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PERIODIC SOLUTIONS OF DUFFING OSCILLATOR WITH VISCOUS DAMPING AND HARDENING NONLINEARITY

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Abstract
The periodic response of a one-degree of freedom system with cubic nonlinearities to a principal resonance is investigated. The modified homotopy perturbation method is used in order to determine the equations that describe the second-order approximate periodic solutions of the system. Results obtained are compared with the numerical integration results and a good agreement is found.
Keywords: Homotopy technique, Duffing oscillator, frequency

1. Introduction

The problem dedicated to Duffing oscillator has received widespread attention in connection with interest in applications such as the rolling motion of a ship and the fact that it is isomorphic with other systems of importance in physics and engineering (e.g. Josephson junction oscillator and Foucault pendulum. Particularly interesting is the response of doffing oscillator to a harmonic excitation in the presence of viscous damping, which has been found to exhibit, among other features, hysteretic and chaotic behaviors. Thus, we consider a system governed by a non-dimensional differential equation of the form:

$$\ddot{u} + \omega^2 u + 2\varepsilon\mu\dot{u} + \varepsilon\alpha u^3 = \varepsilon k \cos \Omega t \qquad (1)$$

where ε is a small parameter, ω, μ, α, k and Ω are positive constant parameters. Primary resonance (i.e. $\Omega \approx \omega$) is considered in the next section. To determine the dependence of u(t) on the parameters $\omega(\Omega)$, α, k and ε we develop an approximate second-order solution using modified homotopy perturbation method.

2. Basic ideas of the modified perturbation method.

We have been considering systems governed by equations having the form

$$\ddot{u} + \omega^2 u = \varepsilon f(\Omega t, u, \dot{u}) \qquad (2)$$

Where, in general f is a nonlinear analytical function, with the period T in the first variable. A periodic solution of Eq.(2) is given by the formula [8]:

$$u(t)=u_0(t)+\sum_{j\ge 1}\frac{\varepsilon^j u_0^{[j]}(t)}{j!} \tag{3}$$

where $u_0^{[j]}(t)$ are obtained from the following equations:

$$\ddot{u}_0(t)+\Omega^2 u_0(t)=0 \tag{4}$$

$$\ddot{u}_0^{[1]}(t)+\Omega^2 u_0^{[1]}(t)=f(\Omega t, u_0(t), \dot{u}_0(t))-\Lambda_0^{[1]}u_0(t) \tag{5}$$

$$\ddot{u}_0^{[2]}(t)+\Omega^2 u_0^{[2]}(t)=2f^{[1]}(\Omega t, u_0, \dot{u}_0(t))-2\Lambda_0^{[1]}u_0^{[1]}(t)-\Lambda_0^{[2]}u_0(t) \tag{6}$$

where

$$f^{[1]}(\Omega t, u_0, \dot{u}_0)=\frac{\partial f}{\partial u}(\Omega t, u_0, \dot{u}_0)u_0^{(1)}(t)+\frac{\partial f}{\partial \dot{u}}(\Omega t, u_0, \dot{u}_0)\dot{u}_0^{(1)}(t) \tag{7}$$

and avoiding the secular term, we obtained $\Lambda_0^{[1]}$ from Eq.(5), respectively $\Lambda_0^{[2]}$ from Eq.(6), and the relationship between the constants of integration. In the same way, we can obtain all the j-th order deformation equations governing $u_0^j(j\ge 2)$.

The frequency Ω is determined from the following equation:

$$\omega^2=\Omega^2+\sum_{j\ge 1}\frac{\varepsilon^j \Lambda_0^{[j]}}{j!} \tag{8}$$

3. Periodic solutions of Eq.(1)

To determine second-order uniform periodic solutions of Eq.(1) we use the Modified Homotopy Perturbation Method and therefore Eqs (4), (5), (6). In this case, the function f becomes:

$$f(\Omega t, u, \dot{u})=-\alpha u^3-2\mu\dot{u}+k\cos\Omega t \tag{9}$$

Eq.(4) can be written as:

$$\ddot{u}_0(t)+\Omega^2 u_0(t)=0 \tag{10}$$

The solutions of Eq.(10) become: $u_0(t)=A\cos\Omega t+B\sin\Omega t$ (11)

where A and B are real unknown constants.

Substituting Eqs.(11) into Eq.(5) yields:

$$\begin{aligned}\ddot{u}_0^{(1)}+\Omega^2 u_0^{(1)}&=(k-2\mu B\Omega-\frac{3}{4}\alpha A^3-\frac{3}{4}\alpha AB^2-A\Lambda_0^{(1)})\cos\Omega t-\\&+(2\mu A\Omega-\frac{3}{4}\alpha B^3-\frac{3}{4}\alpha A^2B-B\Lambda_0^{(1)})\sin\Omega t+\\&+(\frac{3}{4}\alpha AB^2-\frac{1}{4}A^3)\cos 3\Omega t+(\frac{1}{4}\alpha B^3-\frac{3}{4}\alpha A^2B)\sin 3\Omega t\end{aligned} \tag{12}$$

The conditions for elimination of secular terms in Eq.(12) are:

$$\frac{3}{4}\alpha A(A^2+B^2)+A\Lambda_0^{(1)}+2\mu B\Omega=k \tag{13}$$

$$\frac{3}{4}\alpha B(A^2+B^2)+B\Lambda_0^{(1)}-2\mu A\Omega=0 \tag{14}$$

Now, into (13) and (14) we put:

$$A=r\sin\varphi,\ B=r\cos\varphi,\ r,\varphi\in R \tag{15}$$

and we obtain:

$$r=\frac{k\cos\varphi}{2\mu\Omega}\ ;\ A=\frac{k\sin 2\varphi}{4\mu\Omega}\ ;\ B=\frac{k(1+\cos 2\varphi)}{4\mu\Omega}\ ;\ \Lambda_0^1=2\mu\Omega\,\mathrm{tg}\varphi-\frac{3\alpha k^2\cos^2\varphi}{16\mu^2\Omega^2}\ ;\ \cos\varphi\neq 0 \tag{16}$$

Substituting Eqs.(16) in Eq.(11) we obtain the first order solution:

$$u_0(t)=\frac{k\cos\varphi}{2\mu\Omega}\sin(\Omega t+\varphi)\ ;\ \cos\varphi\neq 0 \tag{17}$$

The solution of Eq.(12) can be expressed as:

$$u_0^{(1)}(t)=C\cos\Omega t+D\sin\Omega t+\frac{\alpha A(A^2-3B^2)}{32\Omega^2}\cos 3\Omega t+\frac{\alpha B(3A^2-B^2)}{32\Omega^2}\sin 3\Omega t \tag{18}$$

where C and D are real unknown constants.

Substituting Eqs.(11) and (18) into Eq.(6), yields:

$$\begin{aligned}\ddot{u}_0^{(2)}(t)+\Omega^2u_0^{(2)}(t)=&[-2\mu D\Omega-2C\Lambda_0^{(1)}-A\Lambda_0^{(2)}-\frac{3\alpha^2A(A^2-B^2)(A^2-3B^2)}{128\Omega^2}-\\&-\frac{3\alpha^2AB^2(3A^2-B^2)}{64\Omega^2}-\frac{3}{2}\alpha C(A^2+B^2)-\frac{3}{4}\alpha C(A^2-B^2)-\frac{3}{2}\alpha ABD]\cos\Omega t+\\&+[2\mu C\Omega-2D\Lambda_0^{(1)}-B\Lambda_0^2+\frac{3\alpha^2A^2B(A^2-3B^2)}{64\Omega^2}-\frac{3\alpha^2B(A^2-B^2)(3A^2-B^2)}{128\Omega^2}-\frac{3}{2}\alpha CAB-\\&-\frac{3}{2}\alpha B(A^2+B^2)+\frac{3}{4}\alpha D(A^2-B^2)]\sin\Omega t+\text{N.S.T.}\end{aligned} \tag{1}$$

where N.S.T. stands for terms that do not produce secular terms.

Avoiding the presence of secular terms needs:

$$\left[\frac{3}{4}\alpha(3A^2+B^2)+2\Lambda_0^{(1)}\right]C+\left[\frac{3}{2}\alpha AB+2\mu\Omega\right]D+\Lambda_0^{(2)}A+\frac{3\alpha^2A(A^2+B^2)^2}{128\Omega^2}=0 \tag{20}$$

$$\left[\frac{3}{2}\alpha AB-2\mu\Omega\right]C+\left[\left(\frac{3}{4}\alpha(A^2+3B^2)+2\Lambda_0^{(1)}\right)\right]D+\Lambda_0^{(2)}B+\frac{3\alpha^2B(A^2+B^2)^2}{128\Omega^2}=0 \tag{21}$$

This set of equations can be solved and we obtain:

$$C=0;D=0;\Lambda_0^{(2)}=-\frac{3\alpha^2(A^2+B^2)^2}{128\Omega^2} \tag{22}$$

Substituting Eqs.(16) and (22) in Eq.(18), we obtain:

$$u_0^{(1)} = -\frac{\alpha k^3 \cos^3\varphi}{256\mu^3\Omega^5}\sin(3\Omega t + 3\varphi) \tag{23}$$

Substituting Eqs.(17) and (23) into Eq.(3), we find the second-order approximation to the solution of Eq.(1) for the primary resonant case ($\Omega \approx \omega$) is

$$u(t) = \frac{k\cos\varphi}{2\mu\Omega}\sin(\Omega t + \varphi) - \varepsilon\frac{\alpha k^3\cos^3\varphi}{256\mu^3\Omega^3}\sin(3\Omega t + 3\varphi) + 0(\varepsilon^2) \quad , \cos\varphi \neq 0 \tag{24}$$

From Eqs.(16), (22) and (8) we obtain

$$\omega^2 = \Omega^2 + \varepsilon\left(2\mu\Omega \mathrm{tg}\varphi - \frac{3\alpha k^2\cos^2\varphi}{16\mu^2\Omega^2}\right) - \varepsilon^2\frac{3\alpha^2 k^4\cos^4\varphi}{2048\mu^4\Omega^6} + 0(\varepsilon^3) \quad , \cos\varphi \neq 0 \tag{25}$$

Remark: From Eq.(24) we obtain

$$u(0) = \frac{k\sin 2\varphi}{4\mu\Omega} - \frac{\varepsilon\alpha k^3(\sin 6\varphi + 3\sin 4\varphi + 3\sin 2\varphi)}{2048\mu^3\Omega^3}$$

$$\dot{u}(0) = \frac{k(1+\cos 2\varphi)}{4\mu} - \frac{3\varepsilon\alpha k^3(1+\cos 6\varphi + 3\cos 4\varphi + 3\cos 2\varphi)}{2048\mu^3\Omega^2} \tag{26}$$

$$\ddot{u}(0) = -\frac{k\Omega\sin 2\varphi}{4\mu} + \frac{9\varepsilon\alpha k^3(\sin 6\varphi + 3\sin 4\varphi + 3\sin 2\varphi)}{2048\mu^3\Omega}$$

On the other hand, from Eq.(1) we obtain:

$$\ddot{u}(0) = \varepsilon k - \omega^2 u(0) - 2\varepsilon\mu\dot{u}(0) - \varepsilon\alpha u^3(0) \tag{27}$$

From (26) and (27) we obtain the parameter φ and from Eq.(25) we obtain the frequency Ω. In particular for $\omega = 0{,}736$, $\mu = \frac{1}{2}$, $\alpha = 1$, $k = \frac{1}{2}$, $\varepsilon = \frac{1}{10}$, we obtain: $\Omega = 0{,}741$, $\varphi = \frac{\pi}{10}$.

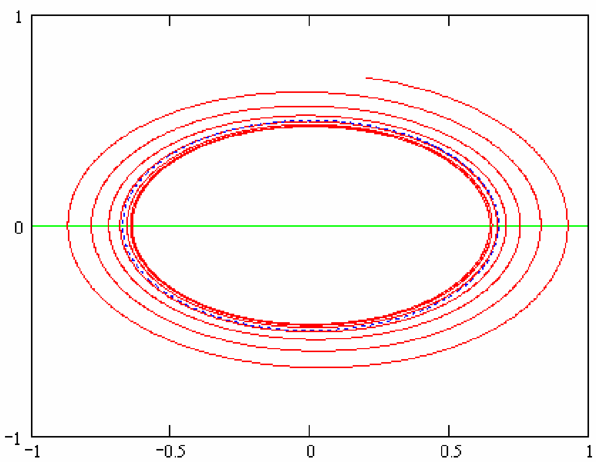

Fig.1: Phase portrait for Eq.(1), $\omega = 0{,}736$, $\mu = \frac{1}{2}$, $\alpha = 1$, $k = \frac{1}{2}$, $\varepsilon = \frac{1}{10}$, $\varphi = \frac{\pi}{10}$

______ *numerical simulation*

_ _ _ _ _ *present method,*

Figure 1 shows the comparison between the present phase portrait obtained from formula (24) and the numerical integration results obtained by using a fourth order Runge-Kutta method. It can be seen that the solution obtained by the present method is nearly identical with that given by numerical method.

4. Conclusions

The Modified Homotopy Perturbation Method has been proved to be effective and has some distinct advantages over usual approximation methods, and a satisfactory result can be obtained.

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