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THE ENERGY – MOMENTUM TENSOR SOFTWARE ASSESSMENT IN CURVED SPACE - TIME

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Abstract
One of the most interesting predicaments which remain unsolved since the birth of General Theory of Relativity (GR), is the energy-momentum localization. All our considerations are within the Lagrange formalism of field theory. The concept of the energy-momentum tensor for gravitational interactions has a long history. To find a generally accepted expression, there have been different attempts. This paper is devoted to the investigation of the energy-momentum problem in the General Relativity theory and its space physics application. In order to cover the huge volume of computation and, bearing in mind to make a general approaching for different space-time configurations, was built a MAPLE application to succeed in studying the energy momentum tensor. In the second part of the paper were presented for two space-time configuration, the comparative results.
Keywords: energy momentum tensor, scalar field, electromagnetic field.

1. Introduction

In the last three decades, field theories on curved manifolds with significant applications to Cosmology have been intensively investigated leading to various exciting results that shad quite a new light on our understanding of the Universe [1]. Besides the historic Einstein's Universe, which continues to play an important role for academic exercises and tests on various semi-classical or quantum field dynamics, [2], exactly solvable models for fields coupled to gravity have been a main target of investigations.

One of the most interesting problems which remain unsolved, since the birth of General Theory of Relativity (GR), is the energy-momentum localization. The notions of energy and momentum play important role in physics [3, 4]. Considering the conservation laws which follow from the equations of motion, we can gain important information about the system even without explicitly solving its equations of motion.

For Lagrangean density based theories, the derivation of the conserved energy-momentum object is closely related to the variational procedure by which the equations of motion are being derived [3, 4].

2. Method and samples

For a mechanical one dimensional system, described by a Lagrange function of the form $L = L(q, \dot{q}, t)$, could be defined the action magnitude by the relation.

$$S[t_1, t_2] = \int_{t_1}^{t_2} L(q, \dot{q}, t)dt \tag{i}$$

The equation for motion could be easily derived imposing the stationary of the considered action $\delta S = 0$ under arbitrary variations of time dependence function $q(t)$. Under this condition, could be derived the classical Euler Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \tag{ii}$$

If the Lagrange function does not depend on the x^{α}, it is derived that the conservation equation are of the form

$$\frac{d}{dx^{\beta}}\left(q_{A,\alpha}\frac{\partial L}{\partial q_{A,\beta}} - \delta_{\alpha}^{\beta} L\right) = 0 \tag{iii}$$

Moving to the relativistic covariant fields' theory, could be considered for a general form of the Lagrange density function depending by a symmetric tensor field $\Phi^{\mu\nu}$, the canonical energy – momentum tensor is [5 - 7]

$$t^{\alpha\beta} = \frac{1}{\sqrt{-g}}\left(g^{\alpha\tau}\Phi^{\mu\nu}{}_{;\tau}\frac{\partial L}{\partial \Phi^{\mu\nu}{}_{;\beta}} - g^{\alpha\beta} L\right) \tag{iv}$$

One of the most interesting problems which remains unsolved, since the birth of General Theory of Relativity (GR), is the energy-momentum localization. To find a generally accepted expression, there have been different attempts [8 - 11].

This paper is devoted to our new proper software approaching results in order to succeed in the investigation of the energy-momentum problem in General Relativity theory. The original forthcoming becomes form new software object structure used in order to succeed in building the correct energy – momentum tensor components, in different approaching.

3. Results and Discussions

The main advantage of the present procedures set is coming from the new approaching strategy [10, 11]. The anterior algorithms [10, 12] was developed in a linear manner, being designed to succeed in writing down the energy – momentum tensor for electromagnetic, scalar and gravitational field, using explicit scalar filed covariant derivative implementation for a general class of Lagrangean configurations. Such approaching was developed, basing on recursive advance, for the first time in the literature.

In this way, we employ one of the best computer algebraic systems for this purpose – the MAPLE platform. This software system itself and its separate libraries are developing with great speeds nowadays. This is related to their popularity and is determined by the need of users to have a more and more efficient interactive programming environment that allows one to solve more complex and more varying problems. The key elements of it, as well as of other similar systems, are the commands for data input and output, mathematical operations, transformations or reduction of complex expressions, drawing graphs, invoking help, etc. They all are grouped into separate libraries according to their purpose, particularity, and the circle of tasks they are meant to do.

In our previous paper [12] we succeed in building consistent software approaching for a fine comparison between energy –momentum tensor construction in General Relativity and teleparalel theory, in different attempts.

A different strategy is adopted now, using a variational calculus procedure in order to obtain the fields equations. This specific approach needs a preliminary structure. Such structure was not trivial and was build after a successive series of tries to succeed in getting a consistent structure.

The first part of the program starts, after initializing the main used package, with a set of definitions for the entire set of necessary objects and global variables.

The next level of the present script is dedicated to the Lagrangean symbolic definitions. This point is essential and represents the gate in order to introduce the considered specifications. The central idea was to declare for each involved field, a symbolic variable which will help in defining the Lagrangean density' terms.

In order to succeed implementing the Euler - Lagrange procedure for fields’ system of equations, it has to differentiate with respect to the symbolic defined variables. The explicit form is finally written down, using a multi-level recursive replacing procedure of the symbolic variables with the specific ones. In a different block instruction, is built the necessary Lorentz condition.

```
> Current_A:=array(1..4);
  Field_A:=A_gauge[compts];
> for i from 1 to 4 do CoordC=Field_A[i] :
  L1A:=subs(Field_A[i]=CoordC,L):
> Momentum _rigid_A:=array(1..4,1..4);
  Force_rigid_A:=array(1..4);
> for l from 1 to 4 do
  for m from 1 to 4 do
  Ltemp[0]:=Momentum_At[l,m]: .....
  for j from 1 to 4 do :
  for k from 1 to 4 do
  Ltemp[(j-1)*4+k]: =subs (Ad[j,k] (coord[1], coord[2], coord[3], coord[4]) =Ac1[j,k],
  Ltemp[(j-1)*4+k-1])  end do :......
  end do:
  for k from 0 to 16 do
  tempq[k]:=expand(Ltemp[k]) end do:
  Momentum_Att[l,m]:=expand(tempq[16]):
   end do end do;
```

For example, we start by considering a Lagrangean density in a $SO(3,1)\times U(1)$ gauge invariance interaction of charged boson of m_0 mass with electromagnetic field, of the form [2, 5, 7, 8]:

$$L = \eta^{ab}\overline{\Phi}_{,a}\Phi_{,b} + m_0{}^2\overline{\Phi}\Phi + \frac{1}{4}F^{ab}F_{ab} \qquad \text{(v)}$$

For spherically symmetric configuration describe by a metric tensor of static conformal type, expressed in Schwarzchild coordinates as [5]

$$ds^2 = e^{2(H(r))}dr^2 + r^2 d\theta^2 + e^{2(G(r))}d\varphi^2 - e^{2(F(r))}dt^2 \qquad \text{(vi)}$$

The Christoffel symbols derived in this frame are

$$\Gamma^2{}_{12} = -\Gamma^1{}_{22} = \frac{1}{r}e^{-(H(r))}, \quad \Gamma^3{}_{13} = -\Gamma^1{}_{33} = G'(r)e^{-(H(r))}, \quad \Gamma^4{}_{14} = \Gamma^1{}_{44} = F'(r)e^{-(H(r))} \quad \text{(vii)}$$

meanwhile the energy-momentum tensor has the form in the Landau prescription

$$\begin{aligned} T_{11} = &-m_0^2\overline{\Phi}\Phi + e^2\overline{\Phi}\Phi\left(A_1{}^2 + A_4{}^2\right) + ie\left[\frac{A_1}{e^{(H(r))}}\left(\overline{\Phi}\Phi_{,r} - \overline{\Phi}_{,r}\Phi\right) + \frac{A_4}{e^{(F(r))}}\left(\overline{\Phi}\Phi_{,t} - \overline{\Phi}_{,t}\Phi\right)\right] + \\ &+\left[\frac{1}{e^{(2H(r))}}\overline{\Phi}_{,r}\Phi_{,r} + \frac{1}{e^{(2F(r))}}\overline{\Phi}_{,t}\Phi_{,t}\right] - \frac{1}{2}\frac{1}{e^{2H(r)}e^{2F(r)}}\left[e^{H(r)}A_{1,t} - F'(r)e^{F(r)}A_4 - e^{F(r)}A_{4,r}\right]^2 \end{aligned} \qquad \text{(viii)}$$

$$T_{22} = -m_0^2\overline{\Phi}\Phi - e^2\overline{\Phi}\Phi\left(A_1{}^2 - A_4{}^2\right) - \left[\frac{A_1}{e^{(H(r))}}\left(\overline{\Phi}\Phi_{,r} - \overline{\Phi}_{,r}\Phi\right) - \frac{A_4}{e^{(F(r))}}\left(\overline{\Phi}\Phi_{,t} - \overline{\Phi}_{,t}\Phi\right)\right] + \\ -\left[\frac{1}{e^{(2H(r))}}\overline{\Phi}_{,r}\Phi_{,r} - \frac{1}{e^{(2F(r))}}\overline{\Phi}_{,t}\Phi_{,t}\right] - \frac{1}{2}\frac{1}{e^{2H(r)}e^{2F(r)}}\left[e^{H(r)}A_{1,t} - F'(r)e^{F(r)}A_4 - e^{F(r)}A_{4,r}\right]^2 \quad \text{(ix)}$$

$$T_{33} = T_{22} \quad \text{(x)}$$

$$T_{44} = m_0^2\overline{\Phi}\Phi + +e^2\overline{\Phi}\Phi\left(A_1{}^2 + A_4{}^2\right) + \left[\frac{A_1}{e^{(H(r))}}\left(\overline{\Phi}\Phi_{,r} - \overline{\Phi}_{,r}\Phi\right) + \frac{A_4}{e^{(F(r))}}\left(\overline{\Phi}\Phi_{,t} - \overline{\Phi}_{,t}\Phi\right)\right] + \\ +\left[\frac{1}{e^{(2H(r))}}\overline{\Phi}_{,r}\Phi_{,r} + \frac{1}{e^{(2F(r))}}\overline{\Phi}_{,t}\Phi_{,t}\right] + \frac{1}{2}\frac{1}{e^{2H(r)}e^{2F(r)}}\left[e^{H(r)}A_{1,t} - F'(r)e^{F(r)}A_4 - e^{F(r)}A_{4,r}\right]^2 \quad \text{(xi)}$$

For the same Lagrangean structure for a space-time configuration of the form [8]

$$\mathrm{ds}^2 = \mathrm{at}^{4/3}\left(\mathrm{dr}^2 + \mathrm{r}^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right)\right) - (\mathrm{dt})^2 \quad \text{(xii)}$$

the derived energy momentum tensor in the same prescription has the components

$$\mathrm{T}_{11} = \left(\overline{\Phi}_{,\mathrm{r}} + \mathrm{ieA}_1\overline{\Phi}\right)\left(\Phi_{,\mathrm{r}} - \mathrm{ieA}_1\Phi\right) + \left(\overline{\Phi}_{,\mathrm{t}} + \mathrm{ieA}_4\overline{\Phi}\right)\left(\Phi_{,\mathrm{t}} - \mathrm{ieA}_4\Phi\right) - \\ - \mathrm{m}_0^2\overline{\Phi}\Phi - \frac{1}{2}\left(\frac{2}{3\mathrm{t}}\mathrm{A}_1 + \mathrm{A}_{1,\mathrm{t}} - \frac{1}{\sqrt{\mathrm{a}}\mathrm{t}^{2/3}}\mathrm{A}_{4,\mathrm{r}}\right)^2 \quad \text{(xiii)}$$

$$\mathrm{T}_{22} = \mathrm{T}_{33} = -\left(\overline{\Phi}_{,\mathrm{r}} + \mathrm{ieA}_1\overline{\Phi}\right)\left(\Phi_{,\mathrm{r}} - \mathrm{ieA}_1\Phi\right) + \left(\overline{\Phi}_{,\mathrm{t}} + \mathrm{ieA}_4\overline{\Phi}\right)\left(\Phi_{,\mathrm{t}} - \mathrm{ieA}_4\Phi\right) - \\ - \mathrm{m}_0^2\overline{\Phi}\Phi + \frac{1}{2}\left(\frac{2}{3\mathrm{t}}\mathrm{A}_1 + \mathrm{A}_{1,\mathrm{t}} - \frac{1}{\sqrt{\mathrm{a}}\mathrm{t}^{2/3}}\mathrm{A}_{4,\mathrm{r}}\right)^2 \quad \text{(xiv)}$$

$$\mathrm{T}_{44} = \left(\overline{\Phi}_{,\mathrm{r}} + \mathrm{ieA}_1\overline{\Phi}\right)\left(\Phi_{,\mathrm{r}} - \mathrm{ieA}_1\Phi\right) + \left(\overline{\Phi}_{,\mathrm{t}} + \mathrm{ieA}_4\overline{\Phi}\right)\left(\Phi_{,\mathrm{t}} - \mathrm{ieA}_4\Phi\right) + \\ + \mathrm{m}_0^2\overline{\Phi}\Phi + \frac{1}{2}\left(\frac{2}{3\mathrm{t}}\mathrm{A}_1 + \mathrm{A}_{1,\mathrm{t}} - \frac{1}{\sqrt{\mathrm{a}}\mathrm{t}^{2/3}}\mathrm{A}_{4,\mathrm{r}}\right)^2 \quad \text{(xv)}$$

$$\mathrm{T}_{14} = \left(\Phi_{,\mathrm{r}} - \mathrm{ieA}_1\Phi\right)\left(\overline{\Phi}_{,\mathrm{t}} + \mathrm{ieA}_4\overline{\Phi}\right) + \left(\overline{\Phi}_{,\mathrm{r}} + \mathrm{ieA}_1\overline{\Phi}\right)\left(\Phi_{,\mathrm{t}} - \mathrm{ieA}_4\Phi\right) \quad \text{(xvi)}$$

which represent the complete magnitude in considered preparation [8].

Conclusions

The problem of energy-momentum localization has been a subject of many researchers but still remains un-resolved. Numerous attempts have been made to explore a quantity which describes the distribution of energy-momentum due to matter, non-gravitational and gravitational fields. This paper continues the investigation of comparing various distributions presented in the literature in the framework of GR. The more information that is assembled on the subject, the better.

These results are obtained with a computational symbolic script which is developed in order to succeed in penetrating the problem of energy-momentum tensor problem, covering the huge volume of computation. The script is made in a reliable manner and represents a good starting point in order to continue this work [9, 12].

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