Analele Universităţii de Vest din Timişoara
Vol. XLVI, 2005
Seria Fizică

DIRECT CONSTRUCTION METHOD FOR CONSERVATION LAWS OF NONLINEAR EQUATIONS*

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Abstract
An effective algorithm method is presented for finding the local conservation laws for partial differential equations with any number of independent and dependent variables. The method does not require the use or existence of a variational principle. As an example, the general approach is applied to the 1D Ricci flow model.
Keywords: nonlinear system, conservation law.

1. Introduction

In the study of differential equations, conservation laws have many significant uses, particularly with regard to integrability and linearization, constants of motion, analysis of solutions, and numerical solution methods. A conservation law attached to a differential equation is usually expressed by a divergence equation which contains some important quantities called conserved densities. The paper presents in the following section a specific algorithm for finding these densities for an arbitrary nonlinear system of equations, proposed by [1]. The third section is devoted to the application of general approach to the 1D Ricci flow model which describes the fast diffusion process. Some concluding remarks will end the paper.

2. The general algorithm

Let us consider a dynamical system described by the nonlinear system of equations:

$$G^{\sigma} = \frac{\partial u^{\sigma}}{\partial t} + g^{\sigma}(t, \mathbf{x}, \mathbf{u}, \partial_x \mathbf{u}, ..., \partial_{\mathbf{x}}^m \mathbf{u}), \sigma = 1, ..., N \quad (1)$$

where $\mathbf{u} \equiv (u^1, ..., u^N) \in \mathbf{R}^N$ and $\mathbf{x} \equiv (x^2, ..., x^p) \in \mathbf{R}^{p-1}$.

* Oral presentation, TIM-05 conference, 24-25 November, 2005, Timisoara

Let $U=\eta^{\sigma}\frac{\partial}{\partial u^{\sigma}}$, the symmetry operator which leaves invariant the system (1). The invariance condition is:

$$U^{(m)}[G^{\alpha}]\equiv D_t\eta^{\sigma}+\frac{\partial g^{\sigma}}{\partial u^{\rho}}\eta^{\rho}+\frac{\partial g^{\sigma}}{\partial u_i^{\rho}}D_i\eta^{\rho}+...+\frac{\partial g^{\sigma}}{\partial u_{i_1...i_m}^{\rho}}D_{i_1...i_m}\eta^{\rho}=0,\ \sigma=1,N \tag{2}$$

where $U^{(m)}$ is the extension of order m for the generator U.

The following step in our algorithm is to check the adjoint invariant condition on the adjoint symmetries $U^{*}=\omega_{\sigma}\frac{\partial}{\partial u_{\sigma}}$. The adjoint of eq. (2) has the expression:

$$-D_t\omega_{\sigma}+\frac{\partial g^{\sigma}}{\partial u^{\rho}}\omega_{\sigma}-D_i(\frac{\partial g^{\sigma}}{\partial u_i^{\rho}}\omega_{\sigma})+...+(-1)^m D_{i_1...i_m}(\frac{\partial g^{\sigma}}{\partial u_{i_1...i_m}^{\rho}}\omega_{\sigma})=0 \tag{3}$$

Without loss of generality, we are free to let η^{σ} and ω_{σ} have no dependence on u_t^{σ} and their derivatives with respect to x^i. A local conservation law of (1) is a divergence expression of the general form:

$$D_t\Phi^t(t,\mathbf{x},\mathbf{u},\partial\mathbf{u},...,\partial^k\mathbf{u})+D_i\Phi^i(t,\mathbf{x},\mathbf{u},\partial\mathbf{u},...,\partial^k\mathbf{u})=0 \tag{4}$$

where the quantities $\{\Phi^t,\Phi^i\}$ are called *conserved densities* attached to the system (1).

In the next considerations, without loss of generality, the functions Φ^t, Φ^i are choosing with no dependences on u_t^{σ} and their x^i derivatives. The law (4) can be put in the form proposed by [1]:

$$D_t\Phi^t+D_i(\Phi^i-\Gamma^i)=(u_t^{\sigma}+g^{\sigma})\Lambda_{\sigma},\sigma=\overline{1,N} \tag{5}$$

where $\Lambda_{\sigma}(t,x,u,\partial_x u,...,\partial_x^k u)$ are called *multipliers* and can be written using the *Euler operators* $\hat{E}_{u^{\sigma}}$ in the form:

$$\Lambda_{\sigma}=\hat{E}_{u^{\sigma}}(\Phi^t)\text{ with }\hat{E}_{u^{\sigma}}=\frac{\partial}{\partial u^{\sigma}}-D_i\frac{\partial}{\partial u_i^{\sigma}}+D_iD_j\frac{\partial}{\partial u_{ij}^{\sigma}}+.. \tag{6}$$

with Γ^i proportional to G^{σ} and their derivatives $D_iG^{\sigma},...,D_{i_1}...D_{i_p}G^{\sigma}$.

The following step is to determine the system of equations for Λ_{σ}.

It is important to note that his action $\hat{E}_{u^{\sigma}}(u_t^{\sigma}+g^{\sigma})\Lambda_{\sigma}$ is zero, because the Euler operator annihilates any divergence. It is convenient to split the left hand side of the resulted equation in two parts: the *leading-part* which contains $G^{\sigma},D_iG^{\sigma}...,D_{i_1}...D_{i_p}G^{\sigma}$ and the *non-leading part*. Both expressions must vanish separately. The obtained *non-leading-equation* is the adjoint eq.(3) with $\omega_{\sigma}=\Lambda_{\sigma}$, $\sigma=\overline{1,N}$, that is:

$$-D_t\Lambda_\sigma + \frac{\partial g^\sigma}{\partial u^\rho}\Lambda_\sigma - D_i(\frac{\partial g^\sigma}{\partial u_i^\rho}\Lambda_\sigma) + .. + +(-1)^m D_{i_1...i_m}(\frac{\partial g^\sigma}{\partial u_{i_1...i_m}^\rho}\Lambda_\sigma) = 0\,. \quad (7)$$

The resulted *leading-equation i*s called *the adjoint invariance condition* on Λ_σ, where $u_\sigma(t,x)$ is an arbitrary function of its arguments. Therefore, this equation splits into separate equations given by the coefficients of $G^\sigma, D_i G^\sigma, ..., D_{i_1}...D_{i_p}G^\sigma$ as follows:

$$(-1)^{p+1}\frac{\partial\Lambda_\sigma}{\partial u_{i_1...i_p}^\rho} + \frac{\partial\Lambda_\rho}{\partial u_{i_1...i_p}^\sigma} = 0,\ \sigma,\rho = \overline{1,N} \quad (8)$$

$$(-1)^{q+1}\frac{\partial\Lambda_\sigma}{\partial u_{i_1...i_q}^\rho} + \frac{\partial\Lambda_\rho}{\partial u_{i_1...i_q}^\sigma} - C_q^{q+1}D_{i_q+1}\frac{\partial\Lambda_\rho}{\partial u_{i_1...i_q+1}^\sigma} + (-1)^{p-q}C_q^p D_{i_q+1}...D_{i_p}\frac{\partial\Lambda_\rho}{\partial u_{i_1...i_p}^\sigma} = 0,\ q = \overline{1,p-1} \quad (9)$$

$$-\frac{\partial\Lambda_\sigma}{\partial u^\rho} + \frac{\partial\Lambda_\rho}{\partial u^\sigma} - D_i\frac{\partial\Lambda_\rho}{\partial u_i^\sigma} + ... + (-1)^p D_{i_1...i_p}\frac{\partial\Lambda_\rho}{\partial u_{i_1...i_p}^\sigma} = 0 \quad (10)$$

By solving the system (7-10) we can obtain the concrete forms for $\Lambda_\rho, \rho = 1,...,N.$ The last step of the algorithm consists in finding, for each Λ_ρ, the conserved densities Φ^t, by inverting the relation (6). The formula to obtain Φ^t is [2]:

$$\Phi^t = \int_0^1 d\lambda(u^\sigma - \bar{u}^\sigma)\Lambda_\sigma[\mathbf{u}_{(\lambda)}] + \int_0^1 d\lambda K(\lambda t, \lambda x) \quad (11)$$

where $u_{(\lambda)}^\sigma = u^\sigma + (1-\lambda)\bar{u}^\sigma$, $\lambda = parameter$; $\Lambda_\sigma[\mathbf{u}_{(\lambda)}] = \Lambda_\sigma(t,\mathbf{x},\mathbf{u}_{(\lambda)},\partial_\mathbf{x}\mathbf{u}_{(\lambda)},...,\partial_\mathbf{x}^p\mathbf{u}_{(\lambda)})$ (12)

$K(t,x) = (u_{(\lambda)t}^\sigma + g^\sigma[\mathbf{u}_{(\lambda)}])\Lambda_\sigma[\mathbf{u}_{(\lambda)}])|_{\lambda=0}$; $g^\sigma[\mathbf{u}_{(\lambda)}] = g^\sigma(t,\mathbf{x},\mathbf{u}_{(\lambda)},\partial_\mathbf{x}\mathbf{u}_{(\lambda)},...,\partial_\mathbf{x}^m\mathbf{u}_{(\lambda)})$ **.** (13)

3. Conservation law for 1D Ricci flow model

The 1D Ricci flow model leads to the *fast diffusion equation* [3]:

$$u_t = (\ln u)_{xx} \Leftrightarrow u_t - \frac{u_{xx}u - u_x^2}{u^2} = 0 \quad (14)$$

The invariance condition for (14) has the form:

$$0 = \eta[2u_t u - u_{xx}] + \eta^x[2u_x] + \eta^t u^2 + \eta^{xx}[-u] \quad (15)$$

where we use the relations [4]: $\eta^x = D_x\eta$, $\eta^t = D_t\eta$, $\eta^{xx} = D_{xx}\eta$. The adjoint of condition (15) has the expression:

$$-2u_x D_x\eta - 2\eta u_{xx} - u^2 D_t\eta + uD_{xx}\eta = 0\,. \quad (16)$$

The conservation law is:

$$div\Phi \equiv D_j\Phi^j = D_t\Phi^t + D_x\Phi^x = 0 \quad (17)$$

and following the general algorithm can be put in the form:

$$D_t\Phi^t + D_x(\Phi^x - \Gamma) = (u_t u^2 - u_{xx}u + u_x^2)\Lambda \tag{18}$$

where Γ is proportional with G and its x derivatives.

We suppose that $\Lambda = \Lambda(x,t,u.u_x,u_{xx})$ and not depends on u_t and its x derivatives. By applying the Euler operator on the right side of (18), results the relation:

$$\begin{aligned} 0 = -u^2 D_t\Lambda - 4u_{xx}\Delta - 4u_x D_x\Lambda - uD_x^2\Lambda + [u_t u^2 - u_{xx}u + u_x^2]\Lambda_u - \\ -D_x[(u_t u^2 - u_{xx}u + u_x^2)\Lambda_{u_x}] + D_x^2[(u_t u^2 - u_{xx}u + u_x^2)\Lambda_{u_{xx}}] \end{aligned} \tag{19}$$

For solving (19), we move off the solution space of the given Ricci equation (14). We will consider that $u(x,t)$ is an arbitrary function of its arguments. Because (19) is linear in u_t, u_{tx}, u_{txx}, the coefficients of these derivatives must be zero. The result is that the only independent equation is:

$$\begin{aligned} 0 = -u^2\Lambda_{u_x} + 2u_x u\Lambda_{u_{2x}} + u^2\Lambda_{xu_{2x}} + u^2\Lambda_{u_{2x}u} u_x + \\ +u^2\Lambda_{u_x u_{2x}} u_{2x} + u^2\Lambda_{u_{2x}u_{2x}} u_{3x} \end{aligned} \tag{20}$$

Because Λ do not depend on u_{3x}, we obtain:

$$\Lambda = a(t,x,u,u_x)u_{2x} + b(t,x,u,u_x) \tag{21}$$

The next step is to write the condition (19) on the solution space of Ricci equation (14). The resulted equation is:

$$\begin{aligned} 0 = -u^4\Lambda_t - (u_{2x}u^3 - u_x^2u^2)\Lambda_u - (u_{3x}u^3 - 3u_x u_{2x}u^2 + 2u_x^3 u)\Lambda_{u_x} - (u_{4x}u^3 - 4u^2u_x u_{3x} + \\ +12u_x^2 u_{2x}u - 3u_{2x}^2u^2 - 6u_x^4)\Lambda_{u_{2x}} - 4u^2u_{2x}\Lambda - 4u^2u_x D_x\Lambda - u^3D_x^2\Lambda \end{aligned} \tag{22}$$

Using (21) in (22) and then vanish the coefficients of x derivatives of $u(x,t)$, starting from the highest order of derivation, we obtain the following results: $a(x,t,u,u_x) = 0, b(x,t,u) =$ $= (c_1x + c_2)u^{-2}\ \forall\, c_1, c_2 = \text{constants}$. The final form of multiplier Λ is: $\Lambda = (c_1x + c_2)u^{-2}$.

It follows that the conserved density Φ^t is computed using the formulas (11). Its expression is: $\Phi^t = \int_0^1 d\lambda[u - \bar{u}]\Lambda[\lambda u + (1-\lambda)\bar{u}] + t\int_0^1 d\lambda K(\lambda t, \lambda x) = (c_1x + c_2)\frac{u - u^0}{u^0},\ \bar{u} \equiv u^0 = const.$

4. Conclusions

The method we presented works for nonlinear systems of differential equations which describe physical processes and allows looking for conserved flows which do not depend on the time-derivatives of the variables. The conservation laws are considered "off-shell" of the systems' solutions, written in the form of weak equalities with the help of some multipliers.

At their turn, the multipliers can be expressed as action of some differential operators (Euler operators) on the density of conservative currents. By inversion formula, the density of conserved quantities can be determinates. Using the general algorithm, we obtain for the fast diffusion equation; a conserved current witch density is linear in the variable and in the coordinate.

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