Analele Universităţii de Vest din Timişoara
Vol. LV, 2011
Seria Fizică

BOSONS GENERATION IN STRONG ELECTRIC FIELDS VIA WKB FORMALISM

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Article Info
Received: 20 December 2010
Accepted: 12 January 2011

Keywords:
Klein-Gordon equation,
particle creation,
WKB formalism

Abstract
For a better understanding of the cosmological particle generation in strong electric fields, recently, a growing interest has been given to the so-called analogue models of gravity, created in condensed matter laboratories. This paper deals with the evolution of relativistic bosons moving in external electric and magnetic static fields. After solving the Klein-Gordon equation, we analyze the boson creation process and compute the main quantities characterizing this phenomenon, within the WKB formalism.

1. Introduction

The particle production in external electric fields by quantum mechanical tunneling has been a challenging topic, ever since Breit and Wheeler have proposed it as a result of two photons collision [1].

This has come soon after Klein found a more intriguing result [2]: for the relativistic particle moving in an external step-function potential, in the case $V_0 > w + m_0 c^2$, the reflected flux is larger than the incident one, although that the total flux is conserved.

After many years, it seams that this so-called *Klein paradox* of unimpeded penetration of relativistic particles through high and wide potentials can finally be tested in experiments involving electrostatic barriers in graphene [3].

Going further to more exotic but related topics, the cosmological particle creation has been studied as a consequence of the spacetime expansion [4]. For a better understanding of this phenomenon, recently, a growing interest has been given to non-trivial quantum effects in laboratories devoted to condensed matter systems, by creating the so-called *analogue models of gravity*, as they are described in [5].

Of course, in this paper, we are referring to ideal bosons moving in external fields, which are not available in nature but, to a very good degree of approximation, one can find measurable properties of mesoscopic systems that do actually behave as it is predicted by such theoretical models [6].

2. Relativistic Bosons

The relativistic complex charged boson of mass m_0, evolving in a static magnetic field orthogonal to a static electric field, is described by the *U(1)*-gauge invariant Lagrangian density

$$\mathcal{L} = \eta^{ij}\left(D_i\psi\right)^* D_j\psi + \frac{m_0^2c^2}{\hbar^2}\psi^*\psi \tag{1}$$

where the gauge covariant derivatives are defined by

$$D_i\psi = \psi_{,i} - \frac{iq}{\hbar}A_i\psi, \;\; D_i\psi^* = \psi_{,i}{}^* + \frac{iq}{\hbar}A_i\psi^*.$$

For a bidimensional thin sample in orthogonal external fields whose intensities are related by $E_0/c > B_0$**,** the components of the four-potential are given by

$$A_x = A_z = 0,\;\; A_y = B_0 x, \quad A_4 = \frac{E_0}{c}x \tag{2}$$

The Euler-Lagrange equation corresponding to the Lagrangean (1) has the explicit form [7]

$$\eta^{ij}\psi_{,ij} - 2i\frac{q}{\hbar}B_0 x\psi_{,y} + 2i\frac{q}{\hbar c^2}E_0 - \left[\frac{m_0^2c^2}{\hbar^2} + \frac{q^2x^2}{\hbar^2}\left(B_0^2 - \frac{E_0^2}{c^2}\right)\right]\psi = 0 \tag{3}$$

and one may employ the standard variables separation

$$\psi = \chi(x)\cdot\exp\left[\frac{i}{\hbar}\left(p_y y + p_z z - wt\right)\right] \tag{4}$$

to come to the following differential equation for the *x*-depending part,

$$\frac{\partial^2\chi}{\partial x^2} + \frac{1}{\hbar^2}\left[\frac{w^2}{c^2} - p_y{}^2 - p_z^2 - m_0^2c^2 + 2qx\left(\frac{E_0 w}{c^2} + B_0 p_y\right) + q^2x^2\left(\frac{E_0^2}{c^2} - B_0^2\right)\right]\chi = 0 \tag{5}$$

Using the notations

$$\alpha \equiv \frac{E_0 w}{c^2} + B_0 p_y, \quad \gamma^2 \equiv \frac{E_0^2}{c^2} - B_0^2 \geq 0, \tag{6}$$

and the new variable

$$\tau = \frac{1}{\sqrt{\hbar q\gamma}}\left(qx\gamma + \frac{\alpha}{\gamma}\right) \tag{7}$$

the Eq. (5) becomes

$$\frac{\partial^2 \chi}{\partial \tau^2} + \left\{\tau^2 + \frac{1}{q\hbar\gamma}\left[-\frac{\left(wB_0 + p_y E_0\right)^2}{c^2\gamma^2} - p_z^2 - m_0^2 c^2\right]\right\}\chi = 0$$

being satisfied by the parabolic cylinder functions [8]

$$\chi(\tau) = \mathcal{N} \cdot D_{-\frac{1\pm i\lambda}{2}}\left[\pm(1\pm i)\tau\right], \tag{8}$$

where the parameter λ is

$$\lambda = \frac{1}{q\hbar\gamma}\left[\frac{\left(wB_0 + p_y E_0\right)^2}{c^2\gamma^2} + p_z^2 + m_0^2 c^2\right] \tag{9}$$

3. Particle Creation within WKB

On the bi-dimensional sample of finite width perpendicular to the magnetic field, the bosons are evolving in the external electric field as one-dimensional bounded harmonic oscillators that can be treated within the phase-integral approach [9].

In our case, the system of bosons is described by the wave equation (3), which admits the solutions of the general form (4). With the gauge (2) and the notations (6), the Klein-Gordon equation in the momentum space is

$$p_x^2 + p_y^2 + p_z^2 + m_0^2 c^2 - \left(\frac{w}{c}\right)^2 - q^2\gamma^2 x^2 - 2qx\alpha = 0 \tag{10}$$

and allows us to write down the momentum p_x as

$$p_x = q\gamma\sqrt{(x+b)^2 - a^2} = iq\gamma\sqrt{a^2 - (x+b)^2} \tag{11}$$

once we have introduced the notations

$$b \equiv \frac{\alpha}{q\gamma^2} \;\; , \;\; a^2 \equiv \frac{1}{q^2\gamma^2}\left[\frac{\left(wB_0 + p_y E_0\right)^2}{c^2\gamma^2} + p_z^2 + m_0^2 c^2\right] \tag{12}$$

As in the case of bounded harmonic oscillators, we define the operator

$$P=\exp\left[\frac{1}{\hbar}\int_{x_1}^{x_2} p_x \ dx\right] \tag{13}$$

satisfying the equation $P^2=-1$, [10]. Explicitly, this means

$$P^2=\exp\left[\frac{2i}{\hbar}q\gamma\int_{x_1}^{x_2}\sqrt{a^2-\left(x+b\right)^2}dx\right]=\exp\left[i\pi(2n+1)\right] \tag{14}$$

where the integration limits are the roots of the equation

$$\left(x+b\right)^2-a^2=0 \tag{15}$$

The relation (14) leads to the following discrete set of energy levels

$$\frac{\left(wB_0+p_yE_0\right)^2}{c^2\gamma^2}+p_z^2+m_0^2c^2=(2n+1)\hbar q\gamma \tag{16}$$

Within WKB formalism, the transmission coefficient for the quantum tunneling of the wave through the one-dimensional potential barrier is defined as

$$\left|T\right|^2 \approx \exp\left[-\frac{2}{\hbar}\int_{x_1}^{x_2}\left|p_x\right| \ dx\right] \tag{17}$$

where

$$\left|p_x\right|=\sqrt{-q^2\gamma^2x^2-2qx\alpha+p_y^2+p_z^2+m_0^2c^2-\left(\frac{w}{c}\right)^2} \tag{18}$$

is the classical momentum. The integration limits are the roots of the equation

$$x^2+2\tfrac{\alpha}{q\gamma^2}x-\frac{p_y^2+p_z^2+m_0^2c^2-\left(\frac{w}{c}\right)^2}{q^2\gamma^2}=0 \tag{19}$$

and they are defining the turning points of the classical trajectory, separating the positive and the negative energy states, once we consider a null electric field outside the range $x\in\left[x_1,x_2\right]$. By performing the integration in (17), the transmission coefficient reads

$$\left|T\right|^2 \approx \exp\left[-\frac{\pi}{\hbar q\gamma}\left(\frac{\left(wB_0+p_yE_0\right)^2}{c^2\gamma^2}+p_z^2+m_0^2c^2\right)\right] \tag{20}$$

In the particular case of a vanishing magnetic induction, the above result becomes

$$\left|T_o\right|^2 \sim \exp\left[-\frac{\pi}{\hbar}\frac{m_0^2c^3}{qE_0}\right]\exp\left[-\frac{\pi c}{\hbar qE_0}\left(p_y^2+p_z^2\right)\right] \tag{21}$$

leading, by integration in the phase-space, to the probability amplitude

$$P_0 = S\,\frac{qE_0}{4\pi^2\hbar c}\exp\left[-\pi\,\frac{E_c}{E_0}\right]\ , \tag{22}$$

where S is the transverse surface. This contains the well-known Sauter exponential [11, 12], expressed in terms of the critical electric field, E_c.

In order to compute the transition rate per unit time, one has to divide the probability amplitude by the total volume, $V = S\Delta x$, and time interval during which quantum tunneling occurs, Δt. For the relativistic particle evolving in an external electric field alone, the so-called tunneling length and the time interval being [6]

$$\Delta x = x_2 - x_1 = \frac{2m_0c^2}{qE_0}\ \ ,\ \ \Delta t = \frac{h}{m_0c^2},$$

the transition rate per unit time and volume reads

$$\Gamma_0 = \frac{P_0}{S\Delta x\Delta t} = \frac{\left(qE_0\right)^2}{4\pi h^2 c}\exp\left[-\pi\,\frac{E_c}{E_0}\right]\ . \tag{23}$$

Finally, for the more general transmission coefficient (20), we write down the probability amplitude as

$$P\ = \exp\left[-\pi\,\frac{m_0^2c^2}{\hbar q\gamma}\right]\left\{\int_{-\infty}^{+\infty} exp\left[-\frac{\pi}{\hbar q\gamma}\,p_z^2\right]\frac{dp_z}{2\pi\hbar}\right\}I. \tag{24}$$

This contains the following integral over the momentum component p_y and the energy w,

$$I = \left[\frac{B_0}{\sqrt{\hbar q\gamma}c\gamma}\right]\int_0^{\infty}\int_{-\infty}^{\infty}\exp\left[-\frac{\pi}{\hbar c^2 q\gamma^3}\left(p_yE_0 + wB_0\right)^2\right]\frac{dp_y}{2\pi\hbar}dw \tag{25}$$

where we have assumed the continuum spectrum approximation, valid for large values of λ, and the integration measure coming from (16). Obviously, the above integral is of the general form

$$I_{ab} = \int_{\mathbb{R}\times\mathbb{R}_+}\exp\left[-(ax+by)^2\right]\ dxdy$$

and has a convergent part which can be computed as $I_{ab}^c = 1/(ab)$.

Putting everything together, one finally gets the following probability amplitude

$$P = S\,\frac{cq\gamma^2}{(2\pi)^3\,\hbar E_0}\exp\left[-\pi\,\frac{m_0^2c^2}{\hbar q\gamma}\right] \tag{26}$$

and the transition rate per unit time and volume

$$\Gamma = \frac{c^2q^2\gamma^3}{8\pi^2h^2E_0}\exp\left[-\pi\,\frac{m_0^2c^2}{\hbar q\gamma}\right] \tag{27}$$

where the tunneling length has been approximated to $\Delta x \approx \dfrac{2m_0c}{q\gamma}$ **.**

Conclusions

The present paper deals with the creation of massive bosons evolving in an external strong electric field orthogonal to a weak constant magnetic induction, oriented along *Ox* and *Oz*, respectively. For the convenient gauge (2), we derive the particle wave functions, solutions to the Klein-Gordon equations, in terms of parabolic cylinder functions. The WKB formalism allows us to compute the transmission coefficient and the transition rate per unit time and volume. As it is expected, for $\gamma = 0$**,** corresponding to $B_0 = E_0 / c$**,** the expressions (26) and (27) are both vanishing. For a zero magnetic induction, $B_0 = 0$, the well-known results from literature (22) and (23) are recovered.

Acknowledgements

OB acknowledges the support of the European Social Fund in Romania, under the responsibility of the Managing Authority for the Sectorial Operational Program for Human Resources Development 2007-2013, Grant POSDRU/88/1.5/S/47646.

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