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REDUCIBLE SECOND-CLASS CONSTRAINTS

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Abstract
An irreducible canonical approach to first- and second-order reducible second-class constraints is given.

1. Introduction

Physical theories of crucial significance, like the ones describing fundamental interactions, are constrained systems at the Hamiltonian level [7,8,3,4]. These are theories pictured by more variables than there are independent physical degrees of freedom. This further implies that the canonical variables are not all independent, there existing some relations among them called constraints. The relevant classification of constraints is that which distinguishes between first-and second-class constraints.

In this talk we expose an irreducible approach to first- and second-order reducible second-class constraints. Our approach is based on the following main steps: (i) we construct the Dirac bracket for the considered reducible second-class constraint systems; (ii) we introduce some supplementary phase-space variables and construct an equivalent set of reducible second-class constraints, whose Dirac bracket coincides with the original one; (iii) we associate an irreducible set of second-class constraints with that from step (ii) and show that the Dirac bracket for the latter set coincides with the Dirac bracket for the former second-class system. These three steps ensure that the original Dirac bracket coincides that constructed with respect to the irreducible second-class constraints.

2. First- and second-class constraints

We consider a dynamical system described by a degenerate Lagrangian

$$\det \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} = 0 \tag{1}$$

For the sake of simplicity we will use the notation for systems with a finite number of degrees of freedom, but the analysis can be straightforwardly extended to field theories. It is known that condition (1) leads to a system with constraints at the Hamiltonian level. The constraints are some relations among the phase space variables, i.e. some relations of the type

$$\Phi_\alpha(z^a) \approx 0, \alpha = 1,...,M, \tag{2}$$

where

$$z^a = (q^i, p_i), \tag{3}$$

are the canonical phase-space coordinates. The symbol "$\approx$" is known as the weak equality and means that (2) are not identities, but merely equations. Constraints (2) are said to be irreducible if the constraint functions $\Phi_\alpha(z^a)$ are independent.

With respect to their behaviour related to the phase-space symplectic structure (the Poisson bracket) the constraints are divided into first- and second-class. Constraints (2) are first-class if

$$[\Phi_\alpha, \Phi_\beta] = C^\gamma{}_{\alpha\beta} \Phi_\gamma \tag{4}$$

where $C^\gamma{}_{\alpha\beta}$ are some phase-space functions. The first-class constraint functions generate some local transformations of the phase-space coordinates called Hamiltonian gauge transformations, so the fixation of the dynamics requires some additional restrictions (canonical gauge conditions) to be imposed on the phase-space coordinates in order to 'kill' the non-physical degrees of freedom. Constraints (2) are second-class if the matrix of elements

$$[\Phi_\alpha, \Phi_\beta] \equiv C_{\alpha\beta}, \tag{5}$$

is invertible, at least weakly. The Hamiltonian equations that govern the dynamics of a purely second-class system read as

$$\dot{z}^a \approx [z^a, H]^*,$$

where H represents the canonical Hamiltonian of the system and $[,]^*$ is the Dirac bracket, defined through

$$[F,G]^* = [F,G] - [F,\Phi_\alpha] C^{\alpha\beta} [\Phi_\beta, G], \tag{6}$$

where $C^{\alpha\beta}$ are the elements of the inverse of $C_{\alpha\beta}$, i.e. $C_{\alpha\beta}C^{\beta\gamma} \approx \delta_{\alpha}{}^{\gamma}$. By contrast to the first-class case, the dynamics of a purely second-class Hamiltonian theory is completely determined by the initial conditions.

3. Reducible approach to first-order reducible second-class constraints

In the sequel we will redenote the second-class constraints by

$$\chi_{\alpha_0}(z^a) \approx 0, \alpha_0 = 1,\ldots,M_0 \tag{7}$$

and for simplicity we will take all the phase-space variables to be bosonic. In addition, we presume that the functions χ_{α_0} are not independent, there existing some nonvanishing functions $Z^{\alpha_0}{}_{\alpha_1}$ such that

$$Z^{\alpha_0}{}_{\alpha_1}\chi_{\alpha_0} = 0, \alpha_1 = 1,\ldots,M_1. \tag{8}$$

Moreover, we assume that $Z^{\alpha_0}{}_{\alpha_1}$ are independent. If these conditions are met, we say that the second-class constraints are first-order reducible and the functions $Z^{\alpha_0}{}_{\alpha_1}$ are called first-order reducibility functions.

The canonical approach to systems with reducible second-class constraints is quite intricate, demanding a modification of the usual rules as the matrix of the Poisson brackets among the constraints is no longer invertible.

A first idea is to isolate a set of independent constraints

$$\chi_A \approx 0, A = 1,\ldots,M_0 - M_1,$$

and then construct the Dirac bracket [7,8] with respect to this set

$$[F,G]^* = [F,G] - [F,\chi_A]M^{AB}[\chi_B,G] \tag{9}$$

In (9), M^{AB} is the inverse of the matrix of elements $C_{AB} = [\chi_A, \chi_B]$ in the sense that $M^{AB}C_{BC} \approx \delta^A{}_C$. The split of the constraints may lead to the loss of important symmetries, so it should be avoided.

Another idea is to construct the Dirac bracket in terms of a non-invertible matrix without separating the independent constraint functions [4,9,9,10,11]. Here, we start with the matrix

$$C_{\alpha_0\beta_0} = [\chi_{\alpha_0}, \chi_{\beta_0}], \tag{10}$$

that is not invertible because

$$Z^{\alpha_0}{}_{\alpha_1} C_{\alpha_0\beta_0} \approx 0. \tag{11}$$

If $d^{\alpha_1}{}_{\alpha_0}$ is solution to the equation

$$d^{\alpha_1}{}_{\alpha_0} Z^{\alpha_0}{}_{\beta_1} \approx \delta^{\alpha_1}{}_{\beta_1} \tag{12}$$

then we can introduce a matrix $M^{\alpha_0\beta_0}$ through the relation

$$M^{\alpha_0\beta_0} C_{\beta_0\gamma_0} \approx \delta^{\alpha_0}{}_{\gamma_0} - Z^{\alpha_0}{}_{\alpha_1} d^{\alpha_1}{}_{\gamma_0} \tag{13}$$

with $M^{\alpha_0\beta_0} = -M^{\beta_0\alpha_0}$. Then, the formula [11]

$$[F,G]^* = [F,G] - [F,\chi_{\alpha_0}] M^{\alpha_0\beta_0} [\chi_{\beta_0}, G], \tag{14}$$

defines the same Dirac bracket like (9) on the surface (7). This result is due to [5].

4. Irreducible approach to first-order reducible second-class constraints

A third possibility is to substitute the reducible second-class constraints by some irreducible ones and further work with the Dirac bracket based on the irreducible constraints. This idea has been inspired by the irreducible approach to first-class constraints and was developed in [6,7].

We start with the matrix of elements

$$D^{\gamma_1}{}_{\alpha_1} = Z^{\alpha_0}{}_{\alpha_1} A_{\alpha_0}{}^{\gamma_1}, \tag{15}$$

where $A_{\alpha_0}{}^{\gamma_1}$ are some functions taken such that

$$rank(D^{\gamma_1}{}_{\alpha_1}) = M_{1.} \tag{16}$$

By means of $D^{\gamma_1}{}_{\alpha_1}$ we build the matrix

$$\mu^{\lambda^0\sigma^0} \approx M^{\lambda^0\sigma^0} + Z^{\lambda_0}{}_{\lambda_1} \overline{D}^{\lambda_1}{}_{\beta_1} \omega^{\beta_1\gamma_1} \overline{D}^{\sigma_1}{}_{\gamma_1} Z^{\sigma_0}{}_{\sigma_0}, \tag{17}$$

where $\overline{D}^{\beta_1}{}_{\gamma_1}$ stands for the inverse of $D^{\gamma_1}{}_{\alpha_1}$, while $\omega^{\beta_1\gamma_1}$ is an arbitrary, invertible antisymmetric matrix.

Theorem 1. The matrix of elements $\mu^{\alpha_0\beta_0}$ is invertible.

Proof. The proof is given in [8].

Theorem 2. The Dirac bracket (14) takes the form

$$[F,G]^* = [F,G] - [F,\chi_{\alpha_0}] \mu^{\alpha_0\beta_0} [\chi_{\beta_0}, G], \tag{18}$$

on the surface $\chi_{\alpha_0} \approx 0$.

Proof. The proof is given in [8].

The inverse of (17) can be written as

$$\mu_{\rho_0\tau_0} \approx C_{\rho_0\tau_0} + A_{\rho_0}{}^{\rho^1} \omega_{\rho_1\tau_0} A_{\tau_0}{}^{\tau_1} \tag{19}$$

where $\omega_{\rho_1\tau_1}$ stands for the inverse of the corresponding upper-indices matrix. Apart from being antisymmetric and invertible, the matrix $\omega_{\rho_1\tau_1}$ is arbitrary. In order to endow this matrix with a concrete significance, we introduce some new variables $(y_{\alpha_1})_{\alpha_1=1,\ldots,M_1}$ with the Poisson brackets

$$[y_{\alpha_1}, y_{\beta_1}] = \omega_{\alpha_1\beta_1}, \tag{20}$$

and consider the system subject to the reducible second-class constraints

$$\chi_{\alpha_0} \approx 0, y_{\alpha_1} \approx 0. \tag{21}$$

The Dirac bracket on the phase-space described by $\left(z^{\alpha}, y_{\alpha 1}\right)$ corresponding to the above second-class constraints reads as

$$[F,G]^*\Big|_{z,y} = [F,G] - [F,\chi_{\alpha_0}]\mu^{\alpha_0\beta_0}[\chi_{\beta_0},G] - [F,y_{\alpha_1}]\omega^{\alpha_1\beta_1}[y_{\beta_1},G] \tag{22}$$

where the Poisson brackets from the right hand-side of (22) contain derivatives with respect to all z^a and y_{α_1}. After some computation we infer that

$$[F,G]^*\Big|_{z,y} \approx [F,G]^*, \tag{23}$$

where $[F,G]^*$ is given by (18) and the weak equality refers to the surface (21). At this point we construct the constraints

$$\tilde{\chi}_{\alpha_0} = \chi_{\alpha_0} + A_{\alpha_0}{}^{\alpha_1} y_{\alpha_1} \approx 0. \tag{24}$$

After some simple computation, from (24) we infer that

$$\chi_{\alpha_0} = D^{\beta_0}{}_{\alpha_0}\tilde{\chi}_{\beta_0}, y_{\alpha_1} = \overline{D}^{\beta_1}{}_{\alpha_1} Z^{\beta_0}{}_{\beta_1}\tilde{\chi}_{\beta_0}, \tag{25}$$

with

$$D^{\beta_0}{}_{\alpha_0} = \delta^{\beta_0}{}_{\alpha_0} - Z^{\beta_0}{}_{\beta_1}\overline{D}^{\beta_1}{}_{\alpha_1} A_{\alpha_0}{}^{\alpha_1}. \tag{26}$$

It is easy to see that if (21) hold, then (24) also hold. From (25) we obtain that if (24) hold, then (21) hold too, so

$$\tilde{\chi}_{\alpha_0} \approx 0 \Leftrightarrow \chi_{\alpha_0} \approx 0, y_{\alpha_1} \approx 0, \tag{27}$$

such that the constraints (29) are equivalent to (21). Finally, if we use (27), then the functions (24) satisfy

$$[\tilde{\chi}_{\alpha_0},\tilde{\chi}_{\beta_0}]\approx\mu_{\alpha_0\beta_0}. \tag{28}$$

The last relation emphasizes the second-class behaviour of constraints (24).

Under these considerations, we can prove the following theorem.

Theorem 3. (i) The second-class constraints

$$\tilde{\chi}_{\alpha_0}\approx 0, \tag{29}$$

are irreducible.

(ii) The Dirac bracket with respect to the irreducible second-class constraints

$$[F,G]^*\big|_{ired}=[F,G]-[F,\tilde{\chi}_{\alpha_0}]\mu^{\alpha_0\beta_0}[\tilde{\chi}_{\beta_0},G], \tag{30}$$

coincides with (22)

$$[F,G]^*\Big|_{ired}=[F,G]^*\Big|_{z,y}, \tag{31}$$

on the surface $\tilde{\chi}_{\alpha_0}\approx 0$.

Proof. The proof is given in [8].

The last theorem proves that we can approach reducible second-class constraints in an irreducible fashion. Thus, starting with the reducible constraints (7), we construct the irreducible constraint functions (24), whose Poisson brackets form an invertible matrix. Formulas (23) and (31) ensure that

$$[F,G]^*\big|_{ired}\approx[F,G]^*, \tag{32}$$

so the fundamental Dirac brackets among the original variables z^a within the irreducible setting coincide with those from the reducible version

$$[z^a,z^b]^*\big|_{ired}\approx[z^a,z^b]^* \tag{33}$$

Moreover, the new variables y_{α_1} do not affect the irreducible Dirac bracket as from (31) we have that $[y_{\alpha_1},F]^*\big|_{ired}\approx 0$. Thus, the equations of motion for the original reducible system can be written as $\dot{z}^a\approx[z^a,H]^*\big|_{ired}$, where H is the canonical Hamiltonian. The equations of motion for y_{α_1} read as $\dot{y}_{\alpha_1}\approx 0$ and lead to $y_{\alpha_1}=0$ by taking some appropriate boundary conditions (vacuum to vacuum) for these unphysical variables. This completes the general procedure.

Let us briefly exemplify the general theory on gauge-fixed two-forms, subject to the second-class constraints

$$\chi_{\alpha_0} \equiv \begin{pmatrix} -2\partial_k \pi_{ki} \\ -\partial_l A^{lj} \end{pmatrix} \approx 0. \tag{34}$$

The constraints (34) are first-stage reducible, with the reducibility functions expressed by

$$Z^{\alpha_0}{}_{\alpha_1} = \begin{pmatrix} \partial^i & 0 \\ 0 & \partial_j \end{pmatrix} \tag{35}$$

Acting along the line exposed in the above, we take the matrix $A_{\alpha_0}{}^{\alpha_1}$ under the form

$$A_{\alpha_0}{}^{\alpha_1} = \begin{pmatrix} -\partial_i & 0 \\ 0 & -\partial_j \end{pmatrix}, \tag{36}$$

so

$$D^{\alpha_1}{}_{\beta_1} = \begin{pmatrix} -\partial_i \partial^i & 0 \\ 0 & -\partial^j \partial_j \end{pmatrix} \tag{37}$$

is invertible. In order to construct the irreducible second-class constraints, we introduce the variables

$$y_{\alpha_1} = \begin{pmatrix} \pi \\ \varphi \end{pmatrix}, \tag{38}$$

and take

$$\omega_{\alpha_1 \beta_1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{39}$$

As it can be seen, the supplementary scalar fields (π, φ) are canonically conjugated, with π the momentum. Then, the irreducible second-class constraints are expressed by

$$\tilde{\chi}_{\alpha_0} \equiv \begin{pmatrix} -2\partial^k \pi_{ki} - \partial_i \pi \\ -\partial_l A^{lj} - \partial^j{}_{\varphi} \end{pmatrix} \approx 0, \tag{40}$$

such that

$$\mu_{\alpha_0 \beta_0} = \begin{pmatrix} 0 & \delta_i{}^j \Delta \\ -\delta^k{}_l \Delta & 0 \end{pmatrix} \tag{41}$$

where $\Delta = \partial^l \partial_l$. By inverting (41) we obtain that the only nonvanishing irreducible Dirac brackets are given by

$$[A^{ij}(x), \pi_{kl}(y)]^* \Big|_{ired} = \frac{1}{2}\left(\delta^{[i}{}_k \delta_l{}^{j]} + \frac{1}{\Delta} \partial^{[i} \delta^{j]}{}_p \partial_{[k} \delta^p{}_{l]} \right) \delta^3(x-y), \tag{42}$$

where the notation $[i_1...i_n]$ means antisymmetry with respect to the indices between brackets.

5. Irreducible approach to second-order reducible second-class constraints

Now, we pass to the case where the second-class constraints are second-order reducible. We say that the second-class constraints are reducible of order two if there are some non-vanishing functions $Z^{\alpha_0}{}_{\alpha_1}$ and $Z^{\alpha_1}{}_{\alpha_2}$ such that

$$Z^{\alpha_0}{}_{\alpha_1}\chi_{\alpha_0} = 0, \alpha_1 = 1,...,M_1, \tag{43}$$

$$Z^{\alpha_1}{}_{\alpha_2}Z^{\alpha_0}{}_{\alpha_1} = 0, \alpha_2 = 1,...,M_2, \tag{44}$$

where, in addition, all the second-order reducibility functions $Z^{\alpha_1}{}_{\alpha_2}$ are assumed to be independent. Along the same line employed at the first-order reducibility case, we construct the irreducible second-class constraints

$$\tilde{\chi}_{\alpha_0} = \chi_{\alpha_0} + A_{\alpha_0}{}^{\alpha_1} y_{\alpha_1} \approx 0, \tag{45}$$

$$\tilde{\chi}_{\alpha_2} = Z^{\alpha_1}{}_{\alpha_2} y_{\alpha_1} \approx 0 \tag{46}$$

equivalent with (21), so we finally obtain that

$$[F,G]^*\big|_{ired} \approx [F,G]^*, \tag{47}$$

where in this situation the irreducible Dirac bracket takes the form

$$\begin{aligned}[F,G]^*\big|_{ired} = {} & [F,G] - [F,\tilde{\chi}_{\alpha_0}]\mu^{\alpha_0\beta_0}[\tilde{\chi}_{\beta_0},G] \\ & - [F,\tilde{\chi}_{\alpha_0}]Z^{\alpha_0}{}_{\alpha_1}\omega^{\alpha_1\beta_1}A_{\beta_1}{}^{\rho_2}\bar{D}^{\beta_2}{}_{\rho_2}[\tilde{\chi}_{\beta_2},G] \\ & - [F,\tilde{\chi}_{\alpha_2}]\bar{D}^{\alpha_2}{}_{\beta_2}A_{\alpha_1}{}^{\beta_2}\omega^{\alpha_1\beta_1}Z^{\beta_0}{}_{\beta_1}[\tilde{\chi}_{\beta_0},G] \\ & - [F,\tilde{\chi}_{\alpha_2}]\bar{D}^{\alpha_2}{}_{\beta_2}A_{\alpha_1}{}^{\beta_2}\omega^{\alpha_1\beta_1}A_{\beta_1}{}^{\rho_2}\bar{D}^{\lambda_2}{}_{\rho_2}[\tilde{\chi}_{\lambda_2},G]\end{aligned} \tag{48}$$

In formula (48) the quantities $\bar{D}^{\alpha_2}{}_{\beta_2}$ stand for the elements of the inverse of

$$D^{\alpha_2}{}_{\beta_2} = A_{\alpha_1}{}^{\alpha_2}Z^{\alpha_1}{}_{\beta_2}, \tag{49}$$

where the functions $A_{\alpha_1}{}^{\alpha_2}$ have been chosen such that

$$rank(D^{\alpha_2}{}_{\beta_2}) = M_{2.}$$

We observe that the terms from the first-line of formula (48) are due to the contribution of the first-order reducibility functions, while the remaining pieces are generated by the existence of the second-order reducibility functions. A detailed example regarding the irreducible approach to second-order reducible second-class constraints is included within this proceeding volume.

6. Conclusion

In this talk we proved that first- and second-order reducible second-class constraints can be approached in an irreducible fashion by respectively transforming them into an equivalent set of irreducible second-class constraints such that the reducible and irreducible Dirac brackets weakly coincide.

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