# SYMMETRIES AND INVARIANTS OF GENERALIZED YANG-MILLS MECHANICAL MODEL 

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#### Abstract

This paper proposes an algorithm for the Lie symmetries investigation in the case of a 2D general mechanical models arising from fields theories. General Lie operators are deduced firstly and, in the next step, the associated Lie invariants are derived. The generalization of 2D Yang-Mills mechanical model is chosen as a test model for this method. Keywords: Generalized symmetries, Yang-Mills system, mechanical models.


## 1. Introduction

The Yang-Mills theory offers one of the most fruitful examples of gauge theories with an intrinsic importance, but also able to act as a test model for various investigation procedures. A very interesting development of the Yang-Mills theory was attended in regard to the constrained dynamical systems, by transforming the field system with an infinite number of degrees of freedom in a mechanical model, with a finite number of them [5]. In recent years a lot of such models were intensively investigated [7, 9].

The symmetries of the original form of Yang-Mills mechanical system

$$
\left\{\begin{array}{l}
\ddot{x}=A x-2 a x y^{2}+4 b x^{3}  \tag{1}\\
\ddot{y}=B y-2 a y x^{2}+4 d y^{3}
\end{array}\right.
$$

(with fixed parameters) was investigated by R. Cimpoiasu and R. Constantinescu in [2] and [3]. This system is obtained from classical Yang-Mills theory for fields with no external sources:

$$
\partial_{\mu} F_{\mu \nu}^{a}+g \varepsilon^{a b c} A_{\mu}^{b} F_{\mu \nu}^{c}=0, \quad \mu, v=0,1,2,3 ; \quad a, b, c=1,2,3
$$

where $\left(A_{\mu}^{a}\right)$ is the quadri-potential and

$$
F_{\mu \nu}^{a}=\partial_{\mu} A_{\nu}^{a}-\partial_{\nu} A_{\mu}^{a}+g \varepsilon^{a b c} A_{\mu}^{b} A_{\nu}^{c} .
$$

In [3] was obtained four cases of integrabilitiy of this system (when $a=3 / 4, b=1 / 16$ and $B=4 A ; a=6, b=1$ and $B=A ; a=1, b=2$ and $B=A$; respectively $a=12, b=16$ and $A=4 B ;$ ). The mechanical system (1) was extended in [4] to a more general system of six equations. In this paper we choused to consider a general 2D system of the form:

$$
\left\{\begin{array}{l}
\ddot{x}=A x-2 a x y^{2}-4 b x^{3}-3 e x^{2} y-f y^{3}  \tag{2}\\
\ddot{y}=B y-2 a y x^{2}-4 d y^{3}-3 f x y^{2}-f x^{3}
\end{array}\right.
$$

where $A, B, a, b, c, e, f$ are parameters, system which is Hamiltonian system with the Hamiltonian

$$
\begin{equation*}
H=\frac{1}{2}\left(\dot{x}^{2}+\dot{y}^{2}\right)-\frac{A}{2} x^{2}-\frac{B}{2} y^{2}+a x^{2} y^{2}+b x^{4}+c y^{4}+e x^{3} y+f x y^{3} . \tag{3}
\end{equation*}
$$

After this introduction, the second section was dedicated to the presentation of a general method to compute the Lie symmetries for the second order 2D general mechanical model. The last section applies this method to the case of generalized 2D Yang-Mills mechanical model, rising up some cases of integrability of this system. Some remarks and conclusions will end the paper.

## 2. General methodology: Lie symmetries for 2D mechanical systems

Consider the general second order system of ordinary equations

$$
\left\{\begin{array}{l}
\ddot{x}=K_{1}(x, y)  \tag{4}\\
\ddot{y}=K_{2}(x, y)
\end{array}\right.
$$

The general theory of the Lie symmetries (see [1] or [7]) affirms that an infinitesimal symmetry operator will have the form:

$$
\begin{equation*}
U=\xi[t, x, y] \frac{\partial}{\partial t}+\varphi_{1}[t, x, y] \frac{\partial}{\partial x}+\varphi_{2}[t, x, y] \frac{\partial}{\partial y} \tag{5}
\end{equation*}
$$

where the notation $f[t, x, y]$ means a dependence of the function $f$ on the independent variable $t$ and the dependent variables $x, y$, with all their derivatives. The Lie symmetries for a PDE system which describes a dynamical system are obtained imposing the Lie invariance condition from Olver ([7]):

$$
\begin{equation*}
\left.p r^{(n)}(U)[\Delta]\right|_{\Delta=0}=0 \tag{6}
\end{equation*}
$$

which in the case of the system (1) is:

$$
\left\{\begin{array}{l}
U^{(2)}\left[K_{1}-\ddot{x}\right]=-\varphi_{1}^{t t}+\varphi_{1} K_{1, x}+\varphi_{2} K_{1, y}=0  \tag{7}\\
U^{(2)}\left[K_{2}-\ddot{y}\right]=-\varphi_{2}^{t t}+\varphi_{1} K_{2, x}+\varphi_{2} K_{2, y}=0
\end{array}\right.
$$

with the notation $K_{i, x}=\frac{\partial K_{i}}{\partial x}, K_{i, y}=\frac{\partial K_{i}}{\partial y}$.
An alternate version of the symmetries generating equation can be obtained by introducing the couple $Q=\left(Q^{1}, Q^{2}\right)$, known as the characteristic of the symmetry operator (2):

$$
\begin{equation*}
Q^{1} \equiv \varphi_{1}-\xi \dot{x}, Q^{2} \equiv \varphi_{2}-\xi \dot{y} \tag{8}
\end{equation*}
$$

The symmetries determining equations (7) become in this case:

$$
\left\{\begin{array}{l}
U^{(2)}\left[K_{1}-\ddot{x}\right]=-D_{t}^{2} Q^{1}+Q^{1} K_{1, x}+Q^{2} K_{1, y}=0  \tag{9}\\
U^{(2)}\left[K_{2}-\ddot{y}\right]=-D_{t}^{2} Q^{2}+Q^{1} K_{2, x}+Q^{2} K_{2, y}=0
\end{array}\right.
$$

Let us consider that $\left\{U_{r,} \mid r=\overline{1, n}\right\}$ represents the set of independent Lie symmetry operators for the system under consideration. The next step of the algorithm consists in finding the invariants $\left\{I_{r}(x, y, \dot{x}, \dot{y}), r=\overline{1, n}\right\}$ associated to independent set of Lie operators which are determined in the previous step. These invariants will be solutions of the equations:

$$
U_{r}^{(1)} I_{r}=0, \quad r=\overline{1, n}
$$

where $U_{r}^{(1)}$ represents the first order extensions of $U_{r}$. The set of second invariants $\left\{I_{r}\right\}$, $r=\overline{1, n}$ will be determined by integrating the equations ([3]):

$$
\begin{align*}
& \frac{\partial I_{r}}{\partial x}=-\dot{Q}^{1},
\end{align*} \frac{\frac{\partial I_{r}}{\partial \dot{x}}=Q^{1}}{\frac{\partial I_{r}}{\partial y}=-\dot{Q}^{2},} \frac{\frac{\partial I_{r}}{\partial \dot{y}}=Q^{2}}{}
$$

It is important to notice that any symmetry and the associated invariant correspond to a different situation. If the system under investigation is Hamiltonian, then an independent symmetry generates a prime integral, and two independent symmetries are sufficient to prove the integrability of the studied system.

## a. The case of zero order symmetries (the classical symmetries)

If we consider that the symmetries characteristics $Q^{1}$ and $Q^{2}$ are independent of the time derivatives of $x$ and $y$, the symmetries determining equations (7) become in this case:

$$
\left\{\begin{array}{l}
Q_{t t}^{1}+Q_{x x}^{1} \dot{x}^{2}+Q_{y y}^{1} \dot{y}^{2}+2 Q_{x y}^{1} \dot{y} \dot{y}+Q_{x}^{1} \ddot{x}+Q_{y}^{1} \ddot{y}+2 Q_{x t}^{1} \dot{x}+2 Q_{t x}^{1} \dot{y}=Q^{1} K_{1, x}+Q^{2} K_{1, y}  \tag{11}\\
Q_{t t}^{2}+Q_{x x}^{2} \dot{x}^{2}+Q_{y y}^{2} \dot{y}^{2}+2 Q_{x y}^{2} \dot{x} \dot{y}+Q_{x}^{2} \ddot{x}+Q_{y}^{2} \ddot{x}+2 Q_{x t}^{2} \dot{x}+2 Q_{t x}^{2} \dot{y}=Q^{1} K_{2, x}+Q^{2} K_{2, y}
\end{array}\right.
$$

Under this observation, by identifying the coefficients of time derivatives of the dependent variables in the left and right side of these equations, the classical symmetries criteria can be formulated as

## Proposition 1

The generalized vector field $U=Q^{1}[t, x, y] \frac{\partial}{\partial x}+Q^{2}[t, x, y] \frac{\partial}{\partial y}$ is a symmetry operator for for the system (4), where the right side members of the equations $K_{1}$ and $K_{2}$ are polynomials in $x$ and $y$ without free terms, if and only if $Q_{i}(i=1,2)$ are polynomials of first degree in all the dependent and independent variables, and the condition
$\left\{\begin{array}{l}Q_{x}^{1} K_{1}+Q_{y}^{1} K_{2}=Q^{1} K_{1, x}+Q^{2} K_{1, y} \\ Q_{x}^{2} K_{1}+Q_{y}^{2} K_{2}=Q^{1} K_{2, x}+Q^{2} K_{2, y}\end{array}\right.$
is verified.
Indeed, the annulations of all coefficients of time derivatives of the dependent variables $x$ and $y$ in (8) produce:

$$
Q_{x x}^{i}=Q_{y y}^{i}=Q_{x y}^{i}=Q_{x t}^{i}=Q_{y t}^{i}=Q_{t t}^{i}=0, \quad i=1,2 .
$$

and the proof of the Proposition 1 is immediate. Note that this criteria is valid for any polynomial 2D mechanical models, not only for the Yang-Mills mechanical model.

## b. The case of generalized symmetries

In this paper we restricted our-self to the case of linear dependence of the generalized symmetry operators on the velocities $\dot{x}$ and $\dot{y}$, then the symmetries characteristics $Q^{1}$ and $Q^{2}$ are given by:

$$
\begin{equation*}
Q^{1} \equiv Q_{11}(t, x, y) \dot{x}+Q_{12}(t, x, y) \dot{y}, Q^{2} \equiv Q_{21}(t, x, y) \dot{x}+Q_{22}(t, x, y) \dot{y} \tag{13}
\end{equation*}
$$

with the coefficients $Q_{i j}$ constants, i.e. the symmetry operator is linear in vitesses. In this case, the symmetry condition (8) rewrite as

$$
\left\{\begin{array}{c}
Q_{12} K_{2, x}=Q_{21} K_{1, y}  \tag{14}\\
Q_{12}\left(K_{1, x}-K_{2, y}\right)=K_{1, y}\left(Q_{11}-Q_{22}\right) \\
Q_{21}\left(K_{1, x}-K_{2, y}\right)=K_{2, x}\left(Q_{22}-Q_{11}\right)
\end{array}\right.
$$

The main result of this section can be formulated as:

## Proposition 2

Consider the system (4) where the right side members of the equations $K_{1}$ and $K_{2}$ are polynomials in $x$ and $y$. Then we have the following assertions:
a) The system (4) always admit the symmetry operator $U_{0}=\dot{x} \frac{\partial}{\partial x}+\dot{y} \frac{\partial}{\partial y}$. which corresponds to the point symmetry $U_{0}^{\prime}=\frac{\partial}{\partial t}$ (the time translation);
b) The operator $U_{1}=Q_{12} \dot{y} \frac{\partial}{\partial x}+Q_{21} \dot{x} \frac{\partial}{\partial y}$ is symmetry for the system (4) if and only if $K_{1, x}=K_{2, y}$ and $Q_{21} K_{1, y}=Q_{12} K_{2, x}$
c) If $K_{1, y}=Q_{12}\left(K_{1, x}-K_{2, y}\right)$ and $K_{2, x}=-Q_{21}\left(K_{1, x}-K_{2, y}\right)$;
then the system (4) admit the symmetry

$$
U_{2}=\left(Q_{12} \dot{y}-\frac{1}{2} \dot{x}\right) \frac{\partial}{\partial x}+\left(Q_{21} \dot{x}+\frac{1}{2} \dot{y}\right) \frac{\partial}{\partial y} .
$$

The proof of this Proposition is a straightforward computation.

## 3. The case of generalized Yang-Mills mechanical model

We applied the methodology exposed in the precedent section to the system (2):

$$
\left\{\begin{array}{l}
\ddot{x}=A x-2 a x y^{2}-4 b x^{3}-3 e x^{2} y-f y^{3}  \tag{15}\\
\ddot{y}=B y-2 a y x^{2}-4 d y^{3}-3 f x y^{2}-f x^{3}
\end{array}\right.
$$

First observation is that this system always posses the symmetry operator $U_{0}=\frac{\partial}{\partial t}$, from the Proposition 2. The associated invariant is the Hamiltonian (3).

Now, if we search the classical symmetries of (15), the Proposition 1 impose to search symmetries with the characteristics polynomials in of first degrees in $x$ and $y$. We can distingue two different cases of the existence of nontrivial symmetries:
a) If $\mathrm{a}=0, \mathrm{e}=0, \mathrm{f}=0$, we obtained the solutions $Q_{1}=m_{1} x+n_{1} y, Q_{2}=m_{2} x+n_{2} y$ where $m_{1}, m_{2}, n_{1}$ and $n_{2}$ are constants. The infinitesimal generators of the symmetries are

$$
\begin{equation*}
\text { i. } \quad U_{1}=x \frac{\partial}{\partial x}, U_{2}=y \frac{\partial}{\partial x}, U_{3}=x \frac{\partial}{\partial y}, U_{4}=y \frac{\partial}{\partial y} ; \tag{16}
\end{equation*}
$$

b) If $A=B, b=c$ and $e=f$, we have the solutions $Q_{1}=m x+n y, Q_{2}=n x+m y$ where
$m$ and $n$ are constants. The generators of the symmetries are:

$$
\begin{equation*}
U_{5}=x \frac{\partial}{\partial x}+y \frac{\partial}{\partial y} ; U_{6}=y \frac{\partial}{\partial x}+x \frac{\partial}{\partial y} \tag{17}
\end{equation*}
$$

If we look now for generalized symmetries linear on the velocities $\dot{x}$ and $\dot{y}$, by applying the Proposition 2 to the system (15), one obtain:
c) If $A=B, a=6 b=6 c$ and $e=f$, then the system (genYM1) admits a second independent generalized symmetry

$$
\begin{equation*}
U_{21}=\dot{y} \frac{\partial}{\partial x}+\dot{x} \frac{\partial}{\partial y} \tag{18}
\end{equation*}
$$

(The associated invariant has the form

$$
\begin{equation*}
\left.I 1=\dot{x} \dot{y}-\frac{A}{2}\left(x^{2}+y^{2}\right)+\frac{2 a}{3}\left(x^{3} y+x y^{3}\right)+\frac{3 e}{2} x^{2} y^{2}+\frac{e}{4}\left(x^{4}+y^{4}\right) .\right) \tag{19}
\end{equation*}
$$

Note that this case is the generalization of the second case of integrability pointed out by R. Cimpoiasu in [3].
d) If $A=B, a=e=f=0$ and $b, c$ are arbitrary, then the system (15) admits the second independent generalized symmetry:

$$
\begin{equation*}
U_{31}=-\dot{x} \frac{\partial}{\partial x}+\dot{y} \frac{\partial}{\partial y} . \tag{20}
\end{equation*}
$$

(The associated invariant has the form

$$
\begin{equation*}
\left.I 2=\frac{A}{2}\left(\dot{x}^{2}+\dot{y}^{2}\right)+\frac{1}{4}\left(-\dot{x}^{2}+\dot{y}^{2}\right)+\frac{1}{2}\left(b x^{4}+c y^{4}\right) .\right) \tag{21}
\end{equation*}
$$

e) If $A=B, a=0, b=c$ and $e=-f$, then the system (15) admits the independent generalized symmetry:
$U_{32}=\left(q \dot{y}-\frac{1}{2} \dot{x}\right) \frac{\partial}{\partial x}+\left(q \dot{x}+\frac{\dot{y}}{2}\right) \frac{\partial}{\partial y}$,
where $q=e / 4 b$.
f) If $A=B, a \neq 0, e \neq f$, and the coefficients $b$ and $c$ verify:

$$
\begin{equation*}
b=\frac{a e}{6(e-f)}+\frac{1}{6}, ; c=\frac{a f}{6(f-e)}+\frac{1}{6} \tag{23}
\end{equation*}
$$

then the system (15) admits the second independent generalized symmetry

$$
\begin{equation*}
U_{23}=\left(q \dot{y}-\frac{1}{2} \dot{x}\right) \frac{\partial}{\partial x}+\left(q \dot{x}+\frac{1}{2} \dot{y}\right) \frac{\partial}{\partial y}, \tag{24}
\end{equation*}
$$

where $q=3(e-f) / 2 a$.

In all other cases, their are no generalized symmetry linear in $\dot{x}$ and $\dot{y}$.

## Conclusions

We investigated the problem of the existence of classical and generalized symmetries of the generalized Yang-Mils Mechanical system (2) using the Lie approach. The main results we obtained could be synthesized as follows:
(i) The generalized YM mechanical model always posses a generalized symmetry $U_{0}=\frac{\partial}{\partial t}$, which corresponds to the Hamiltonian.
(ii) In two cases we have obtained other classical symmetries depending on the variables $x$ and $y$ (the 16-17 formulas). The second case was not trivial, even is a direct extension of the classical Yang-Mills mechanical system. Both cases represent integrable Hamiltonian systems.
(iii) In four cases, we obtained a second independent generalized symmetry - (18) (20) (22) (24). The last three of them pointed out new cases of integrability of the general YangMills mechanical system, distinct from the cases deduced in [3].

The methodological approach exposed here do not require the investigated system to be Hamiltonian and can be easy adapted and applied in the case of other mechanical models of field theories.

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