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THE GENERALIZED ENERGY POLYNOMIAL FOR THE HARPER EQUATION

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Abstract
The derivation of the energy bands for the Harper equation proceeds in terms of a suitable polynomial. The general form of this energy polynomial is presented by accounting for the influence of the anisotropy parameter. The wavefunction can then be written down in an explicit manner.
Keywords: energy bands, Harper equation.

1. Introducere

The Azbel-Hofstadter problem [1,2] is still a fascinating problem. From this problem results the Harper equation [3] which is a second order discret equation exhibiting the form:

$$\phi_{n+1}e^{i\theta_1} + 2\Delta\cos(2\pi\beta n + \theta_2)\phi_n + \phi_{n-1}e^{-i\theta_1} = \varepsilon\phi_n \qquad (1)$$

where n is an integer, α_1 and α_2 are the Brillouin phase, and $\beta = \frac{P}{Q}$ is the commensurability parameter. The energy bands characterizing the Harper equation proceeds in terms of some energy polynomials which has been discussed before for $\Delta = 1$ [5], by using the secular equation [8,9] or the transfer matrix approach [2]; but the wavefunction coefficients was too complex [7], or the definition domains for wavefunction coefficients has been too many (n odd- even or n pozitiv- negative).

2. The polynomial P_n and wavefunction

We start from the generalized equation : $i\left(\frac{1}{z} + \Delta qz\right)\psi(qz) - i\left(\frac{z}{q} + \frac{\Delta}{z}\right)\psi(q^{-1}z) = \varepsilon\psi(z) \qquad (2)$

where $q = \exp(i\pi\beta) = \exp\left(i\pi\frac{P}{Q}\right)$ is the deformation parameter. The wavefunction is expressed in terms of the Laurent sums: $\psi(z) = \sum_{i=-Q}^{Q-1} C_n z^n$

where: $C_{-1} = 0, \quad C_0 = 1$.

Inserting the wavefunction expression into equation (2) we obtain a recurrence equation: $i\left(q^{n+1} - \Delta q^{-(n+1)}\right)C_{n+1} + i\left(\Delta q^n - q^{-n}\right)C_{n-1} = \varepsilon C_n$ which is usefull to calculate a new form of wavefunction.

We introduce the notations [7] : $\,^{\Delta}_{q}[n] = i\left(\Delta q^n - q^{-n}\right)$ (3)

$$_q[n]^{\Delta} = i\left(q^n - \Delta q^{-n}\right)$$

leads us to the new recurrence equation: $_q[n+1]^{\Delta} C_{n+1} + \,^{\Delta}_{q}[n] C_{n-1} = \varepsilon C_n$ (4)

One readily sees that under our notations one has:

$$[n]^2 = \left(\,^{\Delta}_{q}[n]\right)\left(_q[n]^{\Delta}\right) \quad (5)$$

$$\,^{\Delta}_{q}[n] = \left(_q[n]^{\Delta}\right)^*$$

Starting from equation (4) and by n=0 we calculate the coefficients C_n for n>0 and n<0. There are different expressions for this coefficients for different domains [4,5,7]. (n>0 or n<0) and for odd n or even n [7].

Let us introduce the quotations: $m = \left[\frac{n}{2}\right]$ (6)

$$z = \begin{cases} 0, & n > 0 \\ -1, & n < 0 \end{cases}$$

m is the number of coefficients in P_n, and z a step function. Using this quotations we have :

$$P_n = E^n + \sum_{l=0}^{m} (-1)^{2n-l} E^{n-2l} \sum_{k_0=1}^{n-2l+1} \sum_{k_1=k_0+2}^{n-2l+3} \cdots \sum_{k_{m-1}=k_{m-2}+2}^{n-1} \beta_{k_0}\beta_{k_1}\cdots\beta_{k_{n-1}} \quad (7)$$

$$P_n = E^{|n|-2} + \sum_{l=2}^{m} (-1)^{|n|+1-l} E^{|n|-2l} \sum_{k_0=2}^{|n|-2l+1} \sum_{k_1=k_0+2}^{|n|-2l+3} \cdots \sum_{k_{m-1}=k_{m-2}+2}^{|n|-1} \beta_{k_0}\beta_{k_1}\cdots\beta_{k_{n-1}}$$

where : $\beta_i = \left(_{q,\Delta}[i]\right)^2$.

The general expression of the energy coefficients from the wavefunctions is:

$$C_n = \frac{P_n}{\prod_{i=1}^{n} {}_q[i]^{\Delta}}, \text{ n>0} \quad (8)$$

$$C_n = (-1)^n (1-\Delta) \frac{P_n}{\prod_{i=1}^{n-1} {}_q[i]^{\Delta}}, \text{ n<0}$$

and so the wavefunction are: $\psi(z) = \sum_{i=-Q}^{Q-1} \frac{C_n}{\prod_{i=1}^{n+z} {}_q[i]^{\Delta}} z^n$ (9)

In fig. 1 one sees the Δ dependence of the energy bands. So for Δ=0 all energy bands are touching and for Δ between 0 and 1 the formation of the gaps can be observed.

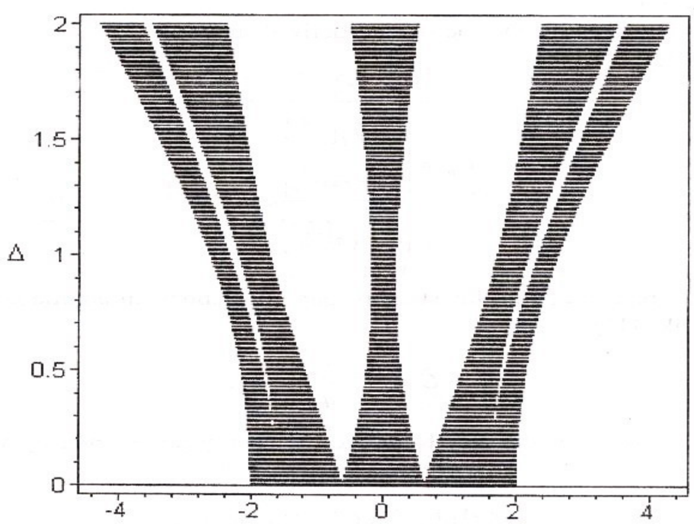

Fig. 1. P=2, Q=5

3. Conclusions

In this paper we find a new expression for the energy polynomial characterizing the energy bands of the anisotropic Harper equation. This new expression can be used to establish thermodynamic properties, the Lyapunov exponent, the Hall conductance without resorting to the explicit knowledge of the energy eigenvalues. The energy polynomial is given by two relations (for n pozitiv and n negative). The unification of this two relations in a more compact one will be done in a further paper.

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