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# ITERATION PROCEDURE FOR DETERMINING APPROXIMATE SOLUTION FOR MOTION OF SIMPLE PENDULUM

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**Abstract**
In this paper, we apply a perturbation technique coupling with the iteration method for determination of approximate solution of a simple pendulum. This method valid for small as well as large values of oscillation amplitude can be applied for non-linear oscillations with single-degree-of-freedom. We compare the approximate period obtained by our procedure with the exact known period and the approximate solution with the numerical integration results obtained using a fourth order Runge-Kutta method.
**Keywords:** Nonlinear oscillations, approximate solution, iterative procedure

## 1. Introduction

Perturbation methods are a kind of powerful tools for treating weakly nonlinear problems, but they are less effective for analysis of strongly nonlinear problems [1]. But, like other nonlinear asymptotic techniques, perturbation methods have their own limitations: almost all perturbative methods are based on such an assumption that a small parameter must exist in an equation. This so-called small parameter assumption greatly restricts applications of perturbation techniques, as well as known, an overwhelming majority of nonlinear problems, especially those having strong nonlinearity, have no small parameter at all.

There exist some alternative analytical asymptotic approaches, such as the nonperturbative method, weighted linearization method, homotopy analysis method, Adomian decomposition method, modified Lindstedt-Poincare method, interpolation perturbation methods. There also exists a wide body of literature dealing with the problem of approximate solutions to nonlinear equations with various different methodologies. It is very difficult to solve nonlinear problems either numerically or theoretically. This is possible due to the fact that various discredited methods or numerical simulations apply iteration techniques to find their numerical solution of nonlinear problems and nearly all iterative methods are sensitive to initial solution.

In this paper we propose a perturbation technique by combining iteration method of He [2] into a new iteration perturbation method. The solutions obtained by the present method are in pretty good agreement with those obtained by exact method or other methods.

## 2. The method of analysis

Consider the following, in general nonlinear oscillation:

$$\ddot{u} + \omega^2 u = f(u), u(0) = A, \dot{u}(0) = 0 \tag{1}$$

We rewrite equation (1) in the following form [8]:

$$\ddot{u} + \Omega^2 u = u\left(\Omega^2 - \omega^2 + \frac{f(u}{u}\right) := ug(u) \tag{2}$$

where $\Omega$ is a priori unknown frequency of the periodic solution, u(t) being sought. The proposed iteration scheme is:

$$\ddot{u}_{n+1} + \Omega^2 u_{n+1} = u_{n-1}[g(u_{n-1}) + g_u(u_{n-1})(u_n - u_{n-1})] \quad , \quad n = 0,1,2,3,... \tag{3}$$

where the inputs of starting functions are:

$$u_{-1}(t) = u_0(t) = A\cos\Omega t \tag{4}$$

It is further required that for each n, the solution to equation (3), is to satisfy the initial conditions

$$u_n(0) = A, \dot{u}_n(0) = 0, n = 1,2,3,... \tag{5}$$

Right side of equation (5) can be expanded into the following Fourier series:

$$\begin{aligned} u_{n-1}[g(u_{n-1}) + g_u(u_{n-1})] &= a_1(A,\Omega,\omega)\cos\Omega t + b_1(A,\Omega,\omega)\sin\Omega t + \\ &+ \sum_{n=2}^{N} a_n(A,\Omega,\omega)\cos n\Omega t + \sum_{n=2}^{N} b_n(A,\Omega,\omega)\sin n\Omega t \end{aligned} \tag{6}$$

where the coefficients $a_n$ and $b_n$ are known, and the integer N depends upon the function $g(u)$ on the right hand side of Eq.(2). In Eq.(6), the requirement of no secular term needs

$$a_1(A,\Omega,\omega) = 0 \quad , \quad b_1(A,\Omega,\omega) = 0 \tag{7}$$

The solution of the equation (3) with the initial conditions (5) is taken to be:

$$\begin{aligned} u_{n+1}(t) &= A\cos\Omega t + \sum_{n=2}^{N} \frac{a_n(A,\Omega,\omega)}{(n^2-1)\Omega^2}(\cos\Omega t - \cos n\Omega t) + \\ &+ \sum_{n=2}^{N} \frac{b_n(A,\Omega,\omega)}{(n^2-1)\Omega^2}(n\sin\Omega t - \sin n\Omega t) \end{aligned} \tag{8}$$

Equation (7) allows the determination of the frequency $\Omega$ as a function of A and $\omega$. This procedure can be performed to any desired iteration step n. An excellent approximate analytical representation to the exact solution, valid for small as well as large values of oscillation amplitude is obtained.

## 3. The motion of a simple pendulum

When damping is neglected, the differential equation governing the free oscillation of the mathematical pendulum is given by

$$m l\ddot{\theta} + mg\sin\theta = 0 \tag{9}$$

or

$$\ddot{\theta} + a\sin\theta = 0 \tag{10}$$

were m is the mass, $l$ length of the pendulum, g the gravitational acceleration and $a = \frac{g}{l}$. The angle θ designates the deviation from the vertical equilibrium position.

We rewrite equation (10) in the form

$$\ddot{\theta} + \Omega^2\theta = \theta\left(\Omega^2 - a\frac{\sin\theta}{\theta}\right) \tag{11}$$

where Ω is an unknown frequency of the periodic solution. Here the initial conditions are $\theta(0) = A$, $\dot{\theta}(0) = 0$, the inputs of starting function are $\theta_{-1}(t) = \theta_0(t) = A\cos\Omega t$ and $g(\theta) = \Omega^2 - a\frac{\sin\theta}{\theta}$. The first iteration is given by the equation:

$$\ddot{\theta}_1 + \Omega^2\theta_1 = \Omega^2 A\cos\Omega t - a\sin(A\cos\Omega t) \tag{12}$$

The term $\sin(A\cos\Omega t)$ can be expanded in the power series:

$$\sin(A\cos\Omega t) = A\cos\Omega t - \frac{A^3\cos^3\Omega t}{3!} + \frac{A^5\cos^5\Omega t}{5!} - \frac{A^7\cos^7\Omega t}{7!} + \frac{A^9\cos^9\Omega t}{9!} + \ldots \tag{13}$$

We rewrite powers of $\cos\Omega t$ in (13) in terms of the cosine of multiples of $\Omega t$ with the aid of the identity [3]

$$\cos^{2n+1}\Omega t = \frac{1}{4^n}\sum_{k=0}^{n}\binom{2n+1}{n-k}\cos(2k+1)\Omega t \tag{14}$$

where

$$\binom{n}{p} = \frac{n!}{p!(n-p)!} \quad ; \quad \binom{n}{0} = 1$$

By using (14), equation (13) may be expressed in the form

$$\sin(A\cos\Omega t) = A\cos\Omega t - \frac{A^3}{24}(\cos 3\Omega t + 3\cos\Omega t) + \frac{A^5}{1920}(\cos 5\Omega t + 5\cos 3\Omega t + 10\cos\Omega t) -$$
$$- \frac{A^7}{322560}(\cos 7\Omega t + 7\cos 5\Omega t + 21\cos 3\Omega t + 35\cos\Omega t) + \frac{A^9}{92897280}(\cos 9\Omega t + \quad (15)$$
$$+ 9\cos 7\Omega t + 36\cos 5\Omega t + 84\cos 3\Omega t + 126\cos\Omega t)$$

Substituting (15) into (12), this can be rewritten as:

$$\ddot{\theta}_1 + \Omega^2\theta_1 = \left[A\Omega^2 - a\left(A - \frac{A^3}{8} + \frac{A^5}{192} - \frac{A^7}{9216} + \frac{A^9}{737280} + \ldots\right)\right]\cos\Omega t - \frac{A^3}{24}\cos 3\Omega t +$$
$$+ \frac{A^5}{1920}(\cos 5\Omega t + 5\cos 3\Omega t) - \frac{A^7}{322560}(\cos 7\Omega t + 7\cos 5\Omega t + 21\cos 3\Omega t) + \quad (16)$$
$$+ \frac{A^9}{92897280}(\cos 9\Omega t + 9\cos 7\Omega t + 36\cos 5\Omega t + 84\cos 3\Omega t) + \ldots$$

No secular terms in $\theta_1$ requires that

$$\Omega_1^2 = a\left(1 - \frac{A^2}{8} + \frac{A^4}{192} - \frac{A^6}{9216} + \frac{A^8}{737280} + \ldots\right) \quad (17)$$

and solving (16) with the initial conditions $\theta_1(0) = A$, $\dot{\theta}_1(0) = 0$, we obtain

$$\theta_1(t) = A\cos\Omega_1 t + \frac{A^3}{192\Omega_1^2}(\cos 3\Omega_1 t - \cos\Omega_1 t) - \frac{A^5}{46080\Omega_1^2}(\cos 5\Omega_1 t - \cos\Omega_1 t) -$$
$$- \frac{A^5}{3072\Omega_1^2}(\cos 3\Omega_1 t - \cos\Omega_1 t) + \frac{A^7}{15482880\Omega_1^2}(\cos 7\Omega_1 t - \cos\Omega_1 t) +$$
$$+ \frac{A^7}{1105920\Omega_1^2}(\cos 5\Omega_1 t - \cos\Omega_1 t) + \frac{A^7}{122880\Omega_1^2}(\cos 3\Omega_1 t - \cos\Omega_1 t) -$$
$$- \frac{A^9}{7431782400\Omega_1^2}(\cos 9\Omega_1 t - \cos\Omega_1 t) - \frac{A^9}{495452160\Omega_1^2}(\cos 7\Omega_1 t - \cos\Omega_1 t) -$$
$$- \frac{A^9}{61931520\Omega_1^2}(\cos 5\Omega_1 t - \cos\Omega_1 t) - \frac{A^9}{8847360\Omega_1^2}(\cos 3\Omega_1 t - \cos\Omega_1 t)$$

(18)

or

$$\theta_1(t) = \left(A - \frac{A^3}{192\Omega_1^2} + \frac{A^5}{2880\Omega_1^2} - \frac{141A^7}{15482880\Omega_1^2} + \frac{61A^9}{464486400\Omega_1^2}\right)\cos\Omega_1 t +$$
$$+ \left(\frac{A^3}{192\Omega_1^2} - \frac{A^5}{3072\Omega_1^2} + \frac{A^7}{122880\Omega_1^2} - \frac{A^9}{8847360\Omega_1^2}\right)\cos 3\Omega_1 t + \quad (19)$$
$$+ \left(-\frac{A^5}{46080\Omega_1^2} + \frac{A^7}{1105920\Omega_1^2} - \frac{A^9}{61931520\Omega_1^2}\right)\cos 5\Omega_1 t +$$

$$+\left(\frac{A^7}{15482880\Omega_1^2}-\frac{A^9}{495452160\Omega_1^2}\right)\cos 7\Omega_1 t-\frac{A^9}{7431782400\Omega_1^2}\cos 9\Omega_1 t$$

The approximate period can be expressed from Eq.(17): $T_{approx}=\frac{2\pi}{\Omega_1}$ and we obtain:

$$T_{approx}=\frac{2\pi}{\sqrt{a}}\left(1+\frac{A^2}{16}+\frac{5A^4}{1536}+\frac{13A^6}{73728}+\frac{239A^8}{23592960}+...\right) \quad (20)$$

while the exact period reads [1], [5]:

$$T_{ex}=\frac{4}{\sqrt{a}}\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} \quad ; \quad k=\sin\frac{A}{2} \quad (21)$$

To compare with the exact period, we have the following Table 1:

| A | $\frac{\pi}{10}$ | $\frac{\pi}{9}$ | $\frac{\pi}{8}$ | $\frac{\pi}{7}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ |
|---|---|---|---|---|---|---|---|
| $\frac{T_{approx}}{T_{ex}}$ | 0,99999681 | 0,99999512 | 0,99999211 | 0,99976763 | 0,99961425 | 0,99951134 | 0,99830437 |

*Table 1: Comparison between the approximate period $T_{approx}$ given by (24) and the exact solution $T_{ex}$ given by (25)*

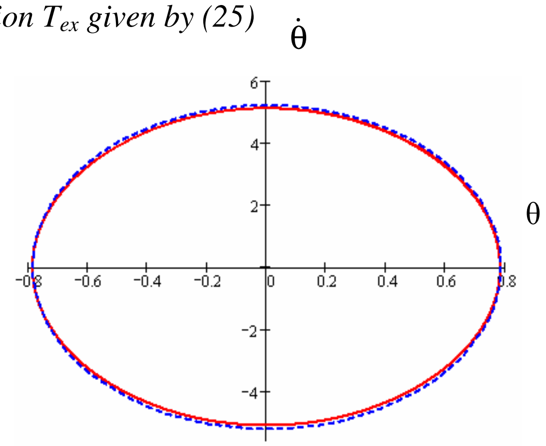


*Fig.1. Limit cycles of equation (19): a=100, A=π/4*
*- - - - - - present method*
*_______ numerical solution*

Figure 1 shows the comparison among the present solution obtained from formulae (17) and (19) and the numerical integration results obtained by using a fourth order Runge-Kutta method.

## 4.Conclusions

The proposed method is effective and has some distinct advantages over usual approximation methods in that the approximate solution obtained in the present paper is valid not only for weakly nonlinear oscillations, but also for strongly nonlinear ones.

## References


[1] A.H.Nayfeh, D.T.Mook, Nonlinear Oscillations, John Willey and Sons, New York (1979)

[2] J.H.He, Journal of Vibration and Control, 2, 121-126 (2002)

[3] I.S.Gradshteyn, I.M.Ryzhik, Tables of Integrals, Series and Products, Academic Press, New York (1980)

[4] C.W.Lim, B.S.Wu, Journal of Sound and Vibration, 257, 202-206 (2002)

[5] P.S.Li, B.S.Wu, Acta Mechanica, 170, 69-75 (2004)

[6] V.Marinca, N.Herişanu, Proceed. XI-th Int. Conf. Mech. Vibr., Sci. Bul. Politehnica Timişoara, 50 (64), pp.99-104 (2005)

[7] V.Marinca, Archives of Mechanics, 58, 3, p.241-256 (2006)

[8] V.Marinca, N.Herişanu, Acta Mechanica, 184, p.231-242 (2006)