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CONSERVATION LAWS FOR CALABI FLOW EQUATION

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Abstract
The paper presents a concrete study on the conserved densities and conservation laws for the 1+1 dimensional version of the Calabi flow equation. Despite the restrained group of generalized local symmetries, the equation accept an infinite set of functional independent conserved densities, so that the assertion of integrability is sustained.
Keywords: Calabi flow, Symmetries, Conservation laws.

1. Introduction

In recent years considerable attention (**Eroare! Fără sursă de referinţă.**, **Eroare! Fără sursă de referinţă.**, **Eroare! Fără sursă de referinţă.**) has been devoted to applications of symmetry group methods to a large variety of two or three order non-linear partial differential equations, but relatively few complete results have been obtained for the fourth order evolution equations.

In this paper we will look for and we will give a description of arbitrary-order conserved densities for the version of Calabi flow equation in 1+1 dimensions. Calabi flow equation is a fourth order differential equation obtained by the deformation of the second order Ricci flow equation. The interest for these types of flow equations comes from their connection with the General Relativity, offering important tools in the study of the black holes and in the attempt of obtaining a quantum theory of gravity ([4]).

The paper has the following structure: after this introductory part, a short presentation of the unidimensional Calabi flow will be given in the second section. The section 3 is dedicated to the conserved quantities and conservation laws which could arise when we restrict our check to their dependence till the first order derivative in the field variable. In the section 4, the study will be extended using an algorithm which could generate conserved quantities depending on the field variable and its derivatives till an arbitrary order. The existence of an infinite number of such quantities sustains the assertion that the Calabi flow equation is an integrable one. Some concluding remarks will end the paper.

2. The 1+1 dimensional version of Calabi flow

In order to obtain a version of the Calabi flow in 1+1 dimensions, we start from the local expression of Calabi flow in 2+1 dimensions, for conformally flat coordinates (Bakas [4]):

$$\partial_t \Phi = -\Delta\Delta\Phi \,, \tag{57}$$

where the symbol Δ is the Laplace-Beltrami operator $\Delta = e^{-\Phi}\partial\bar{\partial}$ for the Kähler metric $g^{z\bar{z}} = e^{\Phi(z,\bar{z};t)}$. With the transformation $u = e^{\Phi}$, the equation (4) become:

$$\partial_t u = -\partial\bar{\partial}(\frac{1}{u}\partial\bar{\partial}\ln(u)) \tag{58}$$

Introducing a linear combination of the variables z and $\bar{z}$ as a new independent variable, by uni-directionalization one obtains an 1+1 dimensional (reduced) equation. For example, if $z + \bar{z} = x$, the equation (5) can be written, modulo a simple scaling factor, as:

$$\partial_t u = -D_{xx}(\frac{1}{u}D_{xx}\ln(u)) \,, \tag{59}$$

or, explicitly:

$$u_t = -\frac{u_{xxxx}}{u^2} + \frac{6u_{xxx}u_x}{u^3} - \frac{21u_{xx}u_x^2}{u^4} + \frac{4u_{xx}^2}{u^3} + \frac{12u_x^4}{u^5} \,, \tag{60}$$

This one-dimensional version of the Calabi flow equation will be used from now on in our paper. It is written in terms of the field (dependent) variable $u(x,t)$ and of two independent variables x and t.

The original Calabi flow equation (5) is supposed to be integrable (Bakas [4]) because it possess a zero curvature representation and an infinite (non-standard) algebraic hierarchy of high order integrable equations. So, the integrability of the equation (7) can be strongly supposed too. By the other hand, in a recent paper (Boldea [7]) we proved that the group of all arbitrary-order (local) generalized symmetries for the equation (7) is generated by three independent symmetry operators. These apparently contradictory results could be due to the limitation to local symmetries in the later case.

3. Low order Conserved Densities and Conservation Laws

A vector function (ρ, θ) on the jet space is called the *conserved current* for a system $u_t = K(u)$ if it solves the equation

$$D_t \rho = D_x \theta, \tag{61}$$

where D_t is the evolutionary derivative and D_x is the total derivative with respect to x. The function ρ is said to be the *conserved density* and θ is said to be *the flux*. One can investigate the equation (5) with the help of the Euler operator E:

$$E = \sum_{n=0}^{\infty} (-D_x)^n \frac{\partial}{\partial u^{(n)}}, \tag{62}$$

where $u^{(n)}$ is the n order spatial derivative of u. The Euler operator E possesses an important property: $Ef = 0$ if and only if $f = D_x(F)$ (**Eroare! Fără sursă de referință.**). Applying the operator E to the equation (6) we obtain the following equation for the conserved densities:

$$E(D_t \rho) = E\left(\frac{\partial \rho}{\partial t} + \sum_{n=0}^{\infty} \frac{\partial \rho}{\partial u^{(n)}} \cdot D_t u^{(n)} \right) = 0. \tag{63}$$

The equation (8) can be solved by splitting it in many equations, following different order in the derivatives of the field variable u. To do it we have to choose the maximum order of derivative involved in the expression of the density: $\rho = \rho(t, x, u, u_x, u_{xx}, \ldots, u_x^{(m)})$.

Let consider the Calabi flow equation case (5). We suppose in first step that the conserved density $\rho = \rho(t, u, u, u_x)$ involve only the independent variables *t*, *x*, the dependent variable *u* and his first derivative. The equation (10) have the form

$$E\left(\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial u} \cdot u_t + \frac{\partial \rho}{\partial u_x} \cdot u_{t,x} \right) = 0. \tag{64}$$

where u_t is given by the Calabi flow equation (5).

The equation (15) contains now 178 terms and, splitting them over the order of the u derivatives, one obtain (using a Maple 9 software) the first equation of the system:

$$2 \frac{u_x^{(6)}}{u^2} \cdot \frac{\partial^2 \rho}{\partial u_x^2} = 0 \tag{65}$$

It must be valid for any u solution of the Calabi flow equation, requirement which imposes that ρ must be linear in u_x: $\rho = f(t, x, u) u_x + g(t, x, u).$

We can now express the equation (8) in terms of f and g and separate it again under different orders of the derivatives of u. The simplest conditions obtained after a reduction procedure are:

$$\begin{cases} \frac{u_{xxxx}}{u^2} g_{uu} & = & \frac{u_{xxxx}}{u^2} f_{ux} \\ \frac{u_{xx}}{u^3} g_{u,xx} & = & \frac{u_{xx}}{u^3} f_{xxxx} \\ g_{u,xx} & = & f_{xxx} \\ f_{t,x} & = & g_{t,u} = 0 \end{cases} \tag{10}$$

for any solution u of the Calabi flow equation (multiple index means a high order derivative).

The general solution of (10) is given by:

$$\begin{cases} f(t,x,u) & = & D_u H(u,x) + a(u) + b_1(t) + cx^2 + dx + e \\ g(t,x,u) & = & D_x H(u,x) + b_2(t) + 2cx + d \end{cases} \tag{11}$$

where $H \in C^2(R^2), a \in C^1(R)$, $b_1 \in C^1(R)$ and $b_2 \in C^1(R)$ are arbitrary functions, and c, d, e are constants. This solution imply the not –trivial conserved density

$$\rho = D_u H(u,x) u_x + D_x H(u,x) \tag{12}$$

Let us note that the corresponding conservation law has a general non-local form for an explicit dependence of H on t.

In particular, the quantities $u^n u_x$ are conserved densities for any n.

4. High order conserved densities

In general case when $\rho = \rho(t,u,u,u_x,..,u_x^{(m)})$, the the complexity of the calculus make very difficult the previous approach. We restrained our investigation to the problem of finding an infinite set of functional independent conserved densities.

In the next, we search for general conserved densities of arbitrary order for Calabi flow with the form

$$\rho(t,x,u,u_x,\ldots,u_x^{(n)}) = f_0(t,x)u + f_1(t,x)u_x + \ldots, f_n(t,x)u_x^{(n)}. \tag{20}$$

The principal result of this section is

Proposition 4 *The expression (20) is a conserved density for unidimensional Calabi flow*

$$u_t = -D_{xx}(\frac{1}{u} D_{xx} \ln(u))$$

if and only if

$$\partial_x\left[f_0-\frac{\partial f_1}{\partial x}+\frac{\partial^2 f_2}{\partial x^2}+\ldots+(-1)^n\frac{\partial^n f_n}{\partial x^n}\right]=\partial_t\left[\sum_{k=0}^{n}(-1)^k\frac{\partial^k f_k}{\partial x^k}\right]=0. \tag{22}$$

The proof is a direct computation of the equation (19). Indeed, if we consider only $\rho_n = f_n(t,x)u_x^{(n)}$, applying the Euler operator one obtain:

$$E(D_t(\rho_n))=(-1)^n\left(\frac{\partial}{\partial t}\cdot\frac{\partial^n f_n}{\partial x^n}+\frac{2u_{xx}}{u^3}\cdot\frac{\partial^{n+1}f_n}{\partial x^{n+1}}-\frac{3u_x^2}{u^2}\cdot\frac{\partial^{n+2}f_n}{\partial x^{n+2}}+\frac{2u_x}{u^3}\cdot\frac{\partial^{n+3}f_n}{\partial x^{n+3}}-\frac{1}{u^2}\cdot\frac{\partial^{n+4}f_n}{\partial x^{n+4}}\right)$$

Then, after substitution of ρ from (21) and some simplifications, the equation (19) reduce to:

$$\frac{\partial}{\partial t}\cdot\left[\sum_{k=0}^{n}(-1)^n\frac{\partial^k f_k}{\partial x^k}\right]+\frac{2u_{xx}}{u^3}\cdot\left[\sum_{k=0}^{n}(-1)^k\frac{\partial^{k+1}f_k}{\partial x^{k+1}}\right]-$$

$$-\frac{3u_x^2}{u^2}\cdot\left[\sum_{k=0}^{n}(-1)^k\frac{\partial^{k+2}f_k}{\partial x^{k+2}}\right]+\frac{2u_x}{u^3}\cdot\left[\sum_{k=0}^{n}(-1)^k\frac{\partial^{k+3}f_k}{\partial x^{k+3}}\right]-\frac{1}{u^2}\cdot\left[\sum_{k=0}^{n}(-1)^k\frac{\partial^{k+4}f_k}{\partial x^{k+4}}\right]=0 \tag{23}$$

for any u solution of Calabi flow equation, then the condition (22) is an evidence.

Remark 5 *Consider the particular case of the precedent Proposition when* $f_k = 0$ *for* $k=1,\ldots,n-1$, *and* $f_n(t,x)=x^n$. *The conditions (22) are obviously verified, then the expression*

$$\eta_n = x^n\cdot D_x^n(u) \tag{66}$$

is a conserved density for Calabi flow, for any natural n, *while the quantity* $x^{n+1}\cdot D_x^n(u)$ *is not a conserved density.*

Note that η_n is functional independent in respect with the set $\Upsilon_{n-1}=\{\eta_0,\ldots,\eta_{n-1}\}$ and cannot be obtained by derivation operations from the elements of Υ_{n-1} or by functional combination of derivatives, then we have determined an infinite set of independents conservation laws and independents conserved quantities $I_n=\int\eta_n(x)dx$ for the uni-dimensional Calabi flow equation.

5. Concluding remarks

We investigated the problem of integrability of the Calabi flow equation, using the method of conservation laws and of their attached conservation currents. Many signs, as for example the existence of a zero curvature representation and of an infinite (non-standard) algebraic hierarchy of high order integrable equations, lead to the idea that we deal with an integrable model. On the other hand, concrete computations of the Lie-type symmetries proved the existence of a finite dimensional symmetry group, fact which do not sustain the previous assertion. Our investigation tried to elucidate these contradictory aspects. The main results we obtained could be synthesized as follows: *(i)* the group of Lie symmetries for the equation (7) is generated by the three operators (given in [6]) which represent "time" translation, "space" translation and a "re-scaling" transformation; *(ii)* conserved densities depending on the field u and its first order spatial derivative u_x are given by the relation (12): $\rho = D_u H(u,x)u_x + D_x H(u,x)$; *(iii)* finally, an infinite set of independent conserved quantities, $I_n = \int \eta_n(x)dx$, arises when we are looking for conserved densities of the form (20): $\rho(t,x,u,u_x,\ldots,u_x^{(n)}) = f_0(t,x)u + f_1(t,x)u_x + \ldots, f_n(t,x)u_x^{(n)}$.and when the conditions imposed by the main theorem from the previous section are observed. Part of the conservation laws attached to these invariants are non-local, but they assure the integrability of the Calabi flow equation for this unidimensional approach.

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