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A TWO DIMENSIONAL LATTICE OF QUANTUM DOTS IN THE PRESENCE OF ELECTRIC AND MAGNETIC FIELDS

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Abstract:
We consider a two dimensional lattice of quantum dots, under the presence of a perpendicular magnetic field and an external electric field oriented along one of the symmetry directions of the lattice. We calculated the electronic spectrum of this lattice in the Hall configuration under a single magnetic-band approximation. By choosing the Landau gauge , a single particle Hamiltonian was formulated, and its eigenfunctions were obtained as appropriately symmetrized, magnetic field- dependent Bloch functions. In the end we achieved a Harper's equation and we calculated the transfer matrix of this equation.
Keywords: Harper equation

1. Introduction

The motion of electrons in two dimensional systems, under the influence of both a periodic potential and external fields has been of interest to theorists and experimentalists very much.[1-6]. In the present paper we modeled a two dimensional lattice of cylindrical quantum dots, by a three dimensional potential, an external perpendicular magnetic field and an electric field which is applied along z direction. We will assume that the electrons in the dot are confined by a very narrow quantum well along the z direction. Therefore, we obtaining a periodic effective potential, depending only on the coordinates over the plane of dots.

2. Harper equation

We consider a two dimensional lattice of quantum dots under the influence of a potential V, which is periodic over the plane of the dots:

$$V(\theta, z) = V(\theta + \mathbf{d}_n, z) \qquad (1)$$

where: $\theta = (x, y)$ - is the dots plane and:

$$\mathbf{d}_n = (n_x d, n_y d)$$

where d is the lattice constant. This potential is given by:

$$V_{eff}(\theta - \mathbf{d}_n) = \sum_{\alpha = x,y} \frac{1}{2} m^* \omega^2 (\alpha - n_\alpha d)^2 \quad , \quad |\alpha - n_\alpha d| \leq \frac{d}{2} \tag{2}$$

In this equation m^* is the effective mass over the plane, and ω is the quantum dot geometric frequency, which is characterized by the length scale:

$$l_{punct} = \sqrt{\hbar / m^* \omega} \tag{3}$$

The electric field is applied along one of the symmetry directions of the lattice, **F=x**F; and the magnetic field is applied normal to the plane of dots **B=z**B . The magnetic field will be included through the vector potential in the Landau gauge. In this gauge the Hamiltonian preserved the translational symmetry of the periodic potential, along the direction perpendicular to the electric field. The single-particle Hamiltonian for the quantum dot array is

$$H(\theta) = \frac{\left(\mathrm{p} + \frac{e}{c} \mathbf{A}(\theta) \right)^2}{2m^*} + \sum_n V_{eff}(\theta - \mathbf{d}_n) + eFx \tag{4}$$

As translational invariance along the y direction is preserved, eigenfunction and energy eigenstates are characterized by fixed values of k_y :

$$H(\theta)\Psi_{k_y}(\theta) = E_{k_y}\Psi_{k_y}(\theta) \tag{5}$$

Applying the Bloch's theorem, the eigenfunctions of the Hamiltonian can be writen:

$$\Psi_{k_y}(\theta) = e^{ik_y y} u_{k_y}(\theta) \tag{6}$$

where $u_{k_y}(\theta)$ is a periodic function along y direction:

$$u_{k_y}(\theta + d\hat{\mathbf{y}}) = u_{k_y}(\theta) \tag{7}$$

and it can be expressed in terms of Wannier functions [7]:

$$u_{k_y}(\theta) = \sum_{\mathbf{d}_n} C_{n_x} \exp\left(-i \left(k_y + \frac{e}{\hbar c} \mathbf{A}(\mathbf{d}_n) \right) (y - n_y d) \right) W(\theta - \mathbf{d}_n) \tag{8}$$

By inserting Eq. (8) in Eq . (6) , we obtain another equation for the eigenfunctions :

$$\Psi_{k_y}(\theta) = \sum_{\mathbf{d}_n} C_{n_x} e^{ik_y n_y d} \exp\left(-i \frac{e}{\hbar c} \mathbf{A}(\mathbf{d}_n)(\theta - \mathbf{d}_n) \right) W(\theta - \mathbf{d}_n) \tag{9}$$

and substituting this equation in Schrodinger Eq. (5) we have the corresponding eigenvalue for the periodic function:

$$\left[\frac{\left(\mathrm{p}+\hat{\mathbf{y}}\hbar k_y+\frac{e}{c}\mathbf{A}(\theta)\right)^2}{2m^*}+\sum_n V_{eff}(\theta-\mathbf{d}_n)+eFx\right]u_{k_y}(\theta)=E_{k_y}u_{k_y}(\theta) \tag{10}$$

Let us apply a discrete translation $T_{\hat{\mathbf{x}}qd}$ to the previous relation, where T denotes the operator of discrete magnetic translations : $T_{\mathbf{A}}(\mathbf{d}_n)=\exp\left[-\frac{i\mathbf{d}_n}{\hbar}\left(\mathbf{p}+\frac{e}{c}\mathbf{A}\right)\right]$

The resulting expression is:

$$\left[\frac{\left(\mathrm{p}+\hat{\mathbf{y}}\hbar\left(k_y+\frac{2\pi q}{d}N_\phi\right)+\frac{e}{c}\mathbf{A}(\theta)\right)^2}{2m^*}+\sum_n V_{eff}(\theta-\mathbf{d}_n)+eFx\right]u_{k_y}(\theta+\hat{\mathbf{x}}qd)= \tag{11}$$

$$=\left(E_{k_y}-qeFd\right)u_{k_y}(\theta+\hat{\mathbf{x}}qd)$$

If N_ϕ is a rational number $N_\phi=\frac{p}{q}$ then $\bar{k}_y=k_y+\left(\frac{2\pi q}{d}\right)N_\phi$ is associated to the same magnetic Bloch function as k_y, as a consequence of the translational symmetry along the y direction. Therefore:

$$E_{\bar{k}_y}=E_{k_y} \tag{12}$$

The Eq. (11) is :

$$\left[\frac{\left(\mathrm{p}+\hat{\mathbf{y}}\hbar\bar{k}_y+\frac{e}{c}\mathbf{A}(\theta)\right)^2}{2m^*}+\sum_n V_{eff}(\theta-\mathbf{d}_n)+eFx\right]u_{k_y}(\theta+\hat{\mathbf{x}}qd)=\left(E_{\bar{k}_y}-qeFd\right)u_{k_y}(\theta+\hat{\mathbf{x}}qd) \tag{13}$$

and the conclusion is that, if E_{k_y} is an eigenvalue belonging to the electronic spectrum, then $E_{\bar{k}_y}-qeFd$ is another eigenvalue corresponding to the same value of k_y .

In this paper we choose W as eigenfunctions of a single-dot Hamiltonian:

$$H_{punct}(\theta)=\frac{\left(\mathbf{p}+\mathbf{y}\frac{eB}{c}x\right)}{2m^*}+\frac{m^*}{2}\omega^2\theta^2 \tag{14}$$

and we will approximate the Wannier functions by the lowest energy eigenstate, which is Φ_0 [8]:

$$\Phi_0(\theta)=\frac{1}{\sqrt{\pi}l_0}\exp\left(-\frac{\theta^2}{2l_0^2}-i\frac{eB}{2\hbar c}xy\right) \tag{15}$$

Therefore, the Eq. (9) become :

$$\Psi_{k_y}(\theta)=\sum_{\mathbf{d}_n} C_{n_x} e^{ik_y n_y d}\exp\left(-i\frac{e}{\hbar c}\mathbf{A}(\mathbf{d}_n)(\theta-\mathbf{d}_n)\right)\Phi_0(\theta-\mathbf{d}_n) \tag{16}$$

We will insert the last equation in the Schrodinger Eq . (5) :

$$\sum_{\mathbf{d}_n} C_{n_x} e^{ik_y n_y d}\left[\frac{\left(\mathbf{p}+\frac{e}{c}\mathbf{A}(\theta)\right)^2}{2m^*}+\sum_n V_{eff}(\theta-\mathbf{d}_n)+eFx-E_{k_y}\right]\exp\left(-i\frac{e}{\hbar c}\mathbf{A}(\mathbf{d}_n)(\theta-\mathbf{d}_n)\right)\Phi_0(\theta-\mathbf{d}_n)=0 \tag{17}$$

which can be writen as:

$$\sum_{\mathbf{d}_n} C_{n_x} e^{ik_y n_y d}\exp\left(-i\frac{e}{\hbar c}\mathbf{A}(\mathbf{d}_n)(\theta-\mathbf{d}_n)\right)\left[\frac{\left(\mathbf{p}+\frac{e}{c}\mathbf{A}(\theta)\right)^2}{2m^*}+\sum_n V_{eff}(\theta-\mathbf{d}_n)+eFx-E_{k_y}\right]\Phi_0(\theta-\mathbf{d}_n)=0 \tag{18}$$

The expression inside the parentheses can be write in terms of the single-dot Hamiltonian:

$$\sum_{\mathbf{d}_n} C_{n_x} e^{ik_y n_y d}\exp\left(-i\frac{e}{\hbar c}\mathbf{A}(\mathbf{d}_n)(\theta-\mathbf{d}_n)\right)\left[H_{punct}(\theta-\mathbf{d}_n)+\Delta V(\theta-\mathbf{d}_n)+eFx-E_{k_y}\right]\Phi_0(\theta-\mathbf{d}_n)=0 \tag{19}$$

where:

$$\Delta V(\theta-\mathbf{d}_n)\equiv\sum_n V_{eff}(\theta-\mathbf{d}_n)-\frac{m^*}{2}\omega^2(\theta-\mathbf{d}_n)^2 \tag{20}$$

Since Φ_0 is an eigenfunction of H_{punct} with eigenvalue Ω, we have :

$$\sum_{\mathbf{d}_n} C_{n_x} e^{ik_y n_y d}\left[\hbar\Omega+\Delta V(\theta-\mathbf{d}_n)+eFx-E_{k_y}\right]\langle\theta|\mathbf{d}_n\rangle=0 \tag{21}$$

where we adopted the Dirac's notation:

$$\langle\theta|\mathbf{d}_n\rangle=\exp\left(-i\frac{e}{\hbar c}\mathbf{A}(\mathbf{d}_n)(\theta-\mathbf{d}_n)\right)\Phi_0(\theta-\mathbf{d}_n) \tag{22}$$

To determine the coefficients C_{n_x} in the last equation we take the internal product with the function $\langle\mathbf{d}_{n'}|\theta\rangle$:

$$\sum_{\mathbf{d}_n} C_{n_x} e^{ik_y n_y d} \left\lfloor (\hbar\Omega - E_{k_y}) \langle \mathbf{d}_{n'} | \mathbf{d}_n \rangle + \langle \mathbf{d}_{n'} | \Delta V(\theta - \mathbf{d}_n) | \mathbf{d}_n \rangle + \langle \mathbf{d}_{n'} | eFx | \mathbf{d}_n \rangle \right\rfloor = 0 \tag{23}$$

In the tight-binding approximation, Eq (32) become::

$$\left[2\cos\left(2\pi N_\phi n_x + k_y d\right) - n_x \frac{eFd}{M}\right] C_{n_x} + C_{n_x+1} + C_{n_x-1} = E_{a\dim} C_{n_x} \tag{24}$$

where
$$E_{a\dim} = \frac{\left\lfloor E_{k_y} - \hbar\Omega - E_F \right\rfloor}{M} \tag{25}$$

$$M = \frac{\hbar\Omega}{2}\left(\frac{l_0}{l_{punct}}\right)^4 \exp\left(-\left(\frac{1}{4}\right)\left(\left(\frac{d}{l_0}\right)^2 + \left(\frac{\pi N_\phi l_0}{d}\right)^2\right)\right)\left(1 - \frac{1}{4}\left(\left(\frac{d}{l_0}\right)^2 + \left(\frac{\pi N_\phi l_0}{d}\right)^2\right)\right)$$

$$E_F = \frac{\hbar\Omega}{2}\left(\frac{l_0}{l_{punct}}\right)^4 Te$$

and *Te* is a function belonging to the matrix elements corresponding to the periodic potential V. This equation is the Harper equation, which determines the energy spectrum.

3. The transfer matrix

The Harper equation (24) can be expressed in an matricial form [1]:

$$\Psi_{n_x+1} = T_{n_x} \Psi_{n_x} \tag{26}$$

where :

$$\Psi_{n_x} = \begin{pmatrix} C_{n_x} \\ C_{n_x-1} \end{pmatrix} \tag{27}$$

and

$$T_{n_x}\left(E_{a\dim}\right) = \begin{pmatrix} E_{a\dim} - 2\cos\left(2\pi N_\phi n_x + k_y d\right) + n_x \dfrac{eFd}{M} & -1 \\ 1 & 0 \end{pmatrix} \tag{28}$$

is the transfer matrix. Because both C_{n_x} and C_{n_x+q} are solutions of the Harper equation (24) we have : $C_{n_x+q} = e^{i\alpha_q} C_{n_x}$

We starting with Ψ_0 and applying the equation (26) :

$$\Psi_q = M_q\left(E_{a\dim}\right)\Psi_0 \tag{29}$$

where :

$$M_q\left(E_{a\dim}\right) = \prod_{i=0}^{q-1} T_i\left(E_{a\dim}\right) \tag{30}$$

must be a unity matrix. Under this condition we have : $e^{i\alpha_q}\Psi_0 = M_q^{11}\Psi_0 + M_q^{12}\Psi_{-1}$

and $$e^{i\alpha_q}\Psi_{-1} = M_q^{21}\Psi_0 + M_q^{22}\Psi_{-1}$$

which has nontrivial solution if we have:

$$\begin{vmatrix} M_q^{11} - e^{i\alpha_q} & M_q^{12} \\ M_q^{21} & M_q^{22} - e^{i\alpha_q} \end{vmatrix} = 0 \qquad (31)$$

The trace of the M matrix must be a real number :

$$M_q^{12} = M_q^{21} = 0 \qquad\qquad M_q^{11} = \frac{1}{M_q^{22}} = e^{\pm i\alpha_q}$$

Therefore, we have: $$Tr\, M_q\left(E_{a\dim}\right) = 2\cos\left(\alpha_q\right) \qquad (32)$$

which become : $$\left|Tr\, M_q\left(E_{a\dim}\right)\right| \le 2 \qquad (33)$$

The equation (33) determines the electronic spectrum of the Harper equation.

4. Numeric result

We present results for a two dimensional array of quantum dots with ldot =30 A and lattice constant d=100 A. We assumed the effective mass =0.067. Fig.1. shows the energy spectrum as a function of k_y. One can see that the band has been split into exactly q=3 subbands. In fig. 2. we have the same results for p=1, q=4.

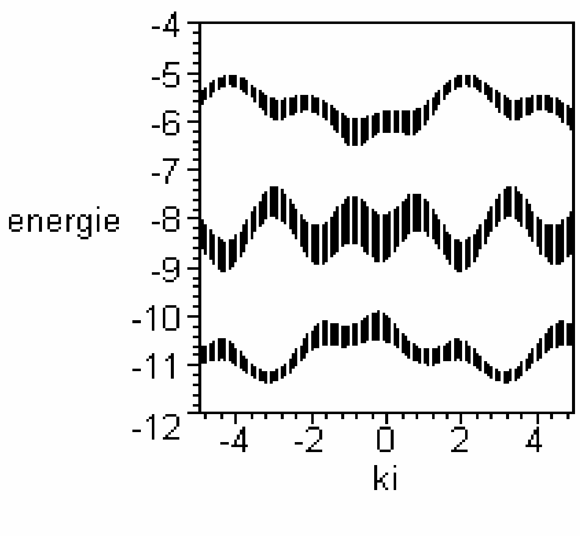

Fig. 1. Energy spectrum for the system, for p=1 , q=3 , ldot=30 A, d=100 A

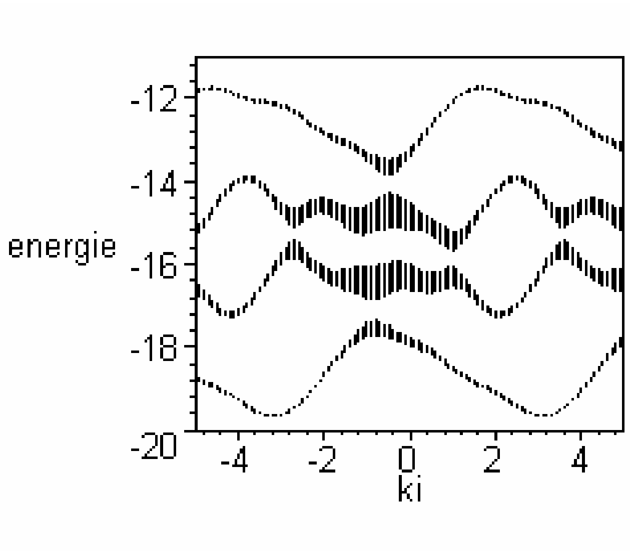

Fig. 2. Energy spectrum for the system, for p=1 , q=3 , ldot=30 A, d=100 A

5. Conclusions

We have studied the energy spectrum of electrons in a two dimensional lattice of quantum dots, subject to an perpendicular magnetic field and an electric field , applied along one of the symmetry directions of the array. The magnetic field is not treated as a perturbation to the band structure, and the external electric field is not included in the wave, its effects are calculated by direct diagonalization of the Hamilton. Despite the approximations involved in this paper (tight-binding, single-band) this method for constructing magnetic Bloch-like functions in the Landau gauge is fairly general.

References

[1] E. Papp, C. Micu, *Ecuatia Harper si aplicatiile ei* (1999)
[2] R. Hofstadter, Phys. Rev. B 14, 2239 (1976)
[3] G. Harper, Proc. Phys. Soc., London Sect A 68, 874 (1955)
[4] B. Wiegmann, Phys. Rev. Lett. 72, 1890 (1994)
[5] M. Torres , Elsevier Science (2004)
[6] M. Tores, Phys. Rev. B 61, 9879 (2000)
[7] H. Wannier, Rev. Mod. Phys. 34, 645 (1962)
[8] E. Munoz, Phys. Rev. B 71, 165301 (2005)