

Rest frame vacuum of the Dirac field on FLRW spacetimes

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Contents

Introduction	3
Dirac's field on FLRW spacetimes	5
Fundamental spinors	9
Adiabatic and rest frame vacua	13
Example I: The Minkowski spacetime	17
Example II: de Sitter expanding universe	18
Example III: Milne-type spacetime	21
Concluding remarks	24
Appendix: Modified Bessel functions	26

Introduction

In general relativity, the problem of defining the vacuum is related to the separation of the positive and negative frequencies of the free quantum fields in the absence of a conserved energy operator commuting with the field equation.

The Dirac field was studied mainly in the local charts with spherical coordinates of the FLRW spacetimes where its time evolution is governed by a pair of time modulation functions that, in general, cannot be solved analytically since they satisfy equations of oscillators with variable frequencies [7–11].

Nevertheless, in the de Sitter case we succeeded to solve the Dirac equation in momentum-helicity [12, 13] or momentum-spin [14] representations, separating the frequencies according to the general prescription of the adiabatic vacuum, useful in some applications [15–17].

Sometime the adiabatic vacuum cannot be defined properly as in the case of the fermions in the accelerated frames where the asymptotic behavior is not relevant. Then the frequencies separation is performed for Gaussian packets localized in the accelerated cavities which play thus the role of rest frames [18–20]. This special conjecture rises the general question concerning the role of the rest frames in separating frequencies.

This problem can be studied in the momentum-spin representation which offers us the opportunity of analyzing the fundamental spinors in the rest frame where the momentum vanishes. We observed that in this limit the fundamental spinors defined according to the adiabatic vacuum do not coincide with the normalized spinors calculated directly in the rest frames [14].

This suggests us that a new vacuum must be introduced in order to assure a complete separation between particles and antiparticles in the rest frames.

Dirac's field on FLRW spacetimes

Let us consider the $(1 + 3)$ -dimensional spatially flat FLRW manifolds, M , where we choose the comoving local FLRW charts $\{t, \mathbf{x}\}$ whose coordinates x^μ (labeled by the natural indices $\mu, \nu, \dots = 0, 1, 2, 3$) are formed by the proper (or cosmic) time, t , and Cartesian space coordinates, x^i ($i, j, k \dots = 1, 2, 3$), for which we may use the vector notation $\mathbf{x} = (x^1, x^2, x^3)$. The FLRW geometry is given by a smooth scale factor $a(t)$ defining the conformal time as,

$$t_c = \int \frac{dt}{a(t)} \rightarrow a(t_c) = a[t(t_c)]. \quad (1)$$

and determining the line elements,

$$\begin{aligned} ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu &= dt^2 - a(t)^2 d\vec{x} \cdot d\vec{x} \\ &= a(t_c)^2 (dt_c^2 - d\vec{x} \cdot d\vec{x}), \end{aligned} \quad (2)$$

of the FLRW chart and of the associated conformal flat charts, $\{t_c, \mathbf{x}\}$.

In these charts, we chose the diagonal tetrad gauge in which the vector fields $e_{\hat{\alpha}} = e_{\hat{\alpha}}^{\mu} \partial_{\mu}$ defining the local orthogonal frames, and the 1-forms $\omega^{\hat{\alpha}} = \hat{e}_{\mu}^{\hat{\alpha}} dx^{\mu}$ of the dual coframes (labeled by the local indices, $\hat{\mu}, \hat{\nu}, \dots = 0, 1, 2, 3$) are defined as

$$e_0 = \partial_t = \frac{1}{a(t_c)} \partial_{t_c}, \quad \omega^0 = dt = a(t_c) dt_c, \quad (3)$$

$$e_i = \frac{1}{a(t)} \partial_i = \frac{1}{a(t_c)} \partial_i, \quad \omega^i = a(t) dx^i = a(t_c) dx^i, \quad (4)$$

in order to preserve the global $SO(3)$ symmetry allowing us to use systematically the $SO(3)$ vectors.

In this tetrad-gauge, the massive Dirac field ψ of mass m satisfies the field equations $(D_x - m)\psi(x) = 0$ given by the Dirac operator

$$D_x = i\gamma^0 \partial_t + i \frac{1}{a(t)} \gamma^i \partial_i + \frac{3i \dot{a}(t)}{2 a(t)} \gamma^0. \quad (5)$$

It is known that the term depending on the Hubble function $\frac{\dot{a}}{a}$ can be removed at any time by substituting $\psi \rightarrow [a(t)]^{-\frac{3}{2}} \psi$. Similar results can be written in the conformal chart.

The general solution of the Dirac equation may be written as a mode integral,

$$\begin{aligned}\psi(t, \mathbf{x}) &= \psi^{(+)}(t, \mathbf{x}) + \psi^{(-)}(t, \mathbf{x}) \\ &= \int d^3p \sum_{\sigma} [U_{\mathbf{p},\sigma}(x)\mathbf{a}(\mathbf{p}, \sigma) + V_{\mathbf{p},\sigma}(x)\mathbf{b}^{\dagger}(\mathbf{p}, \sigma)],\end{aligned}\quad (6)$$

in terms of the fundamental spinors $U_{\mathbf{p},\sigma}$ and $V_{\mathbf{p},\sigma}$ of positive and respectively negative frequencies which are plane waves solutions of the Dirac equation depending on the conserved momentum \mathbf{p} and an arbitrary polarization σ .

These spinors form an orthonormal basis being related through the charge conjugation,

$$V_{\mathbf{p},\sigma}(t, \mathbf{x}) = U_{\mathbf{p},\sigma}^c(t, \mathbf{x}) = C [U_{\mathbf{p},\sigma}(t, \mathbf{x})]^*, \quad C = i\gamma^2, \quad (7)$$

(see the Appendix A), and satisfying the orthogonality relations

$$\langle U_{\mathbf{p},\sigma}, U_{\mathbf{p}',\sigma'} \rangle = \langle V_{\mathbf{p},\sigma}, V_{\mathbf{p}',\sigma'} \rangle = \delta_{\sigma\sigma'} \delta^3(\mathbf{p} - \mathbf{p}') \quad (8)$$

$$\langle U_{\mathbf{p},\sigma}, V_{\mathbf{p}',\sigma'} \rangle = \langle V_{\mathbf{p},\sigma}, U_{\mathbf{p}',\sigma'} \rangle = 0, \quad (9)$$

with respect to the relativistic scalar product [12]

$$\begin{aligned}\langle \psi, \psi' \rangle &= \int d^3x \sqrt{|g|} e_0^0 \bar{\psi}(x) \gamma^0 \psi(x) \\ &= \int d^3x a(t)^3 \bar{\psi}(x) \gamma^0 \psi(x),\end{aligned}\tag{10}$$

where $g = \det(g_{\mu\nu})$ and $\bar{\psi} = \psi^\dagger \gamma^0$ is the Dirac adjoint of ψ . Moreover, this basis is supposed to be complete accomplishing the completeness condition [12]

$$\begin{aligned}\int d^3p \sum_{\sigma} [U_{\mathbf{p},\sigma}(t, \mathbf{x}) U_{\mathbf{p},\sigma}^\dagger(t, \mathbf{x}') + V_{\mathbf{p},\sigma}(t, \mathbf{x}) V_{\mathbf{p},\sigma}^\dagger(t, \mathbf{x}')] \\ = a(t)^{-3} \delta^3(\mathbf{x} - \mathbf{x}').\end{aligned}\tag{11}$$

We obtain thus the orthonormal basis of the momentum representation in which the particle ($\mathbf{a}, \mathbf{a}^\dagger$) and antiparticle ($\mathbf{b}, \mathbf{b}^\dagger$) operators satisfy the canonical anti-commutation relations [12].

Fundamental spinors

In the standard representation of the Dirac matrices (with diagonal γ^0) the general form of the fundamental spinors in momentum representation,

$$U_{\vec{p},\sigma}(t, \vec{x}) = \frac{e^{i\vec{p}\cdot\vec{x}}}{[2\pi a(t)]^{\frac{3}{2}}} \begin{pmatrix} u_p^+(t) \xi_\sigma \\ u_p^-(t) \frac{p^i \sigma_i}{p} \xi_\sigma \end{pmatrix} \quad (12)$$

$$V_{\vec{p},\sigma}(t, \vec{x}) = \frac{e^{-i\vec{p}\cdot\vec{x}}}{[2\pi a(t)]^{\frac{3}{2}}} \begin{pmatrix} v_p^+(t) \frac{p^i \sigma_i}{p} \eta_\sigma \\ v_p^-(t) \eta_\sigma \end{pmatrix}, \quad (13)$$

is determined by the modulation functions $u_p^\pm(t)$ and $v_p^\pm(t)$ that depend only on t and $p = |\vec{p}|$. The Pauli spinors ξ_σ and $\eta_\sigma = i\sigma_2(\xi_\sigma)^*$ must be correctly normalized, $\xi_\sigma^+ \xi_{\sigma'} = \eta_\sigma^+ \eta_{\sigma'} = \delta_{\sigma\sigma'}$, satisfying the completeness condition

$$\sum_{\sigma} \xi_\sigma \xi_\sigma^+ = \sum_{\sigma} \eta_\sigma \eta_\sigma^+ = \mathbf{1}_{2 \times 2}. \quad (14)$$

In Ref. [12] we considered the Pauli spinors of the momentum-helicity basis whose direction of the spin projection is just that of the momentum \vec{p} . However, we can project the spin on an arbitrary direction, independent on \vec{p} , as in the case of the *spin* basis where the spin is projected on the third axis of the rest frame such that $\xi_{\frac{1}{2}} = (1, 0)^T$ and $\xi_{-\frac{1}{2}} = (0, 1)^T$ for particles and $\eta_{\frac{1}{2}} = (0, -1)^T$ and $\eta_{-\frac{1}{2}} = (1, 0)^T$ for antiparticles [14]. In what follows we work exclusively in this basis called the momentum-spin basis.

The modulation functions $u_p^\pm(t)$ and $v_p^\pm(t)$ can be derived by substituting Eqs. (12) and (13) in the Dirac equation. Then, after a few manipulation, we find the systems of the first order differential equations

$$a(t) (i\partial_t \mp m) u_p^\pm(t) = p u_p^\mp(t), \quad (15)$$

$$a(t) (i\partial_t \mp m) v_p^\pm(t) = -p v_p^\mp(t), \quad (16)$$

in the chart with the proper time or the equivalent system in the conformal chart,

$$[i\partial_{t_c} \mp m a(t_c)] u_p^\pm(t_c) = p u_p^\mp(t_c), \quad (17)$$

$$[i\partial_{t_c} \mp m a(t_c)] v_p^\pm(t_c) = -p v_p^\mp(t_c), \quad (18)$$

which govern the time modulation of the free Dirac field on any spatially flat FLRW manifold. Note that these equations are similar to those of the modulation functions of the spherical modes [7–11] but depending on different integration constants.

The solutions of these systems depend on integration constants that must be selected according to the charge conjugation (7) which gives the mandatory condition

$$v_p^\pm(t) = [u_p^\mp(t)]^* . \quad (19)$$

The remaining normalization constants can be fixed since the prime integrals of the systems (15) and (16), $\partial_t(|u_p^+|^2 + |u_p^-|^2) = \partial_t(|v_p^+|^2 + |v_p^-|^2) = 0$, allow us to impose the normalization conditions

$$|u_p^+|^2 + |u_p^-|^2 = |v_p^+|^2 + |v_p^-|^2 = 1 \quad (20)$$

which guarantee that Eqs. (8) and (9) are accomplished.

A special case is that of the rest frame where the Dirac equation in momentum-spin representation for $\mathbf{p} = 0$ can be solved analytically carrying out the normalized fundamental spinors of the rest frame,

$$U_{0,\sigma}(t, \mathbf{x}) = \frac{e^{-imt}}{[2\pi a(t)]^{\frac{3}{2}}} \begin{pmatrix} \xi_\sigma \\ 0 \end{pmatrix}, \quad (21)$$

$$V_{0,\sigma}(t, \mathbf{x}) = \frac{e^{imt}}{[2\pi a(t)]^{\frac{3}{2}}} \begin{pmatrix} 0 \\ \eta_\sigma \end{pmatrix}, \quad (22)$$

which depend on the rest energy $E_0 = m$. In the momentum-helicity representation the rest frame spinors cannot be defined since the helicity is related to a non-vanishing momentum.

Adiabatic and rest frame vacua

The modulation functions, $u_p^\pm(t)$ or $u_p^\pm(t_c)$, can be found by integrating the systems (15) or (17) in each particular case separately and imposing the charge conjugation and normalization conditions, (19) and respectively (20).

However, these conditions are not enough for determining completely these functions such that a supplemental physical hypothesis is required. This is just the criterion of separating the positive and negative frequencies defining thus the vacuum.

The vacuum usually considered in Dirac theories is the traditional adiabatic vacuum intensively studied in the case of the scalar fields [6]. This can be defined for any FLRW manifold for which the scale factor satisfies the condition

$$\lim_{t_c \rightarrow -\infty} a(t_c) = 0. \quad (23)$$

Then the asymptotic form of the system (17),

$$i\partial_{t_c} u_p^\pm(t_c) = p u_p^\mp(t_c), \quad (24)$$

gives the behavior of the modulation functions,

$$u_p^\pm(t_c) \sim c_1 e^{-ipt_c} \pm c_2 e^{ipt_c}, \quad (25)$$

for $t_c \rightarrow -\infty$. According to the common definition, the adiabatic vacuum is set when $c_2 = 0$ since then the modulation functions, $u_p^+(t_c) = u_p^-(t_c)$, describe a massless particle assumed to be of genuine positive frequency. Thus the general condition of selecting the adiabatic vacuum of the Dirac field on FLRW spacetimes takes the simple form

$$u_p^-(t_c, m) = u_p^+(t_c, -m) \quad (26)$$

and similarly for the functions $v_p^\pm(t_c)$.

The major difficulty of the adiabatic vacuum as defined above is that in the momentum-spin representation we cannot reach the rest frame limit.

Indeed, for $p \rightarrow 0$ the condition (26) gives the normalized functions

$$\lim_{p \rightarrow 0} u_p^+(t_c) = \frac{1}{\sqrt{2}} e^{-imt}, \quad \lim_{p \rightarrow 0} u_p^-(t) = \frac{1}{\sqrt{2}} e^{imt}, \quad (27)$$

while the limit of $\frac{p^i \sigma_i}{p}$ for $p \rightarrow 0$ is undetermined. Moreover, if we force this limit to zero we obtain a function u_p^+ with a different normalization factor [14]. Therefore, the corresponding limits of the fundamental spinors will differ from the correct rest spinors (21) and (22) mixing positive and negative frequencies.

The solution is to define a new vacuum able to separate the frequencies in any rest frame of the momentum-spin representation imposing the conditions

$$\lim_{\vec{p} \rightarrow 0} U_{\vec{p}, \sigma}(t, \mathbf{x}) = U_{0, \sigma}(t, \mathbf{x}), \quad (28)$$

$$\lim_{\vec{p} \rightarrow 0} V_{\vec{p}, \sigma}(t, \mathbf{x}) = V_{0, \sigma}(t, \mathbf{x}), \quad (29)$$

according to Eqs. (21) and (22). These are accomplished if we require the normalized modulation functions to satisfy

$$\lim_{p \rightarrow 0} u_p^-(t) = \lim_{p \rightarrow 0} v_p^+(t) = 0, \quad (30)$$

since then the contribution of the matrix $\frac{p^i \sigma_i}{p}$ is eliminated.

We say that these conditions define the *rest frame vacuum* which, in general, is different from the adiabatic one as we will see analyzing few examples.

Concluding we can say that the rest frame vacuum is completely defined by Eqs. (19) and (20) and the limits (28) and (29) for the massive Dirac field on any spatially flat FLRW spacetimes.

Example I: The Minkowski spacetime

This is the simplest example of flat manifold where $t_c = t$ and $a(t) = a(t_c) = 1$. The solutions of the systems (15) and (16) which satisfy the conditions (19) and (20) read

$$u_p^\pm(t) = \sqrt{\frac{E(p) \pm m}{2E(p)}} e^{-iE(p)t} \quad (31)$$

$$v_p^\pm(t) = \sqrt{\frac{E(p) \mp m}{2E(p)}} e^{iE(p)t} \quad (32)$$

where $E(p) = \sqrt{p^2 + m^2}$. Thus we recover the standard fundamental spinors of the Dirac theory on Minkowski spacetime [24] and we can verify that the conditions (26) and (30) are satisfied simultaneously which means that in this case the adiabatic and rest frame vacua *coincide*.

Example II: de Sitter expanding universe

This is a more delicate example since here the rest frame vacuum of the Dirac field is different from the adiabatic one. This spacetime is defined as the expanding portion of the de Sitter manifold where the scale factor has the form $a(t) = \exp(\omega t)$, depending on the de Sitter Hubble constant denoted here by ω [6]. Consequently, in the conformal chart we have

$$t_c = -\frac{1}{\omega}e^{-\omega t} \in (-\infty, 0] \quad \rightarrow \quad a(t_c) = -\frac{1}{\omega t_c}. \quad (33)$$

In this chart, the normalized solutions of the system (17) or (18) can be derived easily obtaining the general form

$$u_p^+(t_c) = \sqrt{-\frac{pt_c}{\pi}} \left[c_1 K_{\nu_-}(ipt_c) + c_2 K_{\nu_-}(-ipt_c) \right], \quad (34)$$

$$u_p^-(t_c) = \sqrt{-\frac{pt_c}{\pi}} \left[c_1 K_{\nu_+}(ipt_c) - c_2 K_{\nu_+}(-ipt_c) \right], \quad (35)$$

where $K_{\nu_{\pm}}$ are the modified Bessel functions [25] of the orders $\nu_{\pm} = \frac{1}{2} \pm i\frac{m}{\omega}$. According to Eq. (A.5) the normalization condition (20) is satisfied only if

$$|c_1|^2 + |c_2|^2 = 1. \quad (36)$$

The functions v_p^{\pm} result from Eq. (19). The adiabatic vacuum can be defined simply by choosing $c_1 = 1$ and $c_2 = 0$ as in Ref. [12]. However, this is different from the rest frame vacuum for which we obtain

$$c_1 = \frac{e^{\frac{\pi m}{\omega}} p^{-i\frac{m}{\omega}}}{\sqrt{1 + e^{\frac{2\pi m}{\omega}}}}, \quad c_2 = \frac{i p^{-i\frac{m}{\omega}}}{\sqrt{1 + e^{\frac{2\pi m}{\omega}}}}, \quad (37)$$

as it results from Eq. (28) and the behavior (A.5) of the modified Bessel functions near $z \sim 0$. Hereby, after using the connection formulas of the modified Bessel functions [25], we obtain the definitive form of the modulation functions of positive frequencies of the rest frame vacuum,

$$u_p^{\pm}(t_c) = \pm \frac{\sqrt{-\pi t_c} p^{\nu_{\mp}}}{\sqrt{1 + e^{\frac{2\pi m}{\omega}}}} I_{\mp \nu_{\mp}}(ipt_c) \quad (38)$$

which have the remarkable property

$$\lim_{t_c \rightarrow 0} |u_p^+(t_c)| = 1, \quad \lim_{t_c \rightarrow 0} u_p^-(t_c) = 0, \quad (39)$$

that may be interpreted as an adiabatic condition at $t \rightarrow \infty$ instead of $t \rightarrow -\infty$. The modulation functions of the negative frequencies have to be calculated according to Eq.(19). Note that the constants (37) can be seen as the Bogoliubov coefficients of the transformation between the orthonormal bases corresponding of the adiabatic and rest frame vacua.

Example III: Milne-type spacetime

3. A *Milne-type spacetime* was studied recently [22, 23]. This is a spatially flat FLRW spacetime having a Milne-type scale factor $a(t) = \omega t$ where ω is a free parameter. Its principal feature is that this is no longer flat as the original Milne's universe, being produced by gravitational sources proportional with $\frac{1}{t^2}$. On this manifold, it is convenient to use the chart of proper time $\{t, \mathbf{x}\}$ for $t > 0$. In this chart and the diagonal tetrad gauge (3), the system (15) can be analytically solved finding the general solutions

$$u_p^+(t) = \sqrt{\frac{|m|t}{2\pi}} \left\{ c_1 \left[K_{\nu_+(p)}(imt) + K_{\nu_-(p)}(imt) \right] + c_2 \left[K_{\nu_+(p)}(-imt) - K_{\nu_-(p)}(-imt) \right] \right\} \quad (40)$$

$$u_p^-(t) = \sqrt{\frac{|m|t}{2\pi}} \left\{ c_1 \left[K_{\nu_+(p)}(imt) - K_{\nu_-(p)}(imt) \right] + c_2 \left[K_{\nu_+(p)}(-imt) + K_{\nu_-(p)}(-imt) \right] \right\} \quad (41)$$

where now the modified Bessel functions have the orders $\nu_{\pm}(p) = \frac{1}{2} \pm i\frac{p}{\omega}$. These solutions comply with the normalization condition (20) calculated according to the identity (A.4) with $\mu = \frac{p}{\omega}$ if

$$|c_1|^2 + |c_2|^2 = 1. \quad (42)$$

In the conformal chart, $a(t_c) = e^{\omega t_c}$ satisfies the asymptotic condition (23) such that we can introduce the adiabatic vacuum imposing the condition (26) which yields $c_1 = c_2 = \frac{1}{\sqrt{2}}$. The rest frame vacuum is given by $c_1 = 1$ and $c_2 = 0$ since then $\lim_{p \rightarrow 0} u_p^-(t) = 0$. It is worth pointing out that in the momentum-helicity basis and chiral representation of the Dirac matrices (with diagonal γ^5) the fundamental spinors of the rest frame vacuum take

the simple form [23]

$$U_{\vec{p},\sigma}(x) = \sqrt{\frac{mt}{\pi}} \frac{e^{i\vec{p}\cdot\vec{x}}}{[2\pi\omega t]^{\frac{3}{2}}} \begin{pmatrix} K_{\sigma-i\frac{p}{\omega}}(imt) \xi_{\sigma}(\vec{p}) \\ K_{\sigma+i\frac{p}{\omega}}(imt) \xi_{\sigma}(\vec{p}) \end{pmatrix} \quad (43)$$

$$V_{\vec{p},\sigma}(x) = \sqrt{\frac{mt}{\pi}} \frac{e^{-i\vec{p}\cdot\vec{x}}}{[2\pi\omega t]^{\frac{3}{2}}} \begin{pmatrix} K_{\sigma-i\frac{p}{\omega}}(-imt) \eta_{\sigma}(\vec{p}) \\ -K_{\sigma+i\frac{p}{\omega}}(-imt) \eta_{\sigma}(\vec{p}) \end{pmatrix}, \quad (44)$$

that can be used in applications. We have thus another example in which the adiabatic and rest frame vacua are different.

Concluding remarks

In the quantum theory of interacting fields the amplitudes of the quantum transitions have to be calculated using perturbations in terms of free fields [26–30, 15]. The problem is how these free fields may be defined when we may choose from many different vacua.

In recent applications of the de Sitter QED only the adiabatic vacuum was used so far [15–17] but it is possible to consider different vacua for defining the free fields involved in perturbation [26, 27].

For example, in a collision process we may take the incident beam in the adiabatic vacuum and the target in the rest frame one. However, for the internal lines of the Feynman diagram the rest frame vacuum is the favorite candidate since this can be defined naturally for the massive fields on any spatially flat FLRW spacetime, without requiring supplemental geometric conditions as that of the adiabatic vacuum given by Eq. (23).

Thus by using many well-defined vacua we could combine the methods of the perturbative quantum field theory with those of the cosmological particle creation.

On the other hand, our rest frame vacuum can be helpful in preparing quantum states formed exclusively by particles or antiparticles, localized in non-inertial frames, generating various quantum effects (as correlation, entanglement, etc.) which are of actual interest [18–20, 31–33].

Appendix: Modified Bessel functions

The modified Bessel functions $I_\nu(z)$ and $K_\nu(z)$ are related as [25]

$$K_\nu(z) = K_{-\nu}(z) = \frac{\pi I_{-\nu}(z) - I_\nu(z)}{2 \sin \pi \nu}, \quad (\text{A.1})$$

$$\begin{aligned} I_{\pm\nu}(z) &= e^{\mp i\pi\nu} I_{\pm\nu}(-z) \\ &= \frac{i}{\pi} \left[K_\nu(-z) - e^{\mp i\pi\nu} K_\nu(z) \right]. \end{aligned} \quad (\text{A.2})$$

Their Wronskians give the identities we need for normalizing the mode functions. For $\nu = i\kappa$ we obtain

$$iI_{i\kappa}(is) \overset{\leftrightarrow}{\partial_s} I_{-i\kappa}(is) = \frac{2 \sinh \pi \kappa}{\pi s}, \quad (\text{A.3})$$

while the identity

$$iK_\nu(-is) \overset{\leftrightarrow}{\partial_s} K_\nu(is) = \frac{\pi}{|s|}, \quad (\text{A.4})$$

holds for any ν .

For $|z| \rightarrow \infty$ and any ν we have,

$$I_\nu(z) \rightarrow \sqrt{\frac{\pi}{2z}} e^z, \quad K_\nu(z) \rightarrow K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}. \quad (\text{A.5})$$

In the limit of $|z| \rightarrow 0$ the functions I_ν behave as

$$I_\nu(z) \sim \frac{1}{\Gamma(\nu + 1)} \left(\frac{z}{2}\right)^\nu, \quad (\text{A.6})$$

while for the functions K_ν we have to use Eq. (A.1).

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