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QUANTUM OBSERVABLES ON THE DE SITTER SPACETIME

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Abstract
The theory of external symmetry in curved spacetimes we have proposed few years ago allows us to correctly define the operators of the quantum field theory on curved backgrounds. Particularly, despite of some doubts appeared in literature, we have shown that a well-defined energy operator can be considered on the de Sitter manifold. With its help new quantum modes were obtained for the scalar, Dirac and vector fields on the de Sitter spacetimes. A short review of these results is presented in this report.
Keywords:

1. Introduction

In general relativity [1 - 3] the development of the quantum field theory in curved spacetimes [4] give rise to many difficult problems related to the physical interpretation of the one-particle quantum modes that may indicate how the quantum fields can be quantized. This is because the form and the properties of the particular solutions of the free field equations [5 - 9] are strongly dependent on the procedure of separation of variables and, implicitly, on the choice of the local chart (natural frame). Moreover, when the fields have spin the situation is more complicated since then the field equations and, therefore, the form of their particular solutions depend, in addition, on the tetrad gauge in which one works [10, 1]. Under such circumstances it would be helpful to use the traditional method of the quantum theory in flat spacetime based on the complete sets of commuting operators that determine the quantum modes as common eigenstates and give physical meaning to the constants of the separation of variables which are just the eigenvalues of these operators.

A good step in this direction could be to proceed like in special relativity looking for the generators of the geometric symmetries similar to the familiar momentum, angular momentum and spin operators of the Poincaré covariant field theories [11]. However, the relativistic covariance in the sense of general relativity is too general to play the same role as the Lorentz or Poincaré covariance in special relativity. In its turn the tetrad gauge invariance of the theories with spin represents another kind of general symmetry that is not able to produce itself conserved quantities [1]. Therefore, one must focus only on the isometry transformations that point out the specific spacetime symmetry related to the presence of the Killing vectors [1, 3, 12].

Our approach is a general theory of tetrad gauge invariant fields defined on curved spacetimes with given external symmetries. This predicts how must transform these fields under isometries in order to leave invariant the form of the field equations and to obtain the general form of the generators of these transformations. The basic idea is that the isometries transformations must preserve the position of the local frames with respect to the natural one. Such transformations can be constructed as isometries *combined* with suitable tetrad gauge transformations necessary for keeping unchanged the tetrad field components. In this way we obtain the external symmetry group showing that it is locally isomorphic with the isometry group.

Furthermore, we show how can be used these results for finding the quantum modes of the Dirac field on dS spacetimes. For quantizing the Dirac field it is convenient to chose the moving charts with Cartesian coordinates where we can identify the components of the momentum operator and normalize the fundamental solutions using the momentum representation [20]. Obviously, to this end our theory of external symmetry is crucial since this gives us the main operators we need as generators of the spinor or vector representations.

2. Relativistic covariance

In the Lagrangian field theory in curved spacetimes the relativistic covariant equations of scalar, vector or tensor fields arise from actions that are invariant under general coordinate transformations. Moreover, when the fields have spin in the sense of the $SL(2,C)$ symmetry then the action must be invariant under tetrad gauge transformations [10].

2. 1. Gauge transformations

Let us consider the curved spacetime M and a local chart (natural frame) of coordinates $x^{\mu}, \mu = 0,1,2,3$. Given a gauge, we denote by $e_{\hat{\mu}}(x)$ the tetrad fields that define the local (unholonomic) frames, in each point x, and by $\hat{e}^{\hat{\mu}}(x)$ those defining the corresponding coframes. These fields have the usual properties $\hat{e}^{\hat{\mu}}_{\alpha} e^{\alpha}_{\hat{\nu}} = \delta^{\hat{\mu}}_{\hat{\nu}}$ $, \hat{e}^{\hat{\mu}}_{\alpha} e^{\beta}_{\hat{\mu}} = \delta^{\beta}_{\alpha}$,

$e_{\hat{\mu}} \cdot e_{\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}}$ and $\hat{e}^{\hat{\mu}} \cdot \hat{e}^{\hat{\nu}} = \eta^{\hat{\mu}\hat{\nu}}$, where $\eta = \mathrm{diag}(1,-1,-1,-1)$ is the Minkowski metric. From the line element:

$$ds^2 = \eta_{\hat{\mu}\hat{\nu}} d\hat{x}^{\hat{\mu}} d\hat{x}^{\hat{\nu}} = g_{\mu\nu}(x) dx^{\mu} dx^{\nu}, \tag{1}$$

expressed in terms of 1-forms, $d\hat{x}^{\hat{\mu}} = \hat{e}^{\hat{\mu}}_{\nu} dx^{\nu}$, we get the components of the metric tensor of the natural frame, $g_{\mu\nu} = \eta_{\hat{\alpha}\hat{\beta}} \hat{e}^{\hat{\alpha}}_{\mu} \hat{e}^{\hat{\beta}}_{\nu}$ and $g^{\mu\nu} = \eta^{\hat{\alpha}\hat{\beta}} e^{\mu}_{\hat{\alpha}} e^{\nu}_{\hat{\beta}}$. These raise or lower the *natural* vector indices, i.e., the Greek ones ranging from 0 to 3, while for the *local* vector indices, denoted by hat Greeks and having the same range, we must use the Minkowski metric. The local derivatives $\hat{\partial}_{\hat{\nu}} = e^{\mu}_{\hat{\nu}} \partial_{\mu}$ satisfy the commutation rules $[\hat{\partial}_{\hat{\mu}}, \hat{\partial}_{\hat{\nu}}] = e^{\alpha}_{\hat{\mu}} e^{\beta}_{\hat{\nu}} (\hat{e}^{\hat{\sigma}}_{\alpha,\beta} - \hat{e}^{\hat{\sigma}}_{\beta,\alpha}) \hat{\partial}_{\hat{\sigma}} = C_{\hat{\mu}\hat{\nu}\cdot}^{\cdot\cdot\hat{\sigma}} \hat{\partial}_{\hat{\sigma}}$ defining the Cartan coefficients which help us to write the *conecttion* components in local frames as

$$\hat{\Gamma}^{\hat{\sigma}}_{\hat{\mu}\hat{\nu}} = e^{\alpha}_{\hat{\mu}} e^{\beta}_{\hat{\nu}} (\hat{e}^{\hat{\sigma}}_{\gamma} \Gamma^{\gamma}_{\alpha\beta} - \hat{e}^{\hat{\sigma}}_{\beta,\alpha}) = \frac{1}{2} \eta^{\hat{\sigma}\hat{\lambda}} (C_{\hat{\mu}\hat{\nu}\hat{\lambda}} + C_{\hat{\lambda}\hat{\mu}\hat{\nu}} + C_{\hat{\lambda}\hat{\nu}\hat{\mu}}). \tag{2}$$

The Minkowski metric $\eta_{\hat{\mu}\hat{\nu}}$ remains invariant under the transformations of the *gauge* group of this metric, $G(\eta) = O(3,1)$. This has as subgroup the Lorentz group, $L^{\uparrow}_{+}$, of the transformations $\Lambda[A(\omega)]$ corresponding to the transformations $A(\omega) \in SL(2,C)$ through the canonical homomorphism [11]. In the standard *covariant* parametrization, with the real parameters $\omega^{\hat{\alpha}\hat{\beta}} = -\omega^{\hat{\beta}\hat{\alpha}}$, we have:

$$A(\omega) = e^{-\frac{i}{2}\omega^{\hat{\alpha}\hat{\beta}} S_{\hat{\alpha}\hat{\beta}}}, \tag{3}$$

where $S_{\hat{\alpha}\hat{\beta}}$ are the covariant basis-generators of the $SL(2,C)$ Lie algebra which satisfy:

$$\left[S_{\hat{\mu}\hat{\nu}}, S_{\hat{\sigma}\hat{\tau}}\right] = i(\eta_{\hat{\mu}\hat{\tau}} S_{\hat{\nu}\hat{\sigma}} - \eta_{\hat{\mu}\hat{\sigma}} S_{\hat{\nu}\hat{\tau}} + \eta_{\hat{\nu}\hat{\sigma}} S_{\hat{\mu}\hat{\tau}} - \eta_{\hat{\nu}\hat{\tau}} S_{\hat{\mu}\hat{\sigma}}). \tag{4}$$

For small values of $\omega^{\hat{\alpha}\hat{\beta}}$ the matrix elements of the transformations Λ can be written as $\Lambda[A(\omega)]^{\hat{\mu}\cdot}_{\cdot\hat{\nu}} = \delta^{\hat{\mu}}_{\hat{\nu}} + \omega^{\hat{\mu}\cdot}_{\cdot\hat{\nu}} + \cdots$.

Now we assume that M is orientable and time-orientable such that $L^{\uparrow}_{+}$ can be considered as the gauge group of the Minkowski metric [3]. Then the fields with spin can be defined as in the case of the flat spacetime, with the help of the finite-dimensional *linear* representations, ρ, of the $SL(2,C)$ group [11]. In general, the fields $\psi_{\rho} : M \to V_{\rho}$ are defined over M with values in the vector spaces V_{ρ} of the representations ρ. In the following we systematically use the bases of V_ρ labeled only by spinor or vector *local* indices defined with respect to the axes of the local frames given by the tetrad fields.

The relativistic covariant field equations are derived from actions [10, 1],

$$S[\psi_\rho, e] = \int d^4x \sqrt{g}\, L(\psi_\rho, D^\rho_{\hat{\mu}}\psi_\rho), \quad g = |\det(g_{\mu\nu})|, \tag{5}$$

depending on the matter fields, ψ_ρ, and the components of the tetrad fields, e, which represent the gravitational degrees of freedom. The covariant derivatives,

$$D^\rho_{\hat{\alpha}} = \hat{e}^\mu_{\hat{\alpha}} D^\rho_\mu = \hat{\partial}_{\hat{\alpha}} + \frac{i}{2}\rho(S^{\hat{\beta}\cdot}_{\cdot\hat{\gamma}})\hat{\Gamma}^{\hat{\gamma}}_{\hat{\alpha}\hat{\beta}}, \tag{6}$$

assure the invariance of the whole theory under the *tetrad gauge* transformations,

$$\begin{aligned} &\hat{e}^{\hat{\alpha}}_\mu(x) \to \hat{e}'^{\hat{\alpha}}_\mu(x) = \Lambda[A(x)]^{\hat{\alpha}\cdot}_{\cdot\hat{\beta}}\hat{e}^{\hat{\beta}}_\mu(x), \\ &e^\mu_{\hat{\alpha}}(x) \to e'^\mu_{\hat{\alpha}}(x) = \Lambda[A(x)]^{\cdot\hat{\beta}}_{\hat{\alpha}\cdot} e^\mu_{\hat{\beta}}(x), \\ &\psi_\rho(x) \to \psi_\rho'(x) = \rho[A(x)]\psi_\rho(x), \end{aligned} \tag{7}$$

determined by the mappings $A: M \to SL(2,C)$ the values of which are the local $SL(2,C)$ transformations $A(x) \equiv A[\omega(x)]$. These mappings can be organized as a group, G, with respect to the multiplication $\times$ defined as $(A' \times A)(x) = A'(x)A(x)$. The notation Id stands for the mapping identity, $Id(x) = 1 \in SL(2,C)$, while A^{-1} is the inverse of A, $(A^{-1})(x) = [A(x)]^{-1}$.

2. 2. Combined transformations

The general coordinate transformations are automorphisms of M which, in the passive mode, can be seen as changes of the local charts corresponding to the same domain of M [3, 12]. If x and x' are the coordinates of a point in two different charts then there is a mapping ϕ between these charts giving the coordinate transformation, $x \to x' = \phi(x)$. These transformations form a group with respect to the composition of mappings, $\circ$, defined as usual, i.e. $(\phi' \circ \phi)(x) = \phi'[\phi(x)]$. We denote this group by A, its identity map by id and the inverse mapping of ϕ by ϕ^{-1}.

The automorphisms change all the components carrying natural indices including those of the tetrad fields [1] changing thus the positions of the local frames with respect to the natural ones. If we assume that the physical experiment makes reference to the axes of the local frame then we arrive to the necessity of introducing the *combined* transformations denoted by (A,ϕ) and defined as gauge transformations, given by $A \in G$, followed by automorphisms, $\phi \in A$. In this new notation the pure gauge transformations will appear as (A,id) while the automorphisms will be denoted from now by (Id,ϕ).

The effect of a combined transformation (A,ϕ) upon our basic fields, ψ_ρ, e and $\hat{e}$ is $x \to x' = \phi(x), e(x) \to e'(x'), \hat{e}(x) \to \hat{e}'(x')$ and $\psi_\rho(x) \to \psi'_\rho(x') = \rho[A(x)]\psi_\rho(x)$ where e' are the transformed tetrads of the components:

$$e'^{\mu}_{\hat{\alpha}}[\phi(x)] = \Lambda[A(x)]^{\cdot\hat{\beta}}_{\hat{\alpha}\cdot}\, e^{\nu}_{\hat{\beta}}(x)\frac{\partial\phi^{\mu}(x)}{\partial x^{\nu}}, \tag{8}$$

which determine the components of $\hat{e}'$ too. Thus we have written down the most general transformation laws that leave the action invariant in the sense that $S[\psi_\rho', e'] = S[\psi_\rho, e]$. The field equations derived from S, written in local frames as $(E_\rho\psi_\rho)(x) = 0$, *covariantly* transform according to the rule $(E_\rho\psi_\rho)(x) \to (E'_\rho\psi'_\rho)(x') = \rho[A(x)](E_\rho\psi_\rho)(x)$ since the operators E_ρ involve covariant derivatives [1].

The association among the transformations of the groups G and A must lead to a new group with a specific multiplication. In order to find how looks this new operation it is convenient to use the composition among the mappings A and ϕ (taken only in this order) giving new mappings, $A \circ \phi \in G$, defined as $(A \circ \phi)(x) = A[\phi(x)]$. The calculation rules $Id \circ \phi = Id$, $A \circ id = A$ and $(A' \times A) \circ \phi = (A' \circ \phi) \times (A \circ \phi)$ are obvious. With these ingredients we define the new multiplication $(A', \phi') * (A, \phi) = \big((A' \circ \phi) \times A, \phi' \circ \phi\big)$. It is clear that now the identity is (Id, id) while the inverse of a pair (A,ϕ) reads $(A,\phi)^{-1} = (A^{-1} \circ \phi^{-1}, \phi^{-1})$. First of all we observe that the operation $*$ is well-defined and represents the composition among the combined transformations since these can be expressed, according to their definition, as $(A,\phi) = (Id,\phi) * (A, id)$. Furthermore, we can convince ourselves that if we perform successively two arbitrary combined transformations, (A,ϕ) and (A',ϕ'), then the resulting transformation is just $(A',\phi') * (A,\phi)$. This means that the combined transformations form a group with respect to the multiplication $*$. It is not difficult to verify that this group, denoted by G, is the semidirect product $G = G \,\&\, A$ where G is the *invariant* subgroup while A is an usual one.

3. External symmetry

In general, the symmetry of any manifold M is given by its isometry group whose transformations leave invariant the metric tensor in any chart. The scalar field transforms under isometries according to the standard scalar representation generated by the orbital generators related to the Killing vectors of M [1, 3, 12]. In the following we present the generalization of this theory of symmetry to fields with spin, for which we have defined the *external* symmetry group and its representations [15].

3. 1. Isometries

There are conjectures when several coordinate transformations, $x \to x' = \phi_\xi(x)$, depend on N independent real parameters, ξ^a ($a,b,c... = 1,2,...,N$), such that $\xi = 0$ corresponds to the identity map, $\phi_{\xi=0} = id$. The set of these mappings is a Lie group [23], $G \in G$, if they accomplish the composition rule $\phi_{\xi'} \circ \phi_\xi = \phi_{f(\xi',\xi)}$, where the functions $f : G \times G \to G$ define the group multiplication. These must satisfy $f^a(0,\xi) = f^a(\xi,0) = \xi^a$ and $f^a(\xi^{-1},\xi) = f^a(\xi,\xi^{-1}) = 0$ where ξ^{-1} are the parameters of the inverse mapping of ϕ_ξ, $\phi_{\xi^{-1}} = \phi_\xi^{-1}$. Moreover, the structure constants of G can be calculated as [24]

$$c_{abc} = \left(\frac{\partial f^c(\xi,\xi')}{\partial \xi^a \partial \xi'^b} - \frac{\partial f^c(\xi,\xi')}{\partial \xi^b \partial \xi'^a} \right)_{|\xi=\xi'=0}. \tag{9}$$

For small values of the group parameters the infinitesimal transformations, $x^\mu \to x'^\mu = x^\mu + \xi^a k_a^\mu(x) + \cdots$, are given by the vectors k_a whose components,

$$k_a^\mu = \frac{\partial \phi_\xi^\mu}{\partial \xi^a}_{|\xi=0}, \tag{10}$$

satisfy the identities $k_a^\mu k_{b,\mu}^\nu - k_b^\mu k_{a,\mu}^\nu + c_{abc} k_c^\nu = 0$, resulting from Eq. (9).

In the following we restrict ourselves to consider only the *isometry* transformations, $x' = \phi_\xi(x)$, which leave invariant the components of the metric tensor [1, 12], i.e.

$$g_{\alpha\beta}(x') \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial x'^\beta}{\partial x^\nu} = g_{\mu\nu}(x). \tag{11}$$

These form the isometry group $G \equiv I(M)$ which is the Lie group giving the symmetry of the spacetime M. We consider that this has N independent parameters and, therefore, $k_a, a = 1,2,...N$, are independent Killing vectors (which satisfy $k_{a\mu;\nu} + k_{a\nu;\mu} = 0$). Then their corresponding Lie derivatives form a basis of the Lie algebra $i(M)$ of the group $I(M)$ [12].

However, in practice we are interested to find the operators of the relativistic quantum theory related to these geometric objects which describe the symmetry of the background. For this reason we focus upon the operator-valued representations [25] of the group $I(M)$ and its algebra. The *scalar* field $\psi : M \to C$ transforms under isometries as $\psi(x) \to \psi'[\phi_\xi(x)] = \psi(x)$. This rule defines the representation $\phi_\xi \to T_\xi$ of the group $I(M)$ whose operators have the action $\psi' = T_\xi \psi = \psi \circ \phi_\xi^{-1}$. Hereby it results that the operators of infinitesimal transformations, $T_\xi = 1 - i\xi^a L_a + \cdots$, depend on the basis-generators, $L_a = -ik_a^\mu \partial_\mu$ with $a = 1,2,...,N$, which are completely determined by the Killing vectors. We find that they obey the commutation rules $[L_a, L_b] = ic_{abc}L_c$, given by the structure constants of $I(M)$. In other words they form a basis of the operator-valued representation of the Lie algebra $i(M)$ in a carrier space of scalar fields. Notice that in the usual quantum mechanics the operators similar to the generators L_a are called often *orbital* generators.

3. 2. The group of external symmetry

It is natural to suppose that the good symmetry transformations we need are combined transformations in which the isometries are preceded by appropriate gauge transformations such that not only the form of the metric tensor should be conserved but the form of the tetrad field components too. Thus we arrive at the main point of our theory. We introduce the *external symmetry* transformations, (A_ξ, ϕ_ξ), as combined transformations involving isometries and corresponding gauge transformations necessary to *preserve the gauge*. We assume that in a fixed gauge, given by the tetrad fields e and $\hat{e}$, A_ξ is defined by

$$\Lambda[A_\xi(x)]^{\hat{\alpha}\cdot}_{\cdot\hat{\beta}} = \hat{e}^{\hat{\alpha}}_\mu[\phi_\xi(x)] \frac{\partial \phi^\mu_\xi(x)}{\partial x^\nu} e^\nu_{\hat{\beta}}(x), \tag{12}$$

with the supplementary condition $A_{\xi=0}(x) = 1 \in SL(2,C)$. Since ϕ_ξ is an isometry Eq.(11) guarantees that $\Lambda[A_\xi(x)] \in L^\uparrow_+$ and, implicitly, $A_\xi(x) \in SL(2,C)$. Then the transformation laws of our fields are

$$(A_\xi, \phi_\xi): \quad \begin{array}{rclcr} x & \to & x' & = & \phi_\xi(x), \\ e(x) & \to & e'(x') & = & e[\phi_\xi(x)], \\ \hat{e}(x) & \to & \hat{e}'(x') & = & \hat{e}[\phi_\xi(x)], \\ \psi_\rho(x) & \to & \psi_\rho{}'(x') & = & \rho[A_\xi(x)]\psi_\rho(x). \end{array} \tag{13}$$

The mean virtue of these transformations is that they leave *invariant* the form of the operators of the field equations, E_ρ, in local frames. This is because the components of the tetrad fields and, consequently, the covariant derivatives in local frames, $D^{\rho}_{\hat{\mu}}$, do not change their form.

For small ξ^a the covariant $SL(2,C)$ parameters of $A_\xi(x) \equiv A[\omega_\xi(x)]$ can be written as $\omega_\xi^{\hat{\alpha}\hat{\beta}}(x) = \xi^a \Omega_a^{\hat{\alpha}\hat{\beta}}(x) + \cdots$ where, according to Eqs.(3) and (12), we have

$$\Omega_a^{\hat{\alpha}\hat{\beta}} \equiv \frac{\partial \omega_\xi^{\hat{\alpha}\hat{\beta}}}{\partial \xi^a}_{|\xi=0} = \left(\hat{e}^{\hat{\alpha}}_\mu k^\mu_{a,\nu} + \hat{e}^{\hat{\alpha}}_{\nu,\mu} k^\mu_a\right) e^\nu_{\hat{\lambda}} \eta^{\hat{\lambda}\hat{\beta}}. \tag{14}$$

We must specify that these functions are antisymmetric if and only if k_a are Killing vectors. This indicates that the association among isometries and the gauge transformations defined by Eq.(12) is correct.

The transformations (A_ξ, ϕ_ξ) form a Lie group related to $I(M)$. Starting with Eq.(12) we find that $(A_{\xi'} \circ \phi_\xi) \times A_\xi = A_{f(\xi',\xi)}$ and we obtain

$$(A_{\xi'}, \phi_{\xi'}) * (A_\xi, \phi_\xi) = (A_{f(\xi',\xi)}, \phi_{f(\xi',\xi)}), \tag{15}$$

and $(A_{\xi=0}, \phi_{\xi=0}) = (Id, id)$. Thus we have shown that the pairs (A_ξ, ϕ_ξ) form a Lie group with respect to the operation $*$. We say that this is the external symmetry group of M and we denote it by $S(M) \subset G$. From Eq.(15) we understand that $S(M)$ is *locally isomorphic* with $I(M)$ and, therefore, the Lie algebra of $S(M)$, denoted by $s(M)$, is isomorphic with $i(M)$ having the same structure constants.

3. 3. Representations

The last of Eqs.(13) which gives the transformation law of the field ψ_ρ defines the operator-valued representation $(A_\xi, \phi_\xi) \to T^\rho_\xi$ of the group $S(M)$,

$$(T^\rho_\xi \psi_\rho)[\phi_\xi(x)] = \rho[A_\xi(x)]\psi_\rho(x). \tag{16}$$

The mentioned invariance under these transformations of the operators of the field equations in local frames reads:

$$T^\rho_\xi E_\rho (T^\rho_\xi)^{-1} = E_\rho. \tag{17}$$

Since $A_\xi(x) \in SL(2,C)$ we say that this representation is *induced* by the representation ρ of $SL(2,C)$ [25, 26]. As we have shown in Sec. 2. 2, if ρ is a vector or tensor representation (having only integer spin components) then the effect of the transformation (16) upon the components carrying natural indices is due only to ϕ_ξ.

The basis-generators of the representations of the Lie algebra $s(M)$ are the operators

$$X_a^\rho = i\frac{\partial T_\xi^\rho}{\partial \xi^a}_{|\xi=0} = L_a + S_a^\rho, \tag{18}$$

which appear as sums among the orbital generators and the *spin terms* which have the action $(S_a^\rho \psi_\rho)(x) = \rho[S_a(x)]\psi_\rho(x)$. This is determined by the form of the *local* $SL(2,C)$ generators,

$$S_a(x) = i\frac{\partial A_\xi(x)}{\partial \xi^a}_{|\xi=0} = \frac{1}{2}\Omega_a^{\hat\alpha\hat\beta}(x) S_{\hat\alpha\hat\beta}, \tag{19}$$

that depend on the functions (14). Furthermore we can verify the expected commutation rules $[X_a^\rho, X_b^\rho] = ic_{abc}X_c^\rho$. Thus we have derived the basis-generators of the operator-valued representation of $s(M)$ induced by the linear representation ρ of $SL(2,C)$. All the operators of this representation commute with the operator E_ρ since, according to Eqs. (17) and (18), we have $[E_\rho, X_a^\rho] = 0$ for all $a = 1,2,...,N$. Therefore, for defining quantum modes we can use the set of commuting operators containing the Casimir operators of $s(M)$, the operators of its Cartan subalgebra and E_ρ.

Finally, we must specify that the basis-generators (18) of the representations of the $s(M)$ algebra can be written in covariant form as:

$$X_a^\rho = -ik_a^\mu D_\mu^\rho + \frac{1}{2}k_{a\mu;\nu}\, e_{\hat\alpha}^\mu e_{\hat\beta}^\nu \rho(S^{\hat\alpha\hat\beta}), \tag{20}$$

generalizing thus the important result obtained in Ref. [14] for the Dirac field.

4. Observables on de Sitter spacetime

Let (M,g) be the de Sitter (dS) spacetime defined as a hyperboloid of radius $R = 1/\omega$ in the $(1+4)$ - dimensional flat spacetime, $({}_5M, {}_5\eta)$, of coordinates $z^A, A,B,... = 0,1,2,3,5$, and metric ${}_5\eta = \mathrm{d}iag(1,-1,-1,-1,-1)$. The hyperboloid equation,

$${}_5\eta_{AB} z^A z^B = -R^2, \qquad R = 1/\omega = \sqrt{3/|\Lambda_c|}. \tag{21}$$

shows that this manifold is the homogeneous space of the pseudo-orthogonal group $SO(4,1)$. This group is in the same time the gauge group of the metric ${}_5\eta$ and the isometry group of the dS spacetime, $G[{}_5\eta] = I(M) = SO(1,4)$. For this reason it is convenient to use the covariant real parameters ${}_5\omega^{AB} = -{}_5\omega^{BA}$ since in this case the orbital basis-generators of the representation of $SO(4,1)$, carried by the space of the scalar functions over M^5, have the standard form:

$$ {}_5L_{AB} = i\left[{}_5\eta_{AC}\, z^C \partial_B - {}_5\eta_{BC}\, z^C \partial_A\right] \tag{22} $$

They will give us directly the orbital basis-generators ${}_5L_{(AB)}$ of the scalar representations of $I(M)$. Indeed, starting with the functions $Z^A(x)$ that solve the equation (21) in a given chart $\{x\}$, one can write down the operators (22) as ${}_5L_{AB} = L_{(AB)} = -ik^{\mu}_{(AB)}\partial_{\mu}$, finding thus the orbital generators $L_{(AB)}$ and implicitly the components $k^{\mu}_{(AB)}(x)$ of the Killing vectors associated to the parameters ${}_5\omega^{AB}$ [15]. Furthermore, one has to calculate the spin parts $S_{(AB)}$, according to Eqs. (19) and (14), arriving to the final form of the basis-generators $X_{(AB)} = L_{(AB)} + \rho(S_{(AB)})$ of any representation ρ of $S(M)$.

4. 1. Static and moving charts

On M there are many static charts for which the time-like Killing vector field $i\partial_{t_s}$ depends on the *static* time t_s. We denote by $\{t_s, \vec{x}_s\}$ the chart with Cartesian coordinates associated to the chart $\{t_s, r_s, \theta, \phi\}$ with spherical coordinates and the conformal spherical line element:

$$ ds^2 = \frac{1}{\cosh^2 \omega r_s}\left[dt_s^2 - dr_s^2 - \frac{1}{\omega^2}\sinh^2 \omega r_s\, (d\theta^2 + \sin^2\theta\, d\phi^2)\right], \tag{23} $$

Another type of static charts with Cartesian coordinates, $\{t_s, \vec{\hat{x}}_s\}$, or spherical ones, $\{t_s, \hat{r}_s, \theta, \phi\}$, have the line element:

$$ ds^2 = (1-\omega^2\hat{r}_s^2)dt_s^2 - \frac{d\hat{r}_s^2}{1-\omega^2\hat{r}_s^2} - \hat{r}_s^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{24} $$

with a finite event horizon at $\omega\hat{r}_s = 1$. The radial coordinates of these two types of charts are related through $\omega\hat{r}_s = \tanh \omega r_S$.

The principal moving charts of physical interest with Cartesian, $\{t, \vec{x}\}$, or spherical coordinates, $\{t, r, \theta, \phi\}$ have FRW line elements as:

$$ ds^2 = dt^2 - e^{2\omega t} d\vec{x}^2. \tag{25} $$

The time $t \in (-\infty, \infty)$ of this chart is interpreted as the *proper* time of an observer at $\vec{x} = 0$. Another moving chart which play here a special role is the chart $\{t_c, \vec{x}\}$ with the *conformal* time t_c and Cartesian spaces coordinates x^i. This chart covers only a half of the manifold M, for $t_c \in (-\infty, 0)$ and $\vec{x} \in D \equiv R^3$. Nevertheless, it has the advantage of a simple conformal flat line element [4],

$$ds^2 = \frac{1}{\omega^2 t_c{}^2}\left(dt_c{}^2 - d\vec{x}^2\right). \tag{26}$$

Moreover, the conformal time t_c is related to the proper time t through

$$\omega t_c = -e^{-\omega t}. \tag{27}$$

The coordinates of the static and moving charts are related by

$$t_s = t - \frac{1}{2\omega}\ln\left(1 - \omega^2 r^2 e^{2\omega t}\right), \quad r_s = \frac{1}{2\omega}\ln\frac{1 + \omega r e^{\omega t}}{1 - \omega r e^{\omega t}}, \quad \hat{r}_s = r e^{\omega t}. \tag{28}$$

4. 2. The $so(4,1)$ basis generators

The next step is to calculate the basis-generators $X_{(AB)}$ of any representation ρ of $S(M)$. However they have different forms which depend on the chart and the gauge we use. Here we restrict ourselves to consider the moving charts and the diagonal Cartesian gauge in the chart $\{t_c, \vec{x}\}$ for which the non-vanishing tetrad components are [8]:

$$e^0_0 = -\omega t_c, \quad e^i_j = -\delta^i_j \omega t_c, \quad \hat{e}^0_0 = -\frac{1}{\omega t_c}, \quad \hat{e}^i_j = -\delta^i_j \frac{1}{\omega t_c}. \tag{29}$$

The group $SO(4,1)$ includes the subgroup $E(3) = T(3) \& SO(3)$ which is just the isometry group of the 3-dimensional Euclidean space of our moving charts, $\{t_c, \vec{x}\}_*$ and $\{t, \vec{x}\}_*$, formed by R^3 translations, $x^i \to x^i + a^i$, and proper rotations, $x^i \to R^{i\cdot}_{\cdot j} x^j$ with $R \in SO(3)$ [11]. Therefore, the basis-generators of its universal covering group, $\widetilde{E}(3) = T(3) \& SU(2) \subset S(M)$, can be interpreted as the components of the momentum, $\vec{P}$, and total angular momentum, $\vec{J}$, operators. The problem of the Hamiltonian (or energy) operator seems to be more complicated, but we know that in the mentioned static central charts with the static time t_s this is $H = \omega X_{(05)} = i\partial_{t_s}$ [15]. Thus the Hamiltonian operator and the components of the momentum and total angular momentum operators (P^i and

$J^i = \varepsilon_{ijk} J_{jk}/2$ respectively) can be identified as being the following basis-generators of the representation ρ of $S(M)$

$$H \equiv \omega X_{(05)} = -i\omega(t_c \partial_{t_c} + x^i \partial_i) \tag{30}$$

$$P^i \equiv \omega\left(X_{(5i)} - X_{(0i)}\right) = -i\partial_i \tag{31}$$

$$J_{ij} \equiv X_{(ij)} = -i(x^i \partial_j - x^j \partial_i) + \rho(S_{ij}) \tag{32}$$

after which one remains with the three basis-generators

$$N^i \equiv X_{(5i)} + X_{(0i)} = \omega({t_c}^2 - r^2)P^i + 2x^i H + 2\omega(\rho(S_{i0})t_c + \rho(S_{ij})x^j), \tag{33}$$

which do not have an immediate physical significance. The $SO(4,1)$ transformations corresponding to these basis-generators and the associated isometries of the chart $\{t_c, \vec{x}\}_*$ are briefly presented in Appendix A.

Starting with above basis-generators, new operators can be constructed according to our physical needs. Thus one sefines the helicity operator:

$$W = \vec{J} \cdot \vec{P} = \rho(\vec{S}) \cdot \vec{P} \tag{34}$$

which is analogous to the time-like component of the four-component Pauli-Lubanski operator of the Poincaré algebra [11].

In the other moving local chart, $\{t, \vec{x}\}$, we have the same operators $\vec{P}$ and $\vec{J} = \vec{L} + \rho(\vec{S})$ (with $\vec{L} = \vec{x} \times \vec{P}$) whose components are the $\widetilde{E}(3)$ generators, while the Hamiltonian operator takes the form:

$$H = i\partial_t + \omega \vec{x} \cdot \vec{P}, \tag{35}$$

where the second term, due to the external gravitational field, leads to the commutation rules:

$$[H, P^i] = i\omega P^i. \tag{36}$$

We observe that in this chart the operators $K^i \equiv X_{(0i)}$ are the analogous of the basis-generators of the Lorentz boosts of $SL(2,C)$ since in the limit of $\omega \to 0$, when () equals the Minkowski line element, the operators $H = P^0, P^i, J^i$ and K^i become the generators of the representation ρ of the group $T(4) \otimes SL(2,C)$ (i.e. the universal covering group of the Poincaré group [11, 19]).

4. 3. The problem of the energy operator

The Hamiltonian operator we defined above is related to the Killing vector $k_{(05)}$ which can be spice-like in some domains of the dS manifold. For this reasons there are some doubts appeared in literature [27] concerning the existence of the operator H. We must specify that this is not an impediment since H has to make sense only inside the light-cones where it is always time-like. In other words, the energy is well-defined wherever an observer can do physical measurements. In the next table we show that the Killing vector $k_{(05)}$ is time-like obeying $k_{(05)}^2 \equiv g(k_{(05)}, k_{(05)}) > 0$ inside the light-cones of the charts we use.

chart	$\{t_s, \vec{\hat{x}}_s\}$	$\{t_s, \vec{x}_s\}$	$\{t, \vec{x}\}$	$\{t_c, \vec{x}\}$
light-cone	$(\omega \hat{r}_s < 1)_{e.h.}$	$r_s < \| t_s \|$	$\omega r e^{\omega t} < 1$	$t_c \ge -\frac{1}{\omega} + \gamma$
$k_{(05)}$	$(-\frac{1}{\omega}, 0, 0, 0)$	$(-\frac{1}{\omega}, 0, 0, 0)$	$(-\frac{1}{\omega}, x^1, x^2, x^3)$	(t_c, x^1, x^2, x^3)
$k_{(05)}^2 > 0$	$1 - \omega^2 \hat{r}_s^2 > 0$	$r_s > 0$	$\frac{1}{\omega^2} - r^2 e^{2\omega t} > 0$	$t_c^2 - r^2 > 0$

Hereby we see that the Killing vector $k_{(05)}$ is time-like inside the light-cone of any given chart such that the energy operator H is well-defined on the entire domain where an observer can measure physical events.

Thus the general conclusion is that the quantum observables we defined on dS space-time are correct.

References

1. S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, Wiley, New York, (1972).
2. C. M. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation*, W. H. Freeman & Co., San Francisco, (1973).
3. R. M. Wald, *General Relativity*, Univ. of Chicago Press, Chicago and London, (1984).
4. N. D. Birrel and P. C. W. Davies, *Quantum Fields in Curved Space*, Cambridge University Press, Cambridge (1982).
5. S. J. Avis, C. J. Isham and D. Storey, *Phys. Rev. D* 10, (1978) 3565;
   P. Breitenlohner and D. Z. Freedman, *Phys. Lett.* 115B, (1982) 197;

D. J. Navarro and J. Navarro-Salas, *J. Math. Phys.* 37, (1996) 6006;
I. I. Cotăescu, *Phys Rev. D* 60, (1999) 107504.
6. V. S. Otchik, *Class. Quant. Grav.* 2, (1985) 539
7. M. Villalba and U. Percoco, *J. Math. Phys.* 31, (1990) 715;
G. V. Shishkin and V. M. Villalba, *J. Math. Phys.* 30, (1989) 2132;
G. V. Shishkin and V. M. Villalba, *J. Math. Phys.* 33, (1992) 2093.
8. G. V. Shishkin, *Class. Quantum Grav.* 8, (1991) 175.
9. A. Lopez-Ortega, *Gen. Rel. Grav.* 38, (2006) 743.
10. R. Utiyama, *Phys. Rev.* 101, (1956) 1597;
T. W. B. Kibble, *J. Math. Phys.* 2, (1961) 212.
11. W. K. Tung, *Group Theory in Physics*, World Sci., Philadelphia, (1985).
12. B. O'Neill, *Semi-Riemannian Geometry*, Academic Press, (1983).
13. D. G. Boulware, *Phys. Rev. D* 12, (1975) 350.
14. B. Carter and R. G. McLenaghan, *Phys. Rev. D* 19, (1979) 1093.
15. I. I. Cotăescu, *J. Phys. A: Math. Gen.* 33, (2000) 9177.
16. I. I. Cotăescu, *Mod. Phys. Lett. A* 13, (1998) 2923;
I. I. Cotăescu, *Phys. Rev. D* 60, (1999) 124006.
17. I. I. Cotăescu, *Mod. Phys. Lett. A* 13, (1998) 2991.
18. J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields*, McGraw-Hill Book Co., NY, (1964).
19. B. Thaller, *The Dirac Equation*, Springer Verlag, Berlin Heidelberg, (1992).
20. I. I. Cotăescu, *Phys. Rev. D* 65, (2002) 084008.
21. I. I. Cotăescu, Radu Racoceanu and Cosmin Crucean, *Remarks on the spherical waves of the Dirac field on de Sitter spacetime*,
22. E. I. Guendelman, *Mod. Phys. Lett. A* 14, (1999) 1043;
E. I. Guendelman, *Mod. Phys. Lett. A* 14, (1999) 1397;
E. I. Guendelman, *Class. Quant. Grav.* 17, (2000) 361;
E. I. Guendelman, *preprints:*gr-qc/0004011, hep-th/005041, hep-th/006079.
23. R. Gilmore, *Lie Groups, Lie Algebras and Some of Their Applications*, Wiley-Interscience, New York, (1974).
24. M. Hamermesh, *Group theory and its applications to physical problems*, Addison-Wesley, Reading MA, (1962).
25. A. O. Barut and R. Rączka, *Theory of Group Representations and Applications*, PWN, Warszawa, (1977).

26. G. Mackey, *Induced Representations of Groups and Quantum Mechanics*, Benjamin, New York, (1968).
27. E. Witten, *preprint hep-th/9802150*