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SYMMETRIES FOR A FOURTH ORDER DIFFERENTIAL EQUATION

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Abstract
The paper intends to present a concrete study of the existence of generalized symmetries for the 1+1 dimensional version of the integrable Calabi flow equation, obtained by uni-directionalization. Some numerical computations was made in order to obtain the explicit form of the symmetries.
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1. Introduction

Point symmetries and generalized symmetries play an important role in the mathematical analysis of differential equations (Ibragimov [1], Blumann [2], Olver [3]). Originating with the work of Lie, symmetry group methods and their recent generalizations have proved useful in understanding conservation laws, in constructing exact solutions, and in establishing complete integrability of certain systems of differential equations.

In recent years considerable attention has been devoted to applications of symmetry group methods to a large variety of two or three order non-linear partial differential equations, but relatively few complete results have been obtained for the fourth order evolution equations.

In this paper we will give a *complete* characterization of all arbitrary-order generalized symmetries for the version of Calabi flow equation in 1+1 dimensions, as a preliminary step to study the Calabi flow in 2+1 dimensions. After a short presentation of the problem of finding generalized symmetries for a given evolution equation in the next, the Section 3 is dedicated to introduction of the equivalent of Calabi flow equation in 1+1 dimension and the last part of the paper present our study about the symmetries of this one.

2. Generalized symmetries for evolution equations

The symmetries encountered in field theory are usually of the type commonly referred to as point, or classical, symmetries. A point symmetry of a system of differential equations is a 1-parameter group of transformations of the underlying space of independent and dependent variables that carries any solution of the equations to another solution. For differential equations derived from a variational principle, the point symmetries which preserve the action lead to conservation laws. However, not all conservation laws stem from point symmetries. To account for all conservation laws in Lagrangian field theory one must enlarge the notion of symmetry to include generalized symmetries.

A *generalized symmetry* is an infinitesimal transformation, constructed locally from the independent variables, the dependent variables, and the *derivatives* of the dependent variables, that carries solutions of the differential equations to nearby solutions. The importance of generalized symmetries is underlined by their role in completely integrable systems of non-linear differential equations. In particular, when a system of differential equations is integrable, it generally admits ``hight orders'' generalized symmetries Olver [3], Fokas [4].

Consider the n-order evolution equation:

$$\Delta = u_t - K(t, x, u, u_x, \cdots, u_x^{(n)}) = 0, \tag{1}$$

where u_t, u_x means the time, respectively space, derivative of the dependent variable $u = u(t, x)$, and $u_x^{(n)}$ is the n-order derivative.

The classical symmetries analysis consider the one-parameter Lie group of infinitesimal transformations in (x, t, u) given by

$$\begin{aligned} x^* &= x + \varepsilon\xi(x,t,u) + O(\varepsilon^2), \\ t^* &= t + \varepsilon\tau(x,t,u) + O(\varepsilon^2), \\ u^* &= u + \varepsilon\phi(x,t,u) + O(\varepsilon^2), \end{aligned} \tag{2}$$

where ε is the group parameter. Then one requires that this transformation leaves invariant the set

$$S_\Delta \equiv \{u(x,t) : \Delta = 0\} \tag{3}$$

of solutions of (1). This yields an overdetermined linear system of equations for the infinitesimals $\xi(x,t,u),\tau(x,t,u),\phi(x,t,u)$. The associated Lie algebra is realised by vector fields of the form

$$v=\xi(x,t,u)\partial_x+\tau(x,t,u)\partial_t+\phi(x,t,u)\partial_u. \tag{4}$$

Having determined the infinitesimals, the symmetry variables are found by solving the characteristic equation

$$dx\xi(x,t,u)=dt\tau(x,t,u)=du\phi(x,t,u), \tag{5}$$

which is equivalent to solving the system

$$\psi\equiv\xi(x,t,u)u_x+\tau(x,t,u)u_t-\phi(x,t,u)=0. \tag{6}$$

The set S_Δ is invariant under the transformation (**Error! Reference source not found.**) provided that

$$\mathrm{p}r^{(n)}v(\Delta)\,|_{\Delta\equiv 0}=0 \tag{7}$$

where $\mathrm{p}r^{(n)}v$ is the n-th prolongation of the vector field (4), which is given explicitly in terms of ξ,τ and ϕ in Ch. 5, Olver [3].

The *generalized symmetries* carried out from the same considerations, but considering all the solutions of the equation (7), depending on t, x, u, and the derivatives of u. The maximum order of derivatives of u give the so-called *order* of the generalized (infinitesimal) symmetry:

$$v=\xi(x,t,u,u_x,\cdots)\partial_x+\tau(x,t,u,u_x,\cdots)\partial_t+\phi(x,t,u,u_x,\cdots)\partial_u.$$

In practice, the equation (7) is very difficult to solve for an initial equation of order highest that two, due to the large numbers of terms involved in the expression of the prolongation. An alternate version of the symmetries equation can be obtained by changing the form of the infinitesimal symmetry $v=\xi\partial_x+\tau\partial_t+\phi\partial_u$ to the equivalent *evolutionary* form

$$v_Q=Q(t,x,u,u_x,\cdots)\partial_u,$$

where $Q=\phi-\xi u_x-\tau u_t$ is the characteristic of the generalized vector field v.

The symmetries equation (7) for the initial equation $u_t=K[u]$ rewrite as:

$$(D_t-K')Q=0, \tag{8}$$

where D_t is the total time derivative (the evolutionary derivative)

$$D_t Q = \partial_t Q + \sum_{i=0}^{\infty} \frac{\partial Q^{\alpha}}{\partial u_x^{(i)}} D_x^i(u_t) = \partial_t Q + Q'(K)$$

and the prime means the Frechet derivative

$$F' = \sum_{i=0}^{\infty} \frac{\partial F^{\alpha}}{\partial u_x^{(i)}} D_x^i.$$

The simplest usual form of the symmetry condition equation is:

$$\partial_t Q + Q'(K) = K'(Q). \tag{9}$$

In practice, one choose an particular order m for the characteristic

$$Q = Q(t, x, u, u_x, ..., u_x^{(m)})$$

and one search all the solution Q of the equation (**Error! Reference source not found.**) by identifying the coefficients of all the corresponding monoms expressed in u and his x-derivatives.

3. The 1+1 dimensional version of Calabi flow

The Calabi flow was defined for $2n$-dimensional Kähler manifolds admitting a Kähler metric g, which is locally expressible in the form

$$ds^2 = 2g_{a\bar{b}} dz^a d\bar{z}^b ,$$

using a system of holomorphic coordinates z^a and their complex conjugates $\bar{z}^a$ with $a = 1,2,\cdots,n$. The general form of Calabi flow equation is (Calabi [5],[6], Bakas [7]):

$$\partial_t g_{a\bar{b}} = \partial^2 R \partial z^a \partial \bar{z}^b \tag{10}$$

where $R = g^{a\bar{b}} R_{a\bar{b}}$ is the Ricci Scalar curvature.

The equation (10) is a parabolic equation for the components of the metric, but it is fourth order in the variables z and $\bar{z}$. Critical points of the flow are called *extremal metrics* and they encompass constant curvature metrics, if they exist on a given Kähler manifold. In this respect, the Calabi flow is used as a tool for studying the conditions for Einstein-Kähler metrics in geometry, and in conjunction with their possible obstructions.

In order to obtain a version of the Calabi flow in 1+1 dimensions, we start from the Bakas ([7]) local expression of Calabi flow in 2+1 dimensions, for conformally flat coordinates

$$ds_t^2 = 2e^{\Phi(z,\bar{z};t)} dz d\bar{z} :$$

is

$$\partial_t \Phi = -\Delta\Delta\Phi\,, \tag{11}$$

where the symbol Δ is the Laplace-Beltrami operator $\Delta = e^{-\Phi}\partial\bar{\partial}$ for the Kähler metric $g^{z\bar{z}} = e^{\Phi(z,\bar{z};t)}$.

Note that this equation can be defined on any Riemann surface, not only on a Kähler manifold. With the transformation $u = e^{\Phi}$, the equation (11) become:

$$\partial_t u = -\partial\bar{\partial}(\frac{1}{u}\partial\bar{\partial}\ln(u)) \tag{12}$$

Introducing a linear combination of the variables z and $\bar{z}$ as a new independent variable, by uni-directionalization one obtain an 1+1 dimensional (reduced) equation. For example, if $z + \bar{z} = x$, the equation (12) rewrite as:

$$\partial_t u = -\partial_{xx}(\frac{1}{u}\partial_{xx}\ln(u))\,, \tag{13}$$

or, explicitly:

$$u_t = -\frac{u_{xxxx}}{u^2} + \frac{6u_{xxx}u_x}{u^3} - \frac{21u_{xx}u_x^2}{u^4} + \frac{4u_{xx}^2}{u^3} + \frac{12u_x^4}{u^5}\,, \tag{14}$$

witch we will use as an unidimensional version of Calabi flow in the next.

4. The Symmetries of Calabi flow in 1+1 dimensions

We look for the generalized symmetries $v_Q = Q[u]\partial_u$ of the equation (14)

$$u_t = K[u] = -\frac{u_{xxxx}}{u^2} + \frac{6u_{xxx}u_x}{u^3} - \frac{21u_{xx}u_x^2}{u^4} + \frac{4u_{xx}^2}{u^3} + \frac{12u_x^4}{u^5}.$$

The symmetry equation (9) write as:

$$\frac{\partial Q}{\partial_t} + \sum_{i=0}^{m}\frac{\partial Q}{\partial u_x^{(i)}} D_x^i(K[u]) = \sum_{i=0}^{4}\frac{\partial K}{\partial u_x^{(i)}} D_x^i(Q[u]). \tag{15}$$

for the m order evolutionary generalized symmetry v_Q. We will search for the solutions of (15) supposing that the order of $Q[u]$ (the maximum order of derivatives of u involved in the expression of Q) is successively 0, 1, 2,...

At order 0, let's take $Q = Q(t, x, u)$. The equation (15) is verified if and only if

$$\frac{\partial Q}{\partial t}=0,\frac{\partial Q}{\partial u}=\frac{Q}{u},\frac{\partial Q}{\partial x}=\frac{Qu_x}{2u}$$

for all the solutions u of the equation (14), from where is simple to conclude that $Q\equiv 0$.

At order 1, consider $Q=Q(t,x,u,u_x)$. The symmetries determining equation (15) imply a system of 15 differential equations in Q, who can be reduced at

$$\frac{\partial Q}{\partial t}=0,\frac{\partial^2 Q}{\partial u_x^2}=0,\frac{\partial Q}{\partial u}=\frac{Q}{u}-\frac{u_x\frac{\partial Q}{\partial u_x}}{u},\frac{\partial Q}{\partial x}=\frac{u_xQ}{2u}-\frac{u_x^2\frac{\partial Q}{\partial u_x}}{2u}$$

for all the solutions u of the equation (14). The first and second equations give $Q\equiv f(x,u)+g(x,u)u_x$, under the supplementary conditions (derived from the others equations):

$$\frac{\partial g}{\partial x}=0,\frac{\partial g}{\partial u}=0,f=0$$

with the solution $g(x,u)=a$, where s is an arbitrary constant. One obtain the generalized evolutionary symmetry:

$$v_1=au_x\partial_u, \tag{16}$$

corresponding to the classical infinitesimal symmetry $v_1=a\partial_x$, witch is a simple translation along the x-axis. For order 2, if we consider $Q=Q(t,x,u,u_x,u_{xx})$, the symmetries equation (15) imply an over-determined system of differential equations for Q that begin with

$$\frac{\partial Q}{\partial u_{xx}}=0,$$

so we have no symmetry of order two. The same result is obtained if we look for the generalized symmetries of order 3. For order 4, if we consider $Q=Q(t,x,u,u_x,u_{xx},u_{xxx},u_{xxxx})$, the symmetries equation (15) determine a system of 245 differential equation involving the partial derivatives of Q. A more simple way to obtain the fourth order evolutionary symmetries is to split Q into tree functions:

$$Q[u]=\phi[u]-\xi[u]u_x-\tau[u]u_t$$

and to solve (15) for ϕ, ξ and τ. Using MAPLE 9 to reduce the system, we have obtained the conditions:

$$\begin{cases} \frac{\partial \xi}{\partial t} &= \frac{\partial \xi}{\partial u} = \frac{\partial^2 \xi}{\partial x} = 0 \\ \frac{\partial \tau}{\partial x} &= \frac{\partial \tau}{\partial u} = 0 \\ \frac{\partial \tau}{\partial t} &= 4\frac{\partial \xi}{\partial x} \\ \phi &= 0 \end{cases} \tag{17}$$

with the solution

$$\xi = a + bx, \quad \tau = c + 4t \tag{18}$$

where a, b and c are arbitrary constants. The corresponding generalized evolutionary symmetries are $v_Q = [(a+bx)u_x + (c+4bt)u_t]\partial_u$, representing the group at tree parameters of classical infinitesimal symmetries:

$$v = -[(a+bx)\partial_x + (c+4bt)\partial_t],. \tag{19}$$

The group (19) is spanned by

$$\begin{aligned} v_1 &= \partial_x \\ v_2 &= \partial_t \\ v_3 &= x\partial_x + 4t\partial_t \end{aligned} \tag{20}$$

witch represent the space and time translation, respectively a scaling transformation ($(x,t) \to (\alpha x, \alpha^4 t)$).

If now we consider $Q = Q(t, x, u, u_x, ..., u_x^{(m)})$ with $m \geq 5$, the equation (15):

$$\frac{\partial Q}{\partial_t} + \sum_{i=0}^{m} \frac{\partial Q}{\partial u_x^{(i)}} D_x^i(K[u]) = \sum_{i=0}^{4} \frac{\partial K}{\partial u_x^{(i)}} D_x^i(Q[u]).$$

generate the condition

$$\frac{\partial Q^\alpha}{\partial u_x^{(m)}} = 0.$$

so, by induction, we do not have generalized evolutionary symmetries of order greater that 4.

Note that the studied equation

$$u_t = -\partial_{xx}(\frac{1}{u}\partial_{xx}\ln(u)),$$

admits a simple Lax representation $L_t = [A, L]$ using

$$\begin{cases} L\psi & = & \psi_x + u\psi \\ A\psi & = & [\partial_x(\frac{1}{u}\partial_{xx}\ln(u))]\psi + \gamma \end{cases} \tag{21}$$

where γ is an arbitrary parameter. (The compatibility condition $\psi_{xt} = \psi_{tx}$ become the (FDE)).

By the other hand, the 1+1 dimensional version of Calabi flow that we studied here is a simple uni-directionalization of the original Calabi flow equation, with is integrable (Bakas [7]) and possess a zero curvature representation and an (algebraic) infinite hierarchy of hight order integrable equations, so the integrability of the equation

$$u_t = -\partial_{xx}(\frac{1}{u}\partial_{xx}\ln(u))$$

can be strongly supposed. The absence of high order generalized symmetries for this equation is a surprising result in this context.

5. Conclusions and future works

In this paper we proved that the group of all arbitrary-order generalized symmetries for the uni-directionalization of Calabi flow equation in 1+1 dimensions:

$$u_t = -\partial_{xx}(\frac{1}{u}\partial_{xx}\ln(u))$$

is spanned by ∂_x ∂_t and $x\partial_x + 3t\partial_t$, witch are geometrical symmetries. A natural extension of this work to the study of Calabi flow in 2+1 dimensions will be treated in a future.

Due to the absence of hight order symmetries for this 1+1 dimensional version of an integrable equation, a study of the existence of some potential (hidden) symmetries, may be non-local, is clearly necessary.

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