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GENERALIZED POTENTIAL SYMMETRIES FOR NON-LINEAR EVOLUTION EQUATIONS

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Abstract. The paper presents a special methodology for obtaining non-local generalized symmetries for nonlinear evolutionary equations. The method put the equation in a new form, as compatibility condition between two equations, and studies the generalized symmetries of the obtained system. This method can allows to discover new non-local symmetries, non-listed when the same equation where studied in a classical way.
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1. Introduction

In recent years considerable attention has been devoted to applications of symmetry group methods to a large variety of two or three order non-linear partial differential equations ([4],[6],[9]), but relatively few complete results have been obtained for the highest order evolution equations.

The purposes of this paper are:

• to generalize the method of Blumann to find non-classical symmetries of a fourth order equation at the evolution equations represented as compatibility condition between two other equations.

• to obtain non-local symmetries of the Calabi flow equation.

2. Classical and non-classical symmetries. Methodological approach

The symmetries encountered in physics are usually of the type commonly referred to as point or Lie-Bäcklund symmetries. For differential equations derived from a variational principle, the Lie-Bäcklund symmetries which preserve the action lead to conservation laws. However, not all conservation laws stem from Lie-Bäcklund symmetries. To account for all conservation laws in Lagrangian field theory one must enlarge the notion of symmetry to include classical and non-classical generalized symmetries.

2.1 Generalized Symmetries for Evolution Equations

Let us consider a n-th order PDE system:

$$\Delta_\nu(t,x,u^{(n)}[x]) = 0 \tag{1}$$

where (t,x) represent the independent variables, while $u \equiv \{u^\alpha, \alpha = \overline{1,q}\} \subset R^q$ the dependent ones. The notation $u^{(n)}$ designates the set of variables which includes u and the partial derivatives of u up to n-th order.

Let us consider the set S_Δ of all the analytic solutions of the system (1). **A symmetry group** associated to the PDE system ((1)) consists in one-parameter group of transformations acting on an open subset $M \subset \chi \times U$ which leave the set S_Δ invariant:

$$\begin{aligned} x^* &= x + \varepsilon\xi(x,t,u) + O(\varepsilon^2), \\ t^* &= t + \varepsilon\tau(x,t,u) + O(\varepsilon^2), \\ u^{\alpha^*} &= u^\alpha + \varepsilon\phi^\alpha(x,t,u) + O(\varepsilon^2), \alpha = \overline{1,q} \end{aligned} \tag{2}$$

where ε is the group parameter, and ξ, τ, ϕ^α are the **infinitesimal generators** of the symmetry group.

The associated infinitesimal symmetry operator is a vector field of the form:

$$U = \xi(t,x,u)\frac{\partial}{\partial x} + \tau(t,x,u)\frac{\partial}{\partial t} + \sum_{\alpha=1}^{q}\phi_\alpha(t,x,u)\frac{\partial}{\partial u^\alpha}.$$

A **Lie-Bäcklund symmetry** for a PDE system which describes a dynamical system is a vector field V verifying the Lie invariance condition is (Olver [9]):

$$pr^{(n)}(U)[\Delta]|_{\Delta=0} = 0 \tag{3}$$

(the symmetries determining equation), where $pr^{(n)}(U)$ is the n order prolongation of U on the Grasmanian. This condition is equivalent to the fact that the coefficients $\xi(t,x,u)$, $\tau(t,x,u)$, $\phi^\alpha(t,x,u)$ are infinitesimal generators of a Lie symmetry group.

A solution $\xi[t,x,u]$, $\tau[t,x,u]$, $\phi^\alpha[t,x,u]$ of the equation (3) , where the notation $f[t,x,u]$ means a dependence of the function f on the independent variables t and x and the dependent variables $u^\alpha, \alpha = \overline{1,q}$ with all their partial derivatives, is called a **generalized symmetry**. The importance of generalized symmetries is underlined by their role in completely integrable systems of non-linear differential equations.

2.2 The Non-local Potential Symmetries

In [4], Bluman introduced a method to find a new class of symmetries for a PDE. By writing a given PDE, denoted by

$$\Delta[x,t,u]=0$$

in a conserved form

$$u_t-\left(K_0(t,x,u,u_x,...)\right)_x=0, \tag{4}$$

a related system denoted by $S[x,t,u,v]=0$ with potentials as additional dependent variables is obtained:

$$\begin{cases} v_x &= u \\ v_t &= K_0(t,x,u,...) \end{cases} \tag{5}$$

If $u(x,t)$, $v(x,t)$ satisfies $S[x,t,u,v]=0$, then $u(x,t)$ solves $\Delta[x,t,w]=0$ and $v(x,t)$ solves an ***integrated related equation*** $T[x,t,v]=0$, obtained by replacing u in the second equation of (5) with v_x. Any Lie group of point transformations admitted by $T[x,t,v]=0$ induces a symmetry for $\Delta[x,t,u]=0$; when at least one of the generators of a group depends explicitly of on potential variable v, then the corresponding symmetry is neither a point nor a Lie-B¨acklund symmetry. These symmetries of $\Delta[x,t,u]=0$ are called ***potential symmetries***.

2.3 Generalized potential symmetries

The method of Blumann presented below can be generalized, with possible usefully applications in the case of evolution equation of order greater that three. By writing the PDE $\Delta[x,t,u]=0$ in a special form:

$$u_t-P(D_x)\left(K_0(t,x,u,u_x,...)\right)=0, \tag{6}$$

where $P(D_x)$ is a derivation operator witch do not depend explicitly on the independent variables x and t, one can rewrite the initial equation as a compatibility condition $P(D_x)(v)_t=[P(D_x)v]_t$ for the related system $PS[x,t,u,v]=0$:

$$\begin{cases} P(D_x)v &= u \\ v_t &= K_1[t,x,u,v] \end{cases} \tag{7}$$

where K_1 is obtained from K_0 under condition that the system (7) will be equivalent with the initial equation $\Delta[x,t,u]=0$.

Any Lie group of point transformations admitted by the associated system $PS[x,t,u,v]=0$ induces a symmetry for the initial equation $\Delta[x,t,u]=0$, by definition of the

invariance of solutions; when at least one of the generators of a group depends explicitly of v, then the corresponding symmetry is a *generalized potential symmetry* for the initial equation, witch have a non-local form if expressed only in u. The generalized symmetries of the associated system $PS = 0$ will determine also non-local generalized potential symmetries for the initial equation.

Note that if $P(D_x) = (1/v)[L - \lambda \cdot Id]$ for an differential operator L, and the expression of $K_1[\lambda, t, x, u, v]$ is linear in v, then the system (7) is a Lax representation of the initial equation $\Delta[x,t,u] = 0$:

$$\begin{cases} Lv - uv & = & \lambda v \\ v_t & = & K_1[\lambda, t, x, u, v] \end{cases}. \tag{8}$$

The Lax representation method is one of the classical way to prove the integrability of an evolution equation.

3. The non-local symmetries of Calabi flow

In this section, I consider a version of the Calabi flow in 1+1 dimensions, obtained in **Error! Reference source not found.** from the local expression of Calabi flow in 2+1 dimensions (see **Error! Reference source not found.**)by uni-directionaliation procedure:

$$\partial_t u = -\partial_{xx}(\frac{1}{u}\partial_{xx}\ln(u)), \tag{9}$$

as a model equation to exemplify the procedure to obtain non-local potential and pseudo-potential symmetries.

The equation (9) write explicitly:

$$u_t = -\frac{u_{xxxx}}{u^2} + \frac{6u_{xxx}u_x}{u^3} - \frac{21u_{xx}u_x^2}{u^4} + \frac{4u_{xx}^2}{u^3} + \frac{12u_x^4}{u^5}. \tag{10}$$

The original Calabi flow equation is supposed to be integrable ([2]) because it possess a zero curvature representation and an infinite (non-standard) algebraic hierarchy of high order integrable equations. So, the integrability of the equation (10) can be strongly supposed too. In a recent paper (A. Boldea, C. Boldea **Error! Reference source not found.**) we investigated the existence of generalized symmetries for this equation, obtaining that that the group of all arbitrary-order (local) generalized symmetries for the equation (10) is generated by three independent symmetry operators. They represent the space and time translation, respectively a scaling transformation ($(x,t) \to (\alpha x, \alpha^4 t)$). The finite number of symmetries

would suggest a limited integrability of the equation. These apparently contradictory results could be due to the limitation to local symmetries in the later case. This is why, the study of the existence of some non-local hidden symmetries is clearly necessary.

3.1 The potential symmetries of Calabi flow

In order to find the potential symmetries of (10), we write the Calabi flow equation in a conserved form:

$$u_t - \left(-\partial_x (\frac{1}{u} \partial_{xx} \ln(u)) \right)_x = 0. \tag{11}$$

The associated auxiliary system $S[x,t,w,v]=0$ will by:

$$\begin{cases} v_x &= u \\ v_t &= -\partial_x (\frac{1}{u} \partial_{xx} \ln(u)) \end{cases} \tag{12}$$

The integrated related equation is obtained by eliminating the u variable from (12):

$$v_t + \partial_x (\frac{1}{v_x} \partial_{xx} \ln(v_x)) =: v_t - K2[v] = 0. \tag{13}$$

The generalized symmetries of the integrated related equation (13) are obtained from the symmetry generating equation (3), explicitely (Olver [9]):

$$\frac{\partial X}{\partial_t} + \sum_{i=0}^{m} \frac{\partial X}{\partial v_x^{(i)}} D_x^i (K2[v]) = \sum_{i=0}^{4} \frac{\partial K2}{\partial v_x^{(i)}} D_x^i (X[v]) = 0 \tag{14}$$

for the m order evolutionary generalized symmetry $v_Q = Q(t,x,u,u_x,\cdots)\partial_v$.

The evolutionary generalized symmetry is

$$X = [(ax+b)v_x + bv + c]\partial_v, \tag{15}$$

then the integrated equation (13) admits an infinite-parameter Lie group of point symmetries spanned by the infinitesimal generators:

$$X_1 = xv_x \frac{\partial}{\partial v}, \qquad X_1 = -\frac{\partial}{\partial x} + v\frac{\partial}{\partial v}, \qquad X_1 = c\frac{\partial}{\partial v} \tag{16}$$

Note that this symmetry generators do not have a local form expressed only in term of u.

3.2 The generalized potential symmetries of Calabi flow

Consider next the case of generalized potential symmetries of Calabi flow (9). In order to apply the method exposed in the Section 2.3, the Calabi flow equation will be considered in the special form:

$$u_t - \left(-(\frac{1}{u}\partial_{xx}\ln(u)) \right)_{xx} = 0, \tag{17}$$

or as compatibility condition between

$$\begin{cases} v_{xx} &= u \\ v_t &= -\frac{1}{u}\partial_{xx}\ln(u) \end{cases} \tag{18}$$

The form of a symmetry operator for the system (20) is

$$X = \xi[t,x,u]\frac{\partial}{\partial x} + \tau[t,x,u]\frac{\partial}{\partial t} + \eta[t,x,u]\frac{\partial}{\partial u} + \phi[t,x,u]\frac{\partial}{\partial v} \tag{19}$$

and the symmetries determining equation will be

$$\begin{cases} pr^{(2)}X(v_{xx} - u) &= 0 \\ pr^{(2)}X(v_t + \frac{1}{u}\partial_{xx}\ln(u)) &= 0 \end{cases} \tag{20}$$

The solutions of the system (22), obtained using a Maple package, are:

$$\begin{cases} \xi &= 0 \\ \tau &= \tau(t) \\ \eta &= -[u^2/(1+2u)]\tau'(t) \\ \phi &= -\iint[u^2/(1+2u)]\tau'(t)dxdx \end{cases} \tag{21}$$

where τ is an arbitrary C^1 function. The generalized potential symmetry operator is

$$X = \tau(t)\frac{\partial}{\partial t} - \frac{u^2}{1+2u}\tau'\frac{\partial}{\partial u} - [\iint \frac{u^2}{1+2u}\tau' dxdx]\frac{\partial}{\partial v}, \tag{22}$$

then the system (20) admits an infinite-parameter Lie group of point symmetries spanned by X. Note that this class of symmetry generators depends on an arbitrary continuous function $\tau = \tau(t)$ and their are obviously non-local.

4. Conclusions

We investigated the problem of the existence of generalized symmetries of the Calabi flow equation, using the classical Lie approach and a new complementary method, based on the Blumann approach. The Calabi Flow represents in the same time an interesting models arising from physics and a good toy models of fourth order differential equation which can be investigated by that technique. The main results we obtained could be synthesized as follows:

(i) The group of classical Lie symmetries for the equation Calabi flow is generated by three geometrical symmetries; the group of non-local potential symmetries is spanned by another three generators X_1, X_2, X_3 from (18).

(ii) If we investigate the symmetries obtained by the *generalization* of the Blumann method, one obtain a group of non-local symmetries spanned by a class of infinite number of generators.

The methodological approach exposed here can be easy adapted and applied in the case of mechanical models of a field theory containing highest order derivative terms (proposed by P. D. Mannheim and A. Davidson [7]), or in the case of the Pais-Uhlenbeck fourth order oscillator (see [8]):

$$\frac{d^4q}{dt^4}+(\omega_1^2+\omega_2^2)\frac{d^2q}{dt^2}+\omega_1\omega_2 q.$$

where $\omega_i = \omega_i(q)$. This subject will be tackled into a forthcoming paper.

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