Analele Universităţii de Vest din Timişoara
Vol. LV, 2011
Seria Fizică

LAGRANGIAN BRST COHOMOLOGY OF MASSLESS MIXED SYMMETRY TENSOR FIELDS. THE CASE (4,1)

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Article Info
Received: 18 December 2010
Accepted: 10 January 2011

Keywords:
quantum field theory,
mixed symmetry tensor field,
BRST symmetry

Abstract
The class of massless tensor fields with the mixed symmetry (4,1) is investigated at the level of its local BRST cohomolgy. The main results regarding the general form of the nontrivial representatives from the local BRST cohomology at a given value of the ghost number and at maximum form degree are exposed. The basic results given here extend those from Refs. [10] and [11] on the tensor field with the mixed symmetry (3,1).

1. Lagrangian formulation

We consider a tensor field $t_{\mu\nu\rho\lambda|\alpha}$ with the mixed symmetry $(4,1)$, corresponding to a Young diagram with 5 cells and 4 rows. This means that $t_{\mu\nu\rho\lambda|\alpha}$ is separately antisymmetric in the first four indices and satisfies the Bianchi I identity

$$t_{[\mu\nu\rho\lambda|\alpha]} \equiv 0. \tag{1}$$

We assume that this tensor field is defined on a Minkowski-flat space-time of dimension D endowed with a metric tensor $\sigma^{\mu\nu}$ of 'mostly plus' signature $\sigma^{\mu\nu} = \sigma_{\mu\nu} = (-,+++++....)$. We denote the trace of this tensor by $t_{\mu\nu\rho}$ and is defined by

$$t_{\mu\nu\rho\lambda} = t_{\mu\nu\rho\lambda|\alpha}\sigma^{\lambda\alpha}. \tag{2}$$

Accordingly, it is a completely antisymmetric tensor field.
The free Lagrangian action of the massless tensor field $t_{\mu\nu\rho\lambda|\alpha}$ reads as

$$S^{\mathrm{L}}\left[t_{\mu\nu\rho\lambda|\alpha}\right] = -\tfrac{1}{2\cdot 4!}\int d^{D}x\Big[\left(\partial_{\theta}t_{\mu\nu\rho\lambda|\alpha}\right)\partial^{\theta}t^{\mu\nu\rho\lambda|\alpha} - \left(\partial^{\alpha}t_{\mu\nu\rho\lambda|\alpha}\right)\partial_{\beta}t^{\mu\nu\rho\lambda|\beta} - 4\left(\partial^{\theta}t_{\theta\mu\nu\rho|\alpha}\right)\partial_{\lambda}t^{\lambda\mu\nu\rho|\alpha}$$
$$-4\left(\partial_{\mu}t_{\nu\rho\lambda}\right)\partial^{\mu}t^{\nu\rho\lambda} - 8\left(\partial^{\alpha}t_{\mu\nu\rho\lambda|\alpha}\right)\partial^{\mu}t^{\nu\rho\lambda} + 12\left(\partial^{\lambda}t_{\lambda\mu\nu}\right)\partial_{\rho}t^{\rho\mu\nu}\Big]. \tag{3}$$

The Lagrangian action is invariant under the (infinitesimal) gauge transformations

$$\delta_{\theta,\varepsilon} t_{\mu\nu\rho\lambda|\alpha} = \partial_{[\mu} \overset{(1)}{\theta}_{\nu\rho\lambda]|\alpha} + \partial_{[\mu} \overset{(1)}{\varepsilon}_{\nu\rho\lambda]\alpha} - \partial_{\alpha} \overset{(1)}{\varepsilon}_{\mu\nu\rho\lambda}. \tag{4}$$

This generating set of gauge transformations is Abelian and off-shell, third-order reducible.

The field equations have the form

$$\frac{\delta S^{L}}{\delta t_{\mu\nu\rho\lambda|\beta}} \equiv \tfrac{1}{4!} T^{\mu\nu\rho\lambda|\beta} \approx 0, \tag{5}$$

where the tensor field $T^{\mu\nu\rho\lambda|\beta}$ is linear in the tensor field $t_{\mu\nu\rho\lambda|\beta}$, second order in its derivatives and displays the mixed symmetry $(4,1)$

$$T^{\mu\nu\rho\lambda|\beta} = \partial_{\theta}\partial^{\theta} t^{\mu\nu\rho\lambda|\beta} + \partial_{\gamma}\left(\partial^{[\mu} t^{\nu\rho\lambda]\gamma|\beta} - \partial^{\beta} t^{\mu\nu\rho\lambda|\gamma}\right) - \partial^{\beta}\partial^{[\mu} t^{\nu\rho\lambda]}$$
$$+ \partial_{\theta}\partial^{\theta}\sigma^{\beta[\mu} t^{\nu\rho\lambda]} - \partial_{\theta}\partial_{\gamma}\sigma^{\beta[\mu} t^{\nu\rho\lambda]\gamma|\theta} - \partial_{\theta}\sigma^{\beta[\mu}\partial^{\nu} t^{\rho\lambda]\theta}. \tag{6}$$

The gauge invariance of the Lagrangian action under the transformation (4) is equivalent to the fact that the functions defining the field equations are not all independent, but satisfy the Noether identities

$$\partial_{\mu} T^{\mu\nu\rho\lambda|\alpha} \equiv 0, \;\; \partial_{\alpha} T^{\mu\nu\rho\lambda|\alpha} \equiv 0. \tag{7}$$

The presence of the reducibility shows that not all the above Noether identities are independent, so the trace of $T^{\mu\nu\rho\lambda|\beta}$ satisfies the identity

$$\partial_{\mu} T^{\mu\nu\rho} \equiv 0. \tag{8}$$

The most general gauge-invariant quantities constructed out of the tensor field $t_{\mu\nu\rho\lambda|\beta}$ and its space-time derivatives is the curvature tensor given below and its derivatives

$$K_{\mu\nu\rho\lambda\sigma|\alpha,\beta} = \partial_{\alpha}\partial_{[\mu} t_{\nu\rho\lambda\sigma]|\beta} - \partial_{\beta}\partial_{[\mu} t_{\nu\rho\lambda\sigma]|\alpha} \equiv \partial_{[\mu} t_{\nu\rho\lambda\sigma]|[\alpha,\beta]} \tag{9}$$

This tensor has the mixed symmetry $(5,2)$ and is described by a Young diagram with five rows and two columns.

2. The BRST symmetry

The construction of the BRST symmetry for the free theory under consideration starts with the identification of the BRST algebra on which the BRST differential s acts. The generators of the BRST algebra and their Grassmann parities are

BRST generator	**ε**	**pgh**	**agh**	**gh**	**BRST generator**	**ε**	**pgh**	**agh**	**gh**
$t_{\mu\nu\rho\lambda\|\alpha}$	0	0	0	0	$t^{*\mu\nu\rho\lambda\|\alpha}$	1	0	1	-1
$\overset{(1)}{C}_{\mu\nu\rho\|\alpha}, \overset{(1)}{\eta}_{\mu\nu\rho\lambda}$	1	1	0	1	$\overset{(1)}{C}{}^{*\mu\nu\rho\|\alpha}, \overset{(1)}{\eta}{}^{*\mu\nu\rho\lambda}$	0	0	2	-2
$\overset{(2)}{C}_{\mu\nu\|\alpha}, \overset{(2)}{\eta}_{\mu\nu\rho}$	0	2	0	2	$\overset{(2)}{C}{}^{*\mu\nu\|\alpha}, \overset{(2)}{\eta}{}^{*\mu\nu\rho}$	0	0	3	-3
$\overset{(3)}{C}_{\mu\nu\|\alpha}, \overset{(3)}{\eta}_{\mu\nu}$	1	3	0	3	$\overset{(3)}{C}{}^{*\mu\|\alpha}, \overset{(3)}{\eta}{}^{*\mu\nu}$	0	0	4	-4
$\overset{(4)}{\eta}_{\mu}$	0	4	0	4	$\overset{(4)}{\eta}{}^{*\mu}$	0	0	5	-5

As both the gauge generators and reducibility functions for this model are field-independent, it follows that the associated BRST differential ($s^2 = 0$) splits into

$$s = \delta + \gamma,$$

where δ represents the Koszul-Tate differential ($\delta^2 = 0$), graded by the antighost number agh ($\mathrm{agh}(\delta) = -1$), while γ is the exterior derivative along the gauge orbits and turns out to be a true differential ($\gamma^2 = 0$) that anticommutes with δ ($\delta\gamma + \gamma\delta = 0$), whose degree is named pure ghost number pgh ($\mathrm{pgh}(\gamma) = 1$). The overall degree that grades the BRST differential is known as the ghost number (gh) and is defined like the difference between the pure ghost number and the antighost number, such that $\mathrm{gh}(s) = \mathrm{gh}(\delta) = \mathrm{gh}(\gamma) = 1$.

3. Cohomology of γ

The most general representative of $H(\gamma)$, α (satisfies $\gamma\alpha = 0$, and $\alpha \neq \gamma\beta$) has the form (up to trivial γ-exact contributions)

$$\alpha = \sum_J \alpha_J([K_{\mu\nu\rho\lambda\alpha|\alpha\beta}],[\Pi^{*\Delta}])\,\omega^J(\Phi_{\mu\nu\rho\lambda\alpha}, \overset{(4)}{\eta}_{\mu}), \tag{10}$$

where

$$\Phi_{\mu\nu\rho\lambda\alpha} = \partial_{[\mu}\overset{(1)}{\eta}_{\nu\rho\lambda\alpha]}.$$

4. Cohomology Theorems

Theorem 1 *The cohomology of d in form degree strictly less than* D *is trivial in the space of invariant polynomials with strictly positive antighost number. This means that the conditions*

$$\gamma\alpha = 0,\ d\alpha = 0,\ agh(\alpha) > 0,\ \deg(a) < D,\ \alpha = \alpha([\Pi^*],[K]),$$

imply

$$\alpha = d\beta,$$

for some invariant polynomials $\beta([\Pi^*],[K])$.

Theorem 2 *The cohomology of d computed in* $H(\gamma)$ *is trivial in form degree strictly less than* D *and in strictly positive antighost number*

$$H_k^{g,p}\big(d, H(\gamma)\big) = 0, \qquad k > 0,\ \ p < D,$$

where p is the form degree, k is the antighost number and g is the ghost number.

Theorem 3 *If* a *satisfies the properties*

$$\mathrm{agh(a)} = k > 0,\quad \mathrm{gh(a)} = g \geq -k,\quad \deg(a) = p \leq D$$

and the equation

$$\gamma a + db = 0,$$

where

$$agh(b) = k > 0,\ \ gh(b) = g + 1 > -k,\ \ \deg(b) = p - 1 < D,$$

then, one can always redefine a

$$a \rightarrow a' = a + d\nu,$$

so that $\gamma a' = 0$.

Theorem 4 *Let* α *be a* δ*-exact invariant polynomial*

$$\alpha = \delta\beta.$$

Then, β can also be taken to be an invariant polynomial.

Theorem 5 *Let* α_k^p *be an invariant polynomial with the form degree* $\deg\left(\alpha_k^p\right) = p$ *and antighost number* $\mathrm{agh}\left(\alpha_k^p\right) = k$ *wich is* δ*-exact modulo* d

$$\alpha_k^p = \delta\lambda_{k+1}^p + d\lambda_k^{p-1} \quad, k \geq 5.$$

Then, we can choose λ_{k+1}^{p} and λ_{k}^{p-1} to be invariant polynomials.

Theorem 6 *From any BRST cocycle from the local cohomology of the BRST differential* s *in form degree* D *and ghost number* g *one can eliminate all the components with antighost number strictly greater then five only through trivial redefinitions.*

Acknowledgements

One of the authors (L.S.-O.) acknowledges partial support from the strategic grant POSDRU/88/1.5/S/49516, Project ID 49516 (2009), co-financed by the European Social Fund - Investing in People, within the Sectoral Operational Programme Human Resources Development 2007 - 2013.

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