is very fast even at the quantum scale, affecting the old particles. This diminishes for younger particles dissapearing for the recently prepared particles for which the r.f.v. tends to stability.

4. The tachyonic states are naturally eliminated having null norms.

Appendix: Modified Bessel functions

The modified Bessel functions $I_\nu(z)$ and $K_\nu(z)$ are related as [51]

$$K_\nu(z) = K_{-\nu}(z) \;=\; \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin\pi\nu}\,, \tag{A.1}$$

$$I_{\pm\nu}(z) = e^{\mp i\pi\nu} I_{\pm\nu}(-z) \;=\; \frac{i}{\pi}\left[K_\nu(-z) - e^{\mp i\pi\nu}K_\nu(z)\right]\;. \tag{A.2}$$

Their Wronskians give the identities we need for normalizing the mode functions. For $\nu = i\mu$ we obtain

$$iI_{i\mu}(is)\stackrel{\leftrightarrow}{\partial_s} I_{-i\mu}(is) = \frac{2\sinh\pi\mu}{\pi s}\,, \tag{A.3}$$

while the identity

$$iK_\nu(-is)\stackrel{\leftrightarrow}{\partial_s} K_\nu(is) = \frac{\pi}{|s|}\,, \tag{A.4}$$

holds for any ν.

For $|z| \to \infty$ and any ν we have,

$$I_\nu(z) \to \sqrt{\frac{\pi}{2z}} e^z\,, \quad K_\nu(z) \to K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}\,. \tag{A.5}$$

In the limit of $|z| \to 0$ the functions I_ν behave as

$$I_\nu(z) \sim \frac{1}{\Gamma(\nu+1)} \left(\frac{z}{2}\right)^\nu\,, \tag{A.6}$$

while for the functions K_ν we have to use Eq. (A.1).

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