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A SECOND-ORDER APPROACH TO FIRST-ORDER SYSTEMS: QUADRATIC ACTIONS

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Abstract

The relationship between the quadratic first- and second-order actions are investigated at both Lagrangian and Hamiltonian levels.

1. Introduction

First-order systems are in general second-class theories, involved in many useful applications, such as self-dual and intermediate models, linearized curved vector Chern-Simons gravity [1 - 3], or in phenomenological theories. The problem of converting second-class systems into some first-class ones attracted much attention lately [4 - 10].

In this paper we investigate the relationship between the quadratic first- and second-order actions at both Lagrangian and Hamiltonian levels. In view of this, starting with a quadratic first-order system subject to purely second-class constraints we implement the following steps: i) we prove that there exists a second-order Lagrangian whose Euler-Lagrange equations describe the same dynamics as the first order-system; ii) the Hamiltonian of the second-order system coincides with the canonical Hamiltonian of the first-order system on the second-class constraint surface. Steps i) and ii) show that a quadratic first-order system endowed only with second-class constraints can be equivalently described by means of a second-order action.

2. Quadratic first-order systems

We take a bosonic first-order system, described by the Lagrangian action

$$S_0\left[q^i\right]=\int dt\left(\frac{1}{2}a_{ij}q^j\dot{q}^i-\frac{1}{2}\mu_{ij}q^iq^j\right)\equiv\int dtL_0, \tag{1}$$

where a_{ij} and μ_{ij} are some antisymmetric, respectively symmetric, constant invertible matrices. The equations of motion read as

$$\dot{q}^i+a^{ik}\mu_{kj}q^j=0, \tag{2}$$

where a^{ij} denote the elements of the inverse of the matrix of elements a_{ij}. Let $\varphi^i(t)$ be the solution of equations (2) in the presence of the initial conditions

$$\varphi^i(t_0)=q_0^i, \tag{3}$$

i.e.

$$\dot{\varphi}^i(t)+a^{ik}\mu_{kj}\varphi^j(t)\equiv 0. \tag{4}$$

System (1) possesses at the Hamiltonian level the irreducible primary constraints

$$\chi_i(q,p)\equiv p_i-\frac{1}{2}a_{ij}q^j\approx 0. \tag{5}$$

Due to the fact that a_{ij} is invertible, it is obvious that constraints (5) are second-class, their consistency leading to no further constraints, but merely determining the associated Lagrange multipliers.

3. Lagrangian second-order approach to first-order systems

The next theorem represents one of our main results.

Theorem 1 *We consider the Lagrangian*

$$\bar{L}_0=\frac{1}{2}a_{lj}\dot{q}^l\left(\mu^{jk}a_{ik}\dot{q}^i-q^j\right), \tag{6}$$

where μ^{ij} *represents the inverse of* μ_{ij}*. Then, the solution* $\Phi^i(t)$ *to the Euler-Lagrange equations* $\delta\bar{L}_0/\delta q^i=0$ *together with the initial conditions*

$$\Phi^i(t_0)=q_0^i,\ \dot{\Phi}^i(t_0)=-a^{ik}\mu_{kj}q_0^i \tag{7}$$

coincides with the solution to equations (2) in the presence of the initial conditions (3), i.e.

$$\Phi^i(t)=\varphi^i(t). \tag{8}$$

Proof. The Euler-Lagrange equations $\delta\bar{L}_0/\delta q^i=0$ take the form

$$\frac{d}{dt}\left(a_{lj}\left(\mu^{jk}a_{ik}\dot{q}^{i}-q^{j}\right)\right)=0. \tag{9}$$

Taking into account that the matrices a_{ij} and μ_{ij} are invertible, from (9) we find that

$$\dot{q}^{i}+a^{ik}\mu_{kj}q^{j}=\alpha^{i}, \tag{10}$$

where α^{i} are some constants determined by the initial conditions of the form (7). Then, from (10) we arrive at

$$\dot{\Phi}^{i}(t)+a^{ik}\mu_{kj}\Phi^{j}(t)\equiv\alpha^{i}. \tag{11}$$

Writing down (11) for $t=t_{0}$ and using (7), we deduce the relations

$$\alpha^{i}\equiv 0, \tag{12}$$

such that (11) leads to

$$\dot{\Phi}^{i}(t)+a^{ik}\mu_{kj}\Phi^{j}(t)\equiv 0. \tag{13}$$

Comparing (13) with (4) we obtain (8). This proves the theorem. Ω

In order to be able to compare the time evolutions described by equations (2) and respectively (9), we must impose in each formulation some initial conditions that are compatible. This means that given the initial conditions (3) for equations (2), we must take (7) as initial conditions for equations (9).

Then, Theorem 1 ensures that the second-order system (6) describes the same dynamics as the quadratic first-order system (1).

4. Hamiltonian relationship between first- and second-order systems

In this section we investigate the Hamiltonian relationship between the systems described by the Lagrangians (1) and respectively (6). Using (5) we get that the canonical Hamiltonian of the first-order system (1) reads as

$$H_{0}=\frac{1}{2}\mu_{ij}q^{i}q^{j}. \tag{14}$$

Now, we construct a Hamiltonian

$$H_{0}^{*}=H_{0}+\text{extra terms in }(q,p), \tag{15}$$

such that

$$\left[H_{0}^{*},\chi_{i}\right]=0\text{ strongly }, \tag{16}$$

where the functions χ_i are given by (5). The construction of H_0^* goes along the same line with the proof of Theorem 1 from [8]. Thus, we take

$$\text{extra terms in } (q,p) = c^j \chi_j + c^{jk} \chi_j \chi_k + \cdots, \tag{17}$$

with $c^{i_1 \cdots i_k}$ unknown coefficients depending on (q,p). Replacing (17) in (16) and identifying the coefficients of the same power order in χ_i, we get a tower of equations for $c^{i_1 \cdots i_k}$, with the solution

$$c^j = -\mu_{lk} a^{jl} q^k, \, c^{jk} = \frac{1}{2} \mu_{li} a^{jl} a^{ki}, c^{j_1 \cdots j_k} = 0, \, k > 2. \tag{18}$$

The above expressions of the coefficients yield the Hamiltonian H_0^* in the form

$$H_0^* = \frac{1}{2} \mu_{ij} q^i q^j - \mu_{lk} a^{jl} q^k \chi_j + \frac{1}{2} \mu_{li} a^{jl} a^{ki} \chi_j \chi_k. \tag{19}$$

On the other hand, the Hamiltonian of the second-order system (6) is expressed by

$$\overline{H}_0 = \frac{1}{2} \mu_{ij} \left(a^{ik} p_k + \frac{1}{2} q^i \right) \left(a^{jl} p_l + \frac{1}{2} q^j \right), \tag{20}$$

while the corresponding Hamilton's equations read as

$$\dot{q}^i = \mu_{kj} a^{ki} \left(a^{jl} p_l + \frac{1}{2} q^j \right), \tag{21}$$

$$\dot{p}_i = -\frac{1}{2} \mu_{ij} \left(a^{jl} p_l + \frac{1}{2} q^j \right). \tag{22}$$

Substituting (21) in (22) we arrive at

$$\dot{p}_i - \frac{1}{2} a_{ij} \dot{q}^j \equiv \dot{\chi}_i = 0. \tag{23}$$

In terms of the Poisson bracket, relations (23) take the form

$$\left[\overline{H}_0, \chi_i \right] = 0. \tag{24}$$

We observe that equations (16) and (24) are identical. This is not a surprise because, if we use (5) in (19), we deduce the formula

$$\overline{H}_0 \equiv H_0^* = H_0 - \mu_{lk} a^{jl} q^k \chi_j + \frac{1}{2} \mu_{li} a^{jl} a^{ki} \chi_j \chi_k. \tag{25}$$

The last formula emphasizes that the Hamiltonian of the second-order system coincides with the

The last formula emphasizes that the Hamiltonian of the second-order system coincides with the canonical Hamiltonian of the first-order system on the second-class constraint surface (5). More precisely, from (25) we find that: a) the Hamiltonian of the second-order system (6) is obtained by adding some liniar and quadratic terms (in the second-class constraints functions χ_i) to the canonical Hamiltonianul of the first-order system (1); b) conversely, the canonical Hamiltonian of the first-order system (1) represents the restriction of the Hamiltonian of the second-order system (6) to the second-class constraint surface (5).

Conclusion

To conclude with, in this paper we have proved that a quadratic first-order system that subject to purely second-class constraints can be equivalently approached in terms of a second-order action.

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