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THE GOWDY T3 COSMOLOGY AS A TEST MODEL FOR NUMERICAL RELATIVITY*

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Abstract
Recent numerical analysis investigations point out several troubles concerning the stability and convergence of equations used in numerical relativity, namely the BSSN method versus the classical ADM one. The Gowdy T3 cosmology proved to be one of the models revealing these problems, thus we concentrate on this model in establishing certain solutions to the pathologies of numerical simulations with the Cactus code.

1. Introduction

Exact solutions of Einstein equations have been for long time used to test the numerical codes for accuracy, stability and convergence [10,11]. Actually we have an entire Cactus code thorn, called Exact with several exact solutions implemented. Certain cosmological solution played a major role in developing numerical cosmology simulations codes too. Recently, the behavior of the codes near the singularities was studied in the context of some inhomogeneous exact solutions, pointing out troubles and pathologies... Here the T3 Gowdy cosmology can play an important role.

The solution to all these is the search for new methods for numerical integration of Einstein eqs. (EE) alternate to ADM and BSSN! An example: harmonic coordinates (HC) [8]. HC were used to prove [9] the local existence of solutions for the vacuum EE. Main advantage of using HC : the EE are in the forms of wave eqs. on curved spacetime - well known in numerical simulations ! Main disadvantage: in HC the time coordinate will not remain timelike during the evolution and this will cause troubles in numerical codes ! Thus

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we have to go back again to exact solutions with known symmetries in HC for testing the numerical evolution systems and codes!

Harmonic coordinates and the T3 Gowdy spacetime

Harmonic coordinates satisfy the wave equation with source:

$$\nabla^a \nabla_a x^\mu = H^\mu \tag{1}$$

Constraints can be introduced defining:

$$C^\mu = g^{\alpha\beta}\Gamma^\mu_{\alpha\beta} + H^\mu \tag{2}$$

Thus it follows that:

$$C^\mu = 0 \tag{3}$$

These are the constraint equations ! Usually we shall consider:

$$H^\mu = 0 \tag{4}$$

The Gowdy T3 space time, has the line element as :

$$ds^2 = e^{(\tau-\lambda)/2}[-e^{-2\tau} d\tau^2 + dz^2] + e^{-\tau}[e^P dx^2 + 2e^P Q dx dy + (e^P Q^2 + e^{-P}) dy^2] \tag{5}$$

where $P = P(\tau, z), Q = Q(\tau, z)$ and $\lambda = \lambda(\tau, z)$. Here the coordinates are all naturally harmonic coordinates (solutions of the wave equation).

Einstein equations have a simple wave equation form [1-4] :

$$\begin{aligned} &[\partial_\tau \partial_\tau - e^{-2\tau} \partial_z \partial_z] P \\ &- e^{2P}[(\partial_\tau Q)^2 - e^{-2\tau}(\partial_z Q)^2] = 0 \\ &\partial_\tau \partial_\tau Q - e^{-2\tau} \partial_z \partial_z Q \\ &+ 2[\partial_\tau P \partial_\tau Q - e^{-2\tau} \partial_z P \partial_z Q] = 0 \end{aligned} \tag{6}$$

In a 3+1 formulation we have the momentum constraint as :

$$-\partial_z \lambda = 2(\partial_\tau P \partial_z P + e^{2P} \partial_\tau Q \partial_z Q) \tag{7}$$

and the hamiltonian constraint as

$$\begin{aligned} &-\partial_\tau \lambda = (\partial_\tau P)^2 + e^{-2\tau}(\partial_z Q)^2 \\ &+ e^{2P}[(\partial_r Q)^2 + e^{-2\tau}(\partial_z Q)^2] \end{aligned} \tag{8}$$

Numerical simulations with Gowdy spacetime

There are three ways to numerically integrate these equations:

- The symplectic integrator (Berger et. al, [1-4]). This method is using the genuine canonical formalism (ADM) with momentum instead of extrinsic curvature,

- Specific codes (Hern, Garfinkle, [5,6]). This method uses specific designed codes only for this metric only. Use of HC is reported as alternate method instead of ADM method
- BSSN method with Cactus code. This method directly integrates EE using Exact thorn.

Metric components are only provided to the code (and appropriate initial data)

Good results, concerning convergence and accuracy were reported with simulations using the first two methods above. As for example in [6] Testing the BSSN method on this metric with Exact thorn gives bad results, strongly dependent on the values of the parameters. we used the Cactus code (http://www.cactuscode.org) with Einstein + BSSN thorns. The metric was implemented through the Exact thorn (as in [10]). Exact thorn is providing to the Cactus code only the initial values of the metric components and parameters, the evolution is done then by the Cactus code alone, thorough solving numerically the full 3D Einstein equation. As for example in the next two figures we have some of our preliminary results.

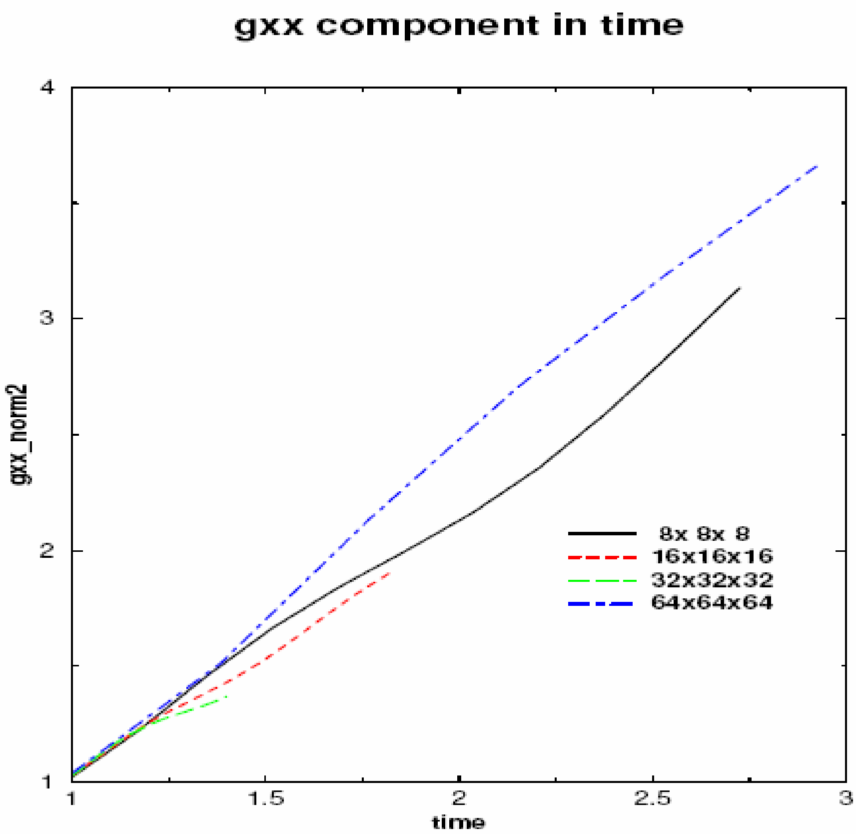

Figure 1: Convergence test for T3 Gowdy metric using Cactus code and BSSN method

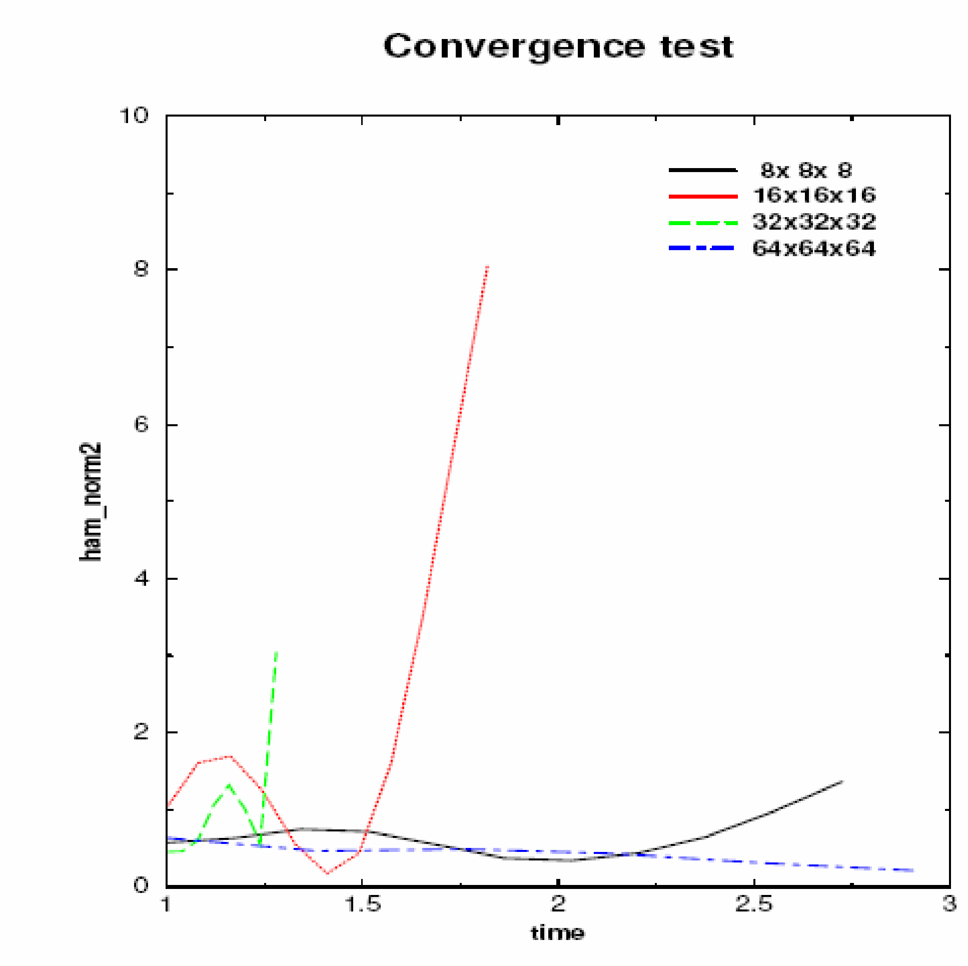

Figure 2: The time evolution of one of the metric components, for different resolutions, using the Cactus code and BSSN method

In figure 1 we have the convergence test (done as described in [10]) for four different resolutions. As it is obvious no convergence is visible, the hamiltonian constraint (it should be zero for all times) is growing fast till the code crashes. In figure 2 we gave the time behavior for the g_{xx} component of the metric, as was output from our Cactus codes. Again no convergence is visible, the code being unstable and inaccurate very fast after the initial time.

Proposed solutions

- Modify the BSSN method. This is not very promising as early attempts (see the japanese group articles) proved to be more difficult to handle (more parameters among other troubles)
- Use the symplectic integrator - based on ADM canonical formalism. This is the most promising, as we have here the entire hamiltonian machinery. But we need to write a complete new code to integrate EE !

- A new code, based on HC (Szylagyi et.al. [7]). As HC proved to be the most accurate in our case (and in other too!) this is the most promising way to develop a new code for solving EE! Actually, at the Albert Einstein Institute in Potsdam, Germany, there exists a code, called ``Generalized harmonic code'' for doing this, as a result of an effort during the last year of the numerical relativity group (mainly based on previous development done by the Pittsburg group).

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