Analele Universităţii de Vest din Timişoara
Vol. XLVIII, 2006
Seria Fizică

BEL-ROBINSON ENERGY TENSORS IN BIANCHI TYPE I SPACETIME

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Abstract
We write the expression for the Bel-Robinson tensor defined in three different ways for a Bianchi type-I cosmological model. In doing so we use the orthonormal basis and compare the results obtained with those for ordinary coordinate basis.
Keywords: Bianchi type-I model, Bel-Robinson tensors

1. Introduction

The search for a well-posed definition of local energy-momentum tensor in gravity led Bel [1,2,3] and independently Robinson [4] to construct a four-index tensor for the gravitational field in vacuum. The properties of the now famous Bel-Robinson (BR) tensor are similar to the traditional energy-momentum tensor and following Senovilla [5,6] can be formulated as follows: (i) it possesses a positive-definite time-like component and a "causal" momentum vector; (ii) its divergence vanishes (in vacuum); (iii) the tensor is zero if and only if the curvature of the spacetime vanishes; (iv) it has positivity property similar to the electromagnetic one; and some others. Construction of BR and the study of its properties were widely considered by a number of authors, e.g., Deser *et. al.* [7,8], Teyssandier [9], Senovilla [5], Bergqvist [10], Andersson [11], Wingbrant [12], Choquet-Bruhat *et. al.* [13] etc.

BR energy in a cosmological setting represents an important tool to investigate the nature of singularities in cosmologies. We mention the relation between geodesic completeness, the existence of closed trapped surfaces and the BR energy tensor[14]. Therefore, in our view it is interesting to consider the BR within the scope of some concrete metric.

The present and early stages of our Universe are described with a good approximation by spatially homogeneous and isotropic Friedman-Robertson-Walker (FRW) models. This isotropy of our Universe represents a puzzle to cosmologists. However, the large scale matter distribution in the observable Universe does not exhibit a high degree of homogeneity. In recent time there was a number of space investigations which detect anisotropies in Universe. There are theoretical arguments that support the existence of an anisotropic phase in the early Universe that approaches later on an isotropic phase. For example Zel'dovich [15] proposed that the Universe may have started anisotropic, but rapidly isotropized as a result of quantum effects.

A Bianchi type I Universe (BI) is a straightforward generalization of the flat Robertson-Walker (RW) Universe. It represents one of the simples models of an anisotropic Universe that describes a homogeneous and spatially flat Universe. Unlike the RW Universe which has the same scale factor for each of the three spatial directions, a BI Universe has a different scale factor in each direction, thereby introducing an anisotropy to the system.

In a recent paper [16] we studied the BR within the framework of BI Universe using two different definitions. In the paper [17] the analysis was extended for some other definitions and investigated the dominant energy property (DEP) and dominant super-energy property (DSEP) within this model.

2. Bel-Robinson tensors: definition and it's general properties

BR tensor first appeared in the endless search for a covariant version of gravitational energy. In general relativity, the energetic content of an electromagnetic field propagating in a region free of charge is described by the well-known symmetric trace-less tensor

$$T_{el}^{\alpha\beta} = -\frac{1}{4\pi}(F^{\alpha\lambda}F_{\lambda}^{\beta} - \frac{1}{4}g^{\alpha\beta}F^{\mu\nu}F_{\mu\nu}), \quad (1)$$

where $F^{\alpha\beta}$ is the electromagnetic field tensor. This tensor satisfies:

$$T_{el;\alpha}^{\alpha\beta} = 0 \quad (2)$$

as a consequence of Maxwell equations with $j^{\mu} = 0$. The tensor $T_{el}^{\alpha\beta}$ enables us to define a local density of electromagnetic energy as measured by an observer moving with the unit 4-velocity u:

$$w_{el}(u) = T_{el}^{\alpha\beta}u_{\alpha}u_{\beta}. \quad (3)$$

It follows from (1) that the energy density is positive definite for any time-like vector u.

Within the scope of general relativity, however, it is well known that the concept of local energy density is meaningless for a gravitational field. To overcome this difficulty one is led to introduce the notion of super-energy tensor constructed with the curvature tensor $R_{\mu\nu\alpha\beta}$. The first example of such a tensor was exhibited by Bel [1], that was further generalized to the case of an arbitrary gravitational field [2]. Note that a similar tensor was also introduced by Robinson [4]. This tensor is now commonly know as the BR tensor as well. Since we are going to compare some distinct definition of BR in this paper, before defining them let us see what kind of properties they should have. In general, the BR tensor has the following symmetry properties:

$$\begin{aligned} B_{\mu\nu\alpha\beta} &= B_{\nu\mu\alpha\beta}, \\ B_{\mu\nu\alpha\beta} &= B_{\mu\nu\beta\alpha}, \\ B_{\mu\nu\alpha\beta} &= B_{\alpha\beta\mu\nu}. \end{aligned} \tag{4)(5)(6}$$

The symmetry property leads to the fact that that in n-dimensional case there are $n(n+1)[n(n+1)+2]/8$ independent components of the BR tensor. In case of $n=4$ out of 256 components only 55 are linearly independent.

In literature there are a few definitions of BR. Here we mention only three.

I. By analogy with the tensor (1) which may be written as

$$T_{\mu\nu} = F_{\mu\alpha}F_{\nu}^{\alpha} + *F_{\mu\alpha} * F_{\nu}^{\alpha}, \tag{7}$$

the BR tensor is defined as [7]:

$$B_{\mu\nu\alpha\beta} = R^{\rho\ \sigma}_{\ \mu\ \alpha}R_{\rho\nu\sigma\beta} + *R^{\rho\ \sigma}_{\ \mu\ \alpha} * R_{\rho\nu\sigma\beta}. \tag{8}$$

Here the dual curvature is $*R^{\mu\nu}{}_{\lambda\sigma} \equiv (1/2)\varepsilon^{\mu\nu}{}_{\alpha\beta}R^{\alpha\beta}{}_{\lambda\sigma}$. It should be noted that this definition is adequate only in 4 dimensions and in vacuum. Otherwise this tensor cannot satisfy the DEP [18] and therefore this expression should not be used in other dimensions or in non-Ricci-flat spacetimes.

Using the definition of dual curvature, from (8) we find

$$B_{\mu\nu\alpha\beta} = R^{\rho\ \sigma}_{\ \mu\ \alpha}R_{\rho\nu\sigma\beta} + R^{\rho\ \sigma}_{\ \mu\ \beta}R_{\rho\nu\sigma\alpha} - \frac{1}{2}g_{\mu\nu}R_{\alpha}{}^{\rho\sigma\tau}R_{\beta\rho\sigma\tau}. \tag{9}$$

The properties (4) and (5) follow immediately from (8) thanks to the symmetry property of the Riemann tensor. The property (6) is straightforward from (8), but for (9) it requires

$$g_{\mu\nu}R_{\alpha}{}^{\rho\sigma\tau}R_{\beta\rho\sigma\tau} = g_{\alpha\beta}R_{\mu}{}^{\rho\sigma\tau}R_{\nu\rho\sigma\tau}. \tag{10}$$

II. The restriction that arises above is due to the fact that in defining the BR tensor we used the dual term with the duality operator acting on the left pair only. To avoid this restrictions the BR tensor can be defined by [9,19]

$$\begin{aligned} 2B_{\mu\nu\alpha\beta} &= R^{\rho\ \sigma}_{\ \mu\ \alpha}R_{\rho\nu\sigma\beta} + {}^*R^{\rho\ \sigma}_{\ \mu\ \alpha}\,{}^*R_{\rho\nu\sigma\beta} \\ &+ R^{*\rho\ \sigma}_{\ \ \mu\ \alpha}\,R^*_{\ \rho\nu\sigma\beta} + {}^*R^{*\rho\ \sigma}_{\ \ \mu\ \alpha}\,{}^*R^*_{\ \rho\nu\sigma\beta}, \end{aligned} \tag{11}$$

where the duality operator acts on the left or on the right pair of indices according to its position. Nowadays this is known as the Bel tensor and was introduced by Bel [2] in a slightly different form.

III. Here we give another definition that gives rise to BR tensor, that is trace-less and totally symmetric. It can be achieved by constructing BR by means of Weyl tensor [10,20].

$$B_{\mu\nu\alpha\beta} = C^{\rho\ \sigma}_{\ \mu\ \alpha}C_{\rho\nu\sigma\beta} + {}^*C^{\rho\ \sigma}_{\ \mu\ \alpha}\,{}^*C_{\rho\nu\sigma\beta}. \tag{12}$$

It can be shown that this BR is totally symmetric, i.e.,

$$B_{ijkl} = B_{(ijkl)}, \tag{13}$$

Moreover, the BR defined through Weyl tensor is trace-free, i.e.,

$$g^{jl}B_{ijkl} \equiv 0. \tag{14}$$

Let us study this case in detail. Using the properties of Levi-Civita tensor we first rewrite (12) in the form

$$B_{\mu\nu\alpha\beta} = C^{\rho\ \sigma}_{\ \mu\ \alpha}C_{\rho\nu\sigma\beta} + C^{\rho\ \sigma}_{\ \mu\ \beta}C_{\rho\nu\sigma\alpha} - \frac{1}{2}g_{\mu\nu}C_{\alpha}^{\ \rho\sigma\tau}C_{\beta\rho\sigma\tau}. \tag{15}$$

In what follows we write the expressions for the components of the BR tensor for a BI metric.

3. BR in BI cosmology

BI cosmological model is the simplest model of an anisotropic cosmology. For it's simplicity and many other outstanding properties BI becomes one of the most investigated cosmological models in recent time. For a detailed review of this metric one can consult [21].

We write the BI metric in the form:

$$ds^2 = dt^2 - a_1^2 dx_1^2 - a_2^2 dx_2^2 - a_3^2 dx_3^2, \tag{16}$$

where a_i are the functions of t only. In order to construct BR tensor first we have to find the Riemann and Weyl tensor for the BI metric.

The nontrivial components of the Riemann tensor in an orthonormal basis take the form:

$$R_{0i0i} = \frac{\ddot{a}_i}{a_i}, \quad R_{ijij} = -\frac{\dot{a}_i}{a_i}\frac{\dot{a}_j}{a_j}, \quad i \neq j = 1,2,3.$$

For the nontrivial components of the Weyl tensor in an orthonormal basis we find:

$$C_{jkjk} = -C_{0i0i} = -\frac{2\ddot{a}_i a_j a_k - \ddot{a}_j a_k a_i - \ddot{a}_k a_i a_j - a_i \dot{a}_j a_k - \dot{a}_k \dot{a}_i a_j + 2\dot{a}_j \dot{a}_k}{6 a_i a_j a_k},$$

$$i \neq j \neq k = 1,2,3.$$

Once we have the components of Riemann and Weyl tensor, we can now write the components for the BR. In what follows we write the nontrivial components of the BR tensor. In doing so we use all the three definitions mentioned above. Here we note that the subscripts i, j, k run from 1 to 3 and they are different, i.e., $i, j, k = 1,2,3,$ and $i \neq j \neq k$.

I. From (9) we now write

$$\begin{aligned}
B_{iijj} &= \frac{\dot{a}_j^2}{a_j^2}\left[\frac{\dot{a}_i^2}{a_i^2} - \frac{\dot{a}_k^2}{a_k^2}\right] - \frac{\ddot{a}_j^2}{a_j^2}, \\
B_{jjjj} &= \frac{\dot{a}_j^2}{a_j^2}\left[\frac{\dot{a}_i^2}{a_i^2} + \frac{\dot{a}_k^2}{a_k^2}\right] + \frac{\ddot{a}_j^2}{a_j^2}, \\
B_{ijij} &= \frac{\dot{a}_i}{a_i}\frac{\dot{a}_j}{a_j}\frac{\dot{a}_k^2}{a_k^2} + \frac{\ddot{a}_i}{a_i}\frac{\ddot{a}_j}{a_j}, \\
B_{00jj} &= \frac{\dot{a}_j^2}{a_j^2}\left[\frac{\dot{a}_i^2}{a_i^2} + \frac{\dot{a}_k^2}{a_k^2}\right] - \frac{\ddot{a}_j^2}{a_j^2}, \qquad (17)(18)(19)(20)(21)(22)(23) \\
B_{jj00} &= -\frac{\ddot{a}_j^2}{a_j^2} + \frac{\ddot{a}_k^2}{a_k^2} + \frac{\ddot{a}_i^2}{a_i^2}, \\
B_{j0j0} &= -\frac{\dot{a}_j}{a_j}\left[\frac{\dot{a}_k}{a_k}\frac{\ddot{a}_k}{a_k} + \frac{\dot{a}_i}{a_i}\frac{\ddot{a}_i}{a_i}\right], \\
B_{0000} &= \sum_{j=1}^{3}\frac{\ddot{a}_j^2}{a_j^2}.
\end{aligned}$$

Comparing these expressions with those of [17] one sees, that the use of an orthonormal basis in this case does not simplify them. As in that case, here too we get the following restrictions on metric functions, that reads

$$\left(\frac{\ddot{a}_i}{a_i}\right)^2 \pm \left(\frac{\ddot{a}_j}{a_j}\right)^2 = \left(\frac{\dot{a}_k}{a_k}\right)^2\left[\left(\frac{\dot{a}_j}{a_j}\right)^2 \pm \left(\frac{\ddot{a}_i}{a_i}\right)^2\right]. \qquad (24)$$

It was shown in [17] that that if one defines BR tensor as (8) or (9), it correspond to the Einstein equations with the source field given by a vacuum.

II. Let us write the nontrivial components of BR defined as (11).

$$\begin{aligned}
B_{0000} &= B_{iiii} = \frac{1}{2}[\frac{\ddot{a}_i^2}{a_i^2} + \frac{\ddot{a}_j^2}{a_j^2} + \frac{\ddot{a}_k^2}{a_k^2} + \frac{\dot{a}_i^2}{a_i^2}\frac{\dot{a}_j^2}{a_j^2} + \frac{\dot{a}_j^2}{a_j^2}\frac{\dot{a}_k^2}{a_k^2} + \frac{\dot{a}_k^2}{a_k^2}\frac{\dot{a}_i^2}{a_i^2}], \\
B_{00kk} &= -B_{iijj} = \frac{1}{2}[\frac{\ddot{a}_i^2}{a_i^2} + \frac{\ddot{a}_j^2}{a_j^2} - \frac{\ddot{a}_k^2}{a_k^2} - \frac{\dot{a}_i^2}{a_i^2}\frac{\dot{a}_j^2}{a_j^2} + \frac{\dot{a}_j^2}{a_j^2}\frac{\dot{a}_k^2}{a_k^2} + \frac{\dot{a}_k^2}{a_k^2}\frac{\dot{a}_i^2}{a_i^2}], \\
B_{ijij} &= \frac{\ddot{a}_i}{a_i}\frac{\ddot{a}_j}{a_j} + \frac{\dot{a}_i}{a_i}\frac{\dot{a}_j}{a_j}\frac{\dot{a}_k^2}{a_k^2}, \\
B_{0i0i} &= -\frac{\dot{a}_i}{a_i}[\frac{\dot{a}_j}{a_j}\frac{\ddot{a}_j}{a_j} + \frac{\dot{a}_k}{a_k}\frac{\ddot{a}_k}{a_k}].
\end{aligned} \tag{25}$$

Comparing those expression with those given in [17] we find that the new basis simplifies our task. In this case we have $B_{00kk} = -B_{iijj}$ which was not the case in ordinary basis.

III. Let us now write the components of BR defined in (12).

$$\begin{aligned}
B_{0000} &= B_{iiii} = \frac{1}{6}[\frac{\ddot{a}_i^2}{a_i^2} + \frac{\ddot{a}_j^2}{a_j^2} + \frac{\ddot{a}_k^2}{a_k^2} + \frac{\dot{a}_i}{a_i}\frac{\dot{a}_j}{a_j}\frac{\dot{a}_k}{a_k}(\frac{\dot{a}_i}{a_i} + \frac{\dot{a}_j}{a_j} + \frac{\dot{a}_k}{a_k}) \\
&- \frac{\dot{a}_i}{a_i}\frac{\dot{a}_j}{a_j}(\frac{\ddot{a}_i}{a_i} + \frac{\ddot{a}_j}{a_j}) - \frac{\dot{a}_j}{a_j}\frac{\dot{a}_k}{a_k}(\frac{\ddot{a}_j}{a_j} + \frac{\ddot{a}_k}{a_k}) - \frac{\dot{a}_k}{a_k}\frac{\dot{a}_i}{a_i}(\frac{\ddot{a}_k}{a_k} + \frac{\ddot{a}_i}{a_i}) \\
&+ \frac{\dot{a}_i^2}{a_i^2}\frac{\dot{a}_j^2}{a_j^2} + \frac{\dot{a}_j^2}{a_j^2}\frac{\dot{a}_k^2}{a_k^2} + \frac{\dot{a}_k^2}{a_k^2}\frac{\dot{a}_i^2}{a_i^2} - \frac{\ddot{a}_i}{a_i}\frac{\ddot{a}_j}{a_j} - \frac{\ddot{a}_j}{a_j}\frac{\ddot{a}_k}{a_k} - \frac{\ddot{a}_k}{a_k}\frac{\ddot{a}_i}{a_i} \\
&+ 2(\frac{\ddot{a}_i}{a_i}\frac{\dot{a}_j}{a_j}\frac{\dot{a}_k}{a_k} + \frac{\ddot{a}_j}{a_j}\frac{\dot{a}_k}{a_k}\frac{\dot{a}_i}{a_i} + \frac{\ddot{a}_k}{a_k}\frac{\dot{a}_i}{a_i}\frac{\dot{a}_j}{a_j})], \\
B_{00kk} &= -B_{iijj} = -\frac{1}{18}(\frac{\dot{a}_i}{a_i}\frac{\dot{a}_j}{a_j} - 2\frac{\dot{a}_j}{a_j}\frac{\dot{a}_k}{a_k} + \frac{\dot{a}_k}{a_k}\frac{\dot{a}_i}{a_i} - 2\frac{\ddot{a}_i}{a_i} + \frac{\ddot{a}_j}{a_j} + \frac{\ddot{a}_k}{a_k}) \\
&\times (\frac{\dot{a}_i}{a_i}\frac{\dot{a}_j}{a_j} + \frac{\dot{a}_j}{a_j}\frac{\dot{a}_k}{a_k} - 2\frac{\dot{a}_k}{a_k}\frac{\dot{a}_i}{a_i} + \frac{\ddot{a}_i}{a_i} - 2\frac{\ddot{a}_j}{a_j} + \frac{\ddot{a}_k}{a_k}) = -B_{ijij}.
\end{aligned} \tag{29)(30}$$

In this case though the expressions for the components of BR does not undergo any radical changes, as it was shown in[17], the dominant property of the BR now fulfills.

4. Conclusions and further questions

In this report we have investigated the Bel-Robinson tensor for the BI spacetime using the orthonormal basis. Comparing the expressions with those found in [17] we see at least for two cases the new basis significantly simplifies the expression and gives clear idea about the dominant property of BR.

The averaging problem in general relativity is an important issue with many implications in cosmology and in the understanding of the recent expansion history of the visible Universe. We must remind that the averaging methods are far from unique and the problem of defining a suitable averaging scheme remains open [22]. The study of the effects of spatial anisotropies on cosmologies by looking at the average properties of BI models could contribute to our understanding of the recent expansion history of the Universe.

As it is known, in curved spacetimes, there is an ambiguity in the construction of a vacuum state, Fock space for quantum fields. In some cases, there may exist coordinates associated with the Killing vectors in analogy with the rectangular coordinates in Minkowski space. However, even if such privileged coordinates do exist, there are problems in the quantization of the fields [23]. That is the case of the quantum fields in BI spacetimes and the Bogolubov coefficients are worthy of being evaluated.

In conclusion the BI cosmology models are of interest deserving further studies.

Acknowledgments

Part of the present work was done with the financial support from a grant afforded by the Representative of Romanian Government to JINR-Dubna within the Hulubei-Meshcheryakov Program on 2006. M. V. has been partially supported by a grant CNCSIS, Romania. M. V. would like to acknowledge the Organizing Committee of the Physics Conference TIM-06, Timisoara, Romania (2006) for hospitality and excellent organization.

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