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THEORETICAL AND EXPERIMENTAL MODELING AND STUDY OF COMPRESSIONAL WAVE MOTION IN PERIODICAL STRUCTURES

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Abstract:
Our work presents a theoretical model and an original experimental method addressed to our eye and brain for understand what is happening to each and every particles that compose a periodical structure when is set into longitudinal oscillations, and to see how the same kind of analysis applies to a system of particles connected by springs along a straight limited line. Physical and mathematical considerations allow us to establish the differential equation and characteristics of compressional traveling waves in helix springs. The particularization of these, for the limited pseudo-continuous medium, gives the time independent wave equation that, by its eigensolutions, can describe not only the standing waves, but also the multiple resonances and normal modes of vibration in this macroscopic periodical structure, as well as in crystalline materials, along a domain selected direction of the lattice. Our experimental method for visual observation and quantitative study of wave motion became possible trough the use of a long helical spring, stretched in vertical position that is excited at its lower end by an electromagnetic audio-vibrator. Obtained results on long helix springs and on beds rubber strings are agree with the theoretical model, being a convincing experiment for perceiving the intricate widespread phenomenon of sound waves that exist in solids, liquids and gases, but that are directly invisible.
Keywords: Mechanics of discrete systems, visualization of compressional standing waves, resonances, normal longitudinal modes of vibration.

1. Introduction

Inside an elastic medium, restoring forces act to reposition the particles on the original equilibrium configuration. The response of one particle to the action of an elastic force is the simple harmonic oscillatory motion. The response of an elastic medium is the traveling undulatory motion, named *wave*. Inside a finite elastic body, a periodical perturbation can excite stationary waves, also named *vibrations*. The elastic wave motion involves small displacements that do not remain localized, but propagate in space and time with a velocity that depends on the elastic properties of the medium. The visualization and study of transversal waves are simple by using the model of waves on a string [1, 2]. Longitudinal waves, also named, after the mechanism of producing, compressional waves, are more difficult to be observed. The existing methods, even the Kundt tube, are indirect methods adequate only for gases and liquids. By these methods, the standing-wave patterns, consisting of antinodes and nodes, can be visualized. The intricate longitudinal compressional vibration and wave motions in periodical structures or in crystal lattice of solids are always studied and modeled by analogy with the traverse vibration of a string fixed at both ends.

2. Theoretical modeling of compressional waves in helical springs

Physical and mathematical considerations upon elasticity and periodical structure of a helix spring allows us to establish the differential equation and characteristics of traveling compressional waves in this macroscopic pseudo-continuous elastic system. In order to obtain the dynamic equation of the spring coils motion along the vertical axis *Ox*, we shall consider a compressional wave pulse excited at the moment $t=0$ in $x=0$, which will arrive in x at the moment t. At the moment $t+dt$, when the wave front arrives in $x+dx$, the infinitely small displacement (coil elongation) will be $d\Psi$. In dynamic regime the local relative compression is:

$$\frac{\Delta l}{l} = \frac{d\Psi}{dx}$$

For the helical spring, which respects the Hooke's law, the relation between force and compression is: $F = k\Delta l = kl\frac{\Delta l}{l} = kl\frac{d\Psi}{dx}$

where k is the elastic constant of the spring of length l.

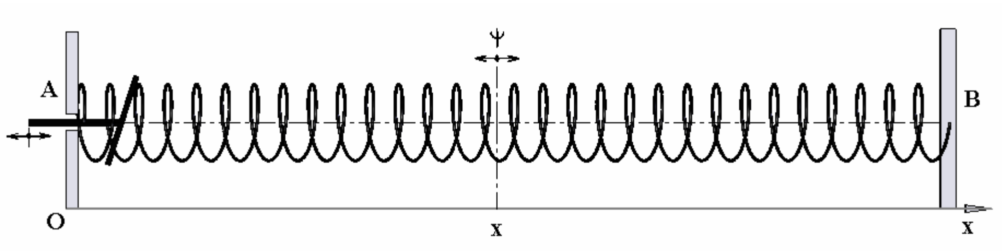

Fig.1 *The compressional pulse travel along a helical spring*

The local restoring force on the wave front will be: $dF = kl\frac{d^2\Psi}{dx^2}dx$

From Newton's law, the value of force, dF, applied to the mass $dm=(m/l)dx$ on the wave front will be: $dF = dm\frac{d^2\Psi}{dx^2} = \frac{m}{l}\frac{d^2\Psi}{dx^2}dx$

From the last two equations, using partial derivatives, we can obtain the differential equation of compressional (longitudinal) waves in helix springs:

$$\frac{d^2\Psi}{dt^2} = \frac{kl^2}{m}\frac{d^2\Psi}{dx^2}\text{, respectively, } \frac{\partial^2\Psi}{\partial t^2} = \mathrm{v}^2\frac{\partial^2\Psi}{\partial x^2} \qquad \textbf{(1)}$$

This represents the classical equation of the one-dimensional waves (plane waves) that propagate in helix springs by compression, in longitudinal direction, with the velocity:

$$\mathrm{v}^2 = \sqrt{\frac{kl^2}{m}} = l\sqrt{\frac{k}{m}} = \sqrt{\frac{kl}{m/l}} = \sqrt{\frac{k_0}{m_0}} \qquad (2)$$

where $k_0=kl$ and $m_0=m/l$ are the elastic constant and the mass of one-unit length of the spring. Equations (1) and (2) described the fact that the helix spring is a carrier for any compressional perturbation as a rod.

The equation of compressional waves in a helix spring is a differential equation with partial derivatives. The general solution depending on variables t and x is:

$$\Psi(t,x) = \Psi_1\left(t - \frac{x}{\mathrm{v}}\right) + \Psi_2\left(t + \frac{x}{\mathrm{v}}\right) \qquad (3)$$

The wave functions Ψ_1 and Ψ_2 describe, in time and space, the traveling of arbitrary dynamic perturbations, which propagate with velocity v that depends on the elastic characteristics of the medium. The first one represents the progressive wave and the second one the regressive wave. For compressional mechanical waves, perturbations depend on the displacements from the equilibrium position of the particles inside the medium, along the propagation velocity. Their values are given by the equation of elongation $\Psi(t,x)$. Due to the elastic characteristics of the restoring forces in the environment, periodic perturbations propagate by near translation like waves representing oscillations distributed in space. In the case of a helix spring harmonically excited at one of its ends with the angular frequency ω, the real solution will be given by the function:

$$\Psi(t,x) = f(x)\cos\omega t \qquad (4)$$

where the time-dependent part has oscillatory form and the space-dependent (time independent) part $f(x)$ is obtained from boundary conditions. The proposed solution has the second order derivatives:

$$\frac{\partial^2\Psi}{\partial t^2} = -\omega^2 f(x)\cos\omega t \quad \text{respectively} \quad \frac{\partial^2\Psi}{\partial x^2} = \frac{d^2 f}{dx^2}\cos\omega t\,.$$

By replacing in the general equation of waves (1), a time-independent equation is obtained:

$$\frac{d^2 f}{dx^2} = -\frac{\omega^2}{\mathrm{v}^2} f \tag{5}$$

The equation (5), named time independent equation of wave, admits the following solution that gives the space undulatory distribution of the standing wave amplitudes along the spring:

$$f(x) = A\sin\frac{\omega}{\mathrm{v}}x$$

At the fixed ends of the spring (*x=0* and *x=l*), the displacements (elongations) being zero, we have:

$$f(0)=f(l)=A\sin\frac{\omega}{\mathrm{v}}l=0 \Rightarrow \frac{\omega_n}{\mathrm{v}}l=n\pi \Rightarrow \omega_n=n\mathrm{v}\frac{\pi}{l}; \quad n=1, 2, 3, \ldots$$

Using this results and the wave speed (2), the natural frequency of oscillation will be:

$$\nu_n = \frac{\omega_n}{2\pi} = n\frac{\mathrm{v}}{2l} = \frac{nl}{2l}\sqrt{\frac{k}{m}} = \frac{n}{2}\sqrt{\frac{k}{m}} = \frac{n}{2l}\sqrt{\frac{k_0}{m_0}} = n\nu_1 \tag{6}$$

The result (6) is that the spring will oscillate or resonate only on certain eigenfrequencies that are integer multiples of the fundamental frequency. All these values, $\nu_n=n\nu_1$, *for* $n=1, 2, 3,\ldots N/2$ where N is the coils number of helix spring, represent *the normal modes* of vibration or *resonances* of the spring. Thus the global stationary vibration motion of the spring, or the standing waves, will be described by the eigenfunctions:

$$\Psi_n = A\sin\frac{\omega_n x}{\mathrm{v}}\cos\omega_n t = A\sin\left(\frac{2\pi}{\lambda_n}x\right)\cos(2\pi\nu_n t) = A\sin\left(n\frac{\pi}{l}x\right)\cos\left(n\pi\sqrt{\frac{k}{m}}t\right) \tag{7}$$

From (7) results the positions of the points where will exist nodes and antinodes and the fact that the distance between two consecutive nodes or antinodes is half of wavelength, $\lambda/2$. This fact limits the existence of standing waves along the spring only to those that satisfy the relation:

$$l = n\frac{\lambda_n}{2} \quad \text{or} \quad \lambda_n = \frac{2l}{n} \quad n=1, 2, 3,\ldots \tag{8}$$

The wave motion inside a finite medium, like the helix spring fixed at its ends, may be considered as an oscillation of the whole spring, every coil having a harmonic oscillation with the same common frequency, but with constant amplitude which depends on the coil position. The elastic coupling between coils and the mass of the spring distributed on its length induces, by the interference of direct and reflected wave, the appearance of a standing wave, with an integer number of antinodes. Each number of antinodes has a correspondent natural eigenfrequence (or resonance); all these patterns of this stationary motion are known as normal modes of vibration.

3. Experimental device for visualization of standing longitudinal waves in solids

The experimental method and the physical model meant to the direct observation and the study of the stationary compressional waves have been realized. The principle of our method [3, 4] is shown in Fig. 2. A helix spring stretched in vertical position is excited at its lower end by an electro-magnetic audio-vibrator with variable measurable frequency. On this pseudo-continuous elastic system, one may directly observe the minimums and maximums of oscillations (nodes and antinodes) for tens modes, corresponding to certain discrete excitation frequencies, $\nu_n = n\nu_1$, $n = 1, 2, 3\ldots$, multiples of a fundamental frequency ν_1. These vibration modes appear due to the undulatory phenomena that take place along the spring fixed at its ends. The sinusoidal (harmonic) oscillations $\Psi(t)$, having the angular frequency $\omega = 2\pi\nu$, produced by displacements at the lower end of the spring, propagate with velocity **v** through direct successive wave fronts (progressive waves). At the upper end they are reflected, generating regressive waves. The two compressional wave motions can be described by the following equations:

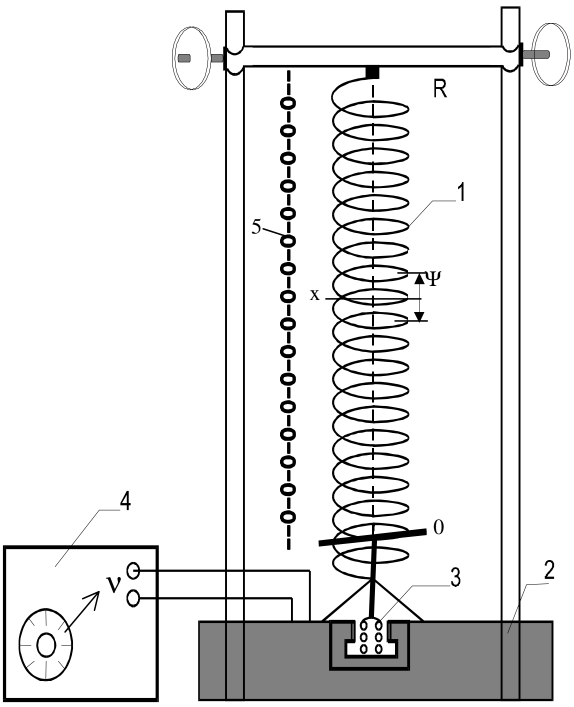

***Fig. 2.** Array to visualize and to study the longitudinal standing waves*

$$\Psi_p = a\sin\omega\left(t - \frac{x}{\mathrm{v}}\right) \quad \text{respectively} \quad \Psi_r = a\sin\left[\omega\left(t + \frac{x}{\mathrm{v}}\right) + \pi\right]$$

The resultant wave motion of the spring is obtained by composing the two waves:

$$\Psi = \Psi_p + \Psi_r = 2a\cos\left[\frac{\omega}{\mathrm{v}}x + \frac{\pi}{2}\right]\sin\left[\omega t + \frac{\pi}{2}\right] = 2a\sin(\kappa x)\cos(\omega t) \tag{9}$$

This equation gives the elongation of a motion named standing (stationary) wave, or vibration motion, which is oscillatory in time, and undulatory in space, having the local amplitude:

$$A(x) = 2a\sin(\omega x/\mathrm{v}) = 2a\sin(\kappa x)$$

constant in time, for a spring coil situated at a given distance x. It has the values between $A=0$ and $A=2a$ for the coils situated in the points named *nodes* respectively in *antinodes*, at distances:

$$x = p\pi\frac{\mathrm{v}}{\omega} = 2p\frac{\lambda}{4} \quad \text{respectively} \quad x = (2p+1)\frac{\pi}{2}\frac{\mathrm{v}}{\omega} = (2p+1)\frac{\lambda}{4} \qquad p=0,1,2...$$

where $\lambda=\mathrm{v}T=\mathrm{v}/\nu$ is the wave length and $\kappa=\omega/\mathrm{v}=2\pi/\lambda$ is the wave number. From these, it results that between two consecutive nodes exist a length $\lambda/2$. The spring of length l will have an integer number of antinodes between them. Thus we can see along the spring n parts where the oscillating coils are visible as a translucent grating. From, $l=n\lambda_n/2$ results that only the stationary waves having the wavelength $\lambda_n=21/n$, or wave number $\kappa_n=n\,\pi/l$, can exist along the spring.

The wave motion resulting from the superposition (constructive interference) of the direct and reflected waves which propagate through a finite medium, such as a helix spring having fixed ends, can be considered as a global motion of the entire spring, each coil executing harmonic oscillations with the same frequency and with an amplitude depending on the coil position. Each discrete number of antinodes has a correspondent natural frequency, $\nu_n=n\nu_1$, $(n=1, 2, 3,...)$, named *harmonics or overtone*, or to each pattern of standing wave corresponds a *normal vibration mode.*

The propagation speed of waves along the spring, **v,** can be calculated with the following relation:

$$\mathrm{v} = \nu_n\lambda_n = n\nu_1\frac{2l}{n} = 2l\nu_1$$

The standing waves equation describing the stationary compressional vibratory motion of whole helical spring, longitudinal excited, is:

$$\Psi_n = 2a\sin\left(n\frac{\pi}{l}x\right)\cos(n\omega_1 t) \quad n=1,2,3... \tag{10}$$

Eigenfunctions $\boldsymbol{\Psi_n}$, describe the standing wave and give all eigenvalues of this sound motion that are transmitted to the air and perceived as simple or composed musical sounds. The amplitudes and frequencies of upper harmonics (overtone) that overlap the fundamental harmonic ν_1 make up the discrete spectrum of the emitted sound, similar to vibrating strings.

Direct observation (visualization) of the natural vibration modes as cloudy grating is based on the light reflection or the optical transmittance of the space through where each coil passes during its longitudinal oscillation. The local cloudiness is bigger or smaller, like is the density of the probability to find the spring coil in the considered point *x*. This is inverse proportional with the velocity, $d\Psi/dt$, of the coil passing through a considered position *x*. Mathematically, we can write: $w = \frac{dP}{dx} \sim \frac{1}{|d\Psi/dt|} = \frac{1}{n\omega_1 \Psi_n(x)\sin(n\omega_1 t)} = \frac{1}{n\omega_1 \sqrt{\Psi_{n,max}^2 - \Psi_n^2}}$ (11)

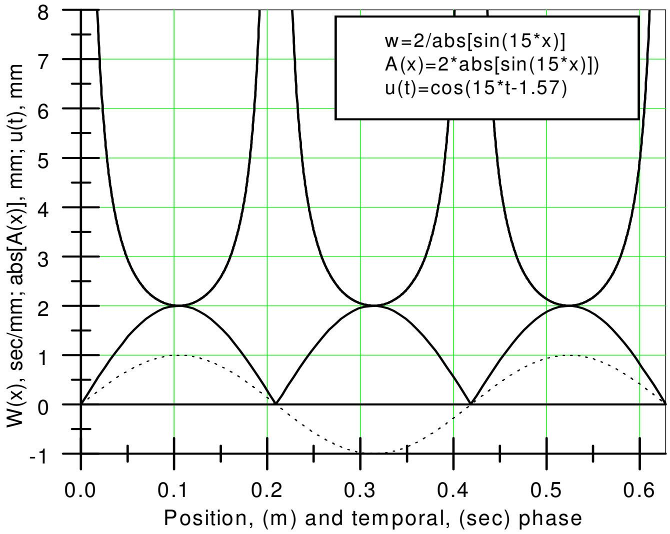

Fig. 3. The optical transmittance (cloudiness) through oscillating helical spring

Analyzing this function from spatial distribution point of view, we can conclude that the probability has maximum value in nodes and minimum value in antinodes. The time-analyze shows that it has minimum value when coils pass through equilibrium position and maximum value in return points. All these can be visually observed in Fig. 3. The wavelengths and also the local amplitudes in the antinodal positions of standing waves can be measured. Using these values, one can calculate all characteristics of helix spring wave's carrier.

4. Results and conclusion

The method and the experimental equipment [4] for visual observation and study of compressional waves may be understood from figure 2. The helix spring 1 is placed in vertical position, stretched between points 0 and R of the stand 2. The diaphragm 3 of a loudspeaker (or headphone) acts the spring in 0 through a rod T. When the loudspeaker is connected to a generator 4 with variable frequency in audio domain, the diaphragm transmits periodical impulses to the spring. By increasing the frequency, when it reaches a value ν_1, the coils from the middle of the spring are situated in an antinode, where longitudinal vibrating with visible amplitude of several millimeters can be measured. At the ends of the spring we can observe two nodes. This value of frequency is the first resonance, or the fundamental vibration mode. By increasing the audio frequency to $2\nu_1$, we can observe, along the spring, 2 antinodes and 3 nodes, at the middle and ends. Our studies have been made on springs with different lengths and elastic constants, being possible to observe up to 50 vibration modes. In all these cases the amplitude of coils vibration in antinodes was observed like a diffuse spliced image of the coil. This fact is illustrated in Fig. 4, by a photo of number n=8 compressional mode having a ventral amplitude that can be measured with a sliding calipers.

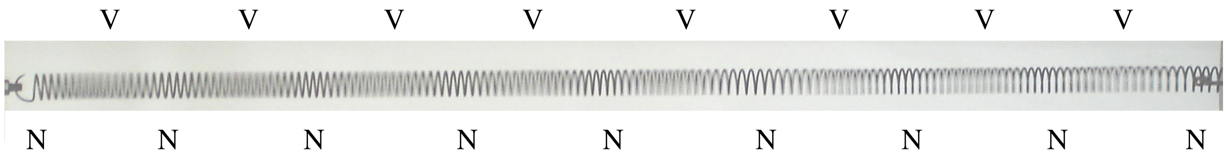

Fig.4 Distribution of nodes and antinodes along the spring

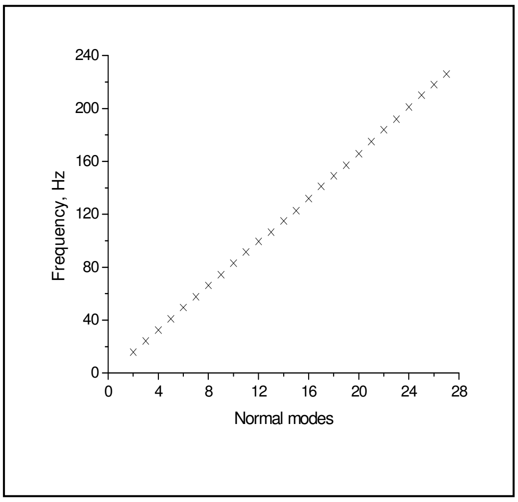

***Fig. 5**. Normal modes of helix spring*

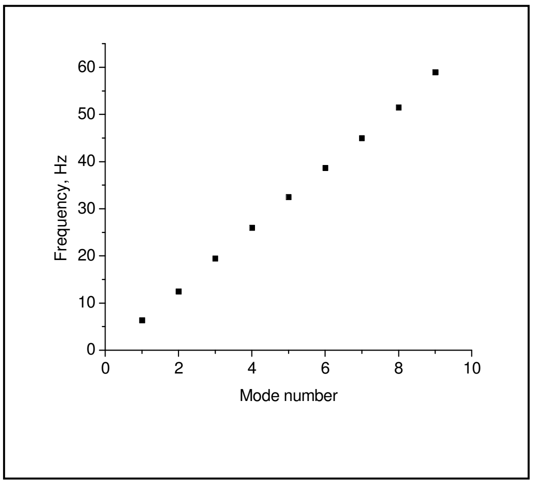

***Fig. 6**. Resonances of rubber string beads*

Experiments made on other periodical structures, consisting of necklaces of spherical beads equal displaced on elastic threads made of rubber, highlighted a similar vibratory behavior, like springs. In figure 2, the necklace drawn at the left can replace the helical spring and the resonances of these periodical systems are presented in figure 5 and 6.

All these facts permit an intuitive real representation, in our minds, of collisions [5] and of vibration modes of atoms in a crystalline row of a solid. Experiments [6, 7] related to the possible vibration modes confirm the existence of one maximum resonant frequency for which the normal compressional mode consists in the fact that a coil (atom) vibrates and the nearest neighbor coils (atoms) rest. The frequencies corresponding to resonances were obtained by experiment and also they were calculated using equation (6), results being in good accord.

From the theoretical point of view, the visualization of diverse patterns of compressional vibration motion or multiple resonances confirms the fact that discrete increases of undulatory amounts are caused by wave's propagation in limited media. From practical point of view, the finding out of the spring's resonances is important in the manufacturing industry, for vehicles, elastic drivers, design of resonant boxes, musical and room acoustics, reduction of environmental noise. In conclusion we highlight that our patented method and stated physical model of study for the standing longitudinal waves, presented in this paper, represents a didactic and scientific acquirement, suitable for direct visual perception and study of the widespread phenomenon of compressional waves that exist in solids, liquids and gases, but that are directly invisible. The results of this research may be also be used in drafting and manufacture of helix springs used for different purposes.

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