Analele Universităţii de Vest din Timişoara

Vol. XLVI, 2005
Seria Fizică

A GÖDEL TYPE SOLUTION IN A GAUGE THEORY OF GRAVITATION*

S. Babeti[1], G. Zet [2]

[1]*Department of Physics, "Politehnica" University, Timisoara, Romania*
[2]*Department of Physics, "Gh. Asachi" Technical University, Iasi, Romania*

Abstract
In a gauge theory of gravitation, with a particular form of gauge fields on obtain a Gödel type solution in the case of null torsion. The correspondent field strengths for the general case are presented and, using the resulting constraints of null torsion case, the Gödel type model are developed in a gauge theory of gravitation with de Sitter group as symmetry group. All the tensorial calculus are performed with analytical programs conceived in GRTensorII for Maple 8.

1. Introduction

The gauge theory of gravitation allows us to describe the gravity in a similar way with other interactions (electromagnetic, weak, strong). As gauge group of gravitation we use the de-Sitter group in order to obtain a model with cosmological constant for the gravitational field. The Poincaré gauge theory is obtained as a limit of de-Sitter model when the cosmological constant vanishes. The *Section 2* is devoted to the formulation of the de-Sitter gauge model on a spherical symmetric Minkowski space-time. The general expressions for the components $F^{A}_{\mu\nu}$ of the strength tensor of the gauge fields are obtained. Particular ansatzs for the gauge fields are chosen and the corresponding components are presented in *Section 3*. The model lead to a Gödel type model in the gauge theory of gravitation, a nonsingular solution. The case of null torsion is considered in *Section 4* with the resulting constraints. With these constraints on obtain the components of the $F^{ab}_{\mu\nu}$ tensor and the scalar *F*. Using these results on obtain Einstein type equations in which the matter takes the form of a pressure free perfect fluid with no vanishing density. The tensorial operations involve a great number of calculations, and that imposes computer implementation. From this point of view, the

* Oral presentation, TIM-05 conference, 24-25 November, 2005, Timisoara

symbolic programs, as Maple, are appropriate. In this paper the calculations are performed using the GRTensorII computer algebra package, running on the Maple 8 platform. The basic analytical program for the Gödel type model in the gauge theory of gravitation is presented and also a model program for the case of Gödel metric in spherical coordinates which verifies Einstein equations.

2. Theory with DS gauge group for gravitation

We consider a gauge theory of gravitation having de-Sitter (*DS*) group as local symmetry. Let X_A, A = 1, 2,..., 10 be a basis of DS Lie algebra with the corresponding equations of structure given by [3]:

$$[X_A, X_B] = if_{AB}^C X_C, \tag{2.1}$$

where $f_{AB}^C = -f_{BA}^C$ are the constants of structure whose concrete expressions will be given below [eq.(2.4)]. We introduce, as usually, 10 gauge fields $h_\mu^A(x)$, *A = 1, 2, ..., 10, μ = 0, 1, 2, 3*. Then, we construct the tensor of the gauge fields (strength tensor) $F_{\mu\nu} = F_{\mu\nu}^A X_A$ which takes its values in the Lie algebra of the DS group (Lie algebra-valued). The components of this tensor are given by:

$$F_{\mu\nu}^A = \partial_\mu h_\nu^A - \partial_\nu h_\mu^A + f_{BC}^A h_\mu^B h_\nu^C. \tag{2.2}$$

In order to write the constants of structure f_{AB}^C, we use the following notation for the index A:

$$A = \begin{cases} a = 0, 1, 2, 3, \\ [ab] = [01], [02], [03], [12], [13], [23]. \end{cases} \tag{2.3}$$

This means that A can stand for a single index as well as for a pair of indices. The infinitesimal generators X_A are interpreted as: $X_a \equiv P_a$ (energy-momentum operators) and $X_{[ab]} \equiv M_{ab}$ (angular momentum operators) with property $M_{ab} = -M_{ba}$, *a, b = 0, 1, 2, 3* [1]. For the constants of structure f_{AB}^C we find the following expressions:

$$f_{bc}^a = f_{c[de]}^{[ab]} = f_{[bc][de]}^a = 0, \qquad f_{cd}^{[ab]} = 4\lambda^2(\delta_c^b\delta_d^a - \delta_c^a\delta_d^b) = -f_{dc}^{[ab]},$$

$$f_{b[cd]}^a = -f_{[cd]b}^a = \frac{1}{2}(\eta_{bc}\delta_d^a - \eta_{bd}\delta_c^a), \tag{2.4}$$

$$f_{[ab][cd]}^{[ef]} = \frac{1}{4}(\eta_{bc}\delta_a^e\delta_d^f - \eta_{ac}\delta_b^e\delta_d^f + \eta_{ad}\delta_b^e\delta_c^f - \eta_{bd}\delta_a^e\delta_c^f) - e \leftrightarrow f,$$

where λ is a real parameter, and $\eta_{ab} = diag(1,-1,-1,-1)$ is the Minkowski metric. In fact here we have a deformation of DS Lie algebra having λ as parameter. When $\lambda \to 0$, we obtain the Poincaré Lie algebra, i.e. the DS group contracts to the Poincaré group.

We will denote the gauge fields (or potentials) $h_\mu^A(x)$ by $e_\mu^a(x)$ (tetrad fields) if $A = a$ and by $\omega_\mu^{ab}(x) = -\omega_\mu^{ba}(x)$ (spin connection) if $A = [ab]$. Then, introducing the relations (2.4) into the definition (2.2), we find the following expressions of the strength tensor components:

$$F_{\mu\nu}^a = \partial_\mu e_\nu^a - \partial_\nu e_\mu^a + (\omega_\mu^{ab} e_\nu^c - \omega_\nu^{ab} e_\mu^c)\eta_{bc}, \tag{2.5}$$

$$F_{\mu\nu}^{ab} = \partial_\mu \omega_\nu^{ab} - \partial_\nu \omega_\mu^{ab} + (\omega_\mu^{ac}\omega_\nu^{db} - \omega_\nu^{ac}\omega_\mu^{db})\eta_{cd} - 4\lambda^2(e_\mu^a e_\nu^b - e_\nu^a e_\mu^b). \tag{2.6}$$

The quantity $F_{\mu\nu}^a$ is interpreted as the torsion tensor and $F_{\mu\nu}^{ab}$ as the curvature tensor of a Riemann-Cartan space-time defined by the gravitational gauge fields e_μ^a and ω_μ^{ab}.

The action associated to the gravitational gauge fields, quadratic in the components $F_{\mu\nu}^A$, is written in the form [4]:

$$S_g = \int d^4x \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^A F_{\rho\sigma}^B Q_{AB}, \tag{2.7}$$

where $\varepsilon^{\mu\nu\rho\sigma}$ is the Levi-Civita symbol of rang four. This action is independent of any specific metric on M_4. The quantities Q_{AB} are constants, symmetric with respect to the indices A, B: $Q_{AB} = Q_{BA}$. If we chose [1]:

$$Q_{AB} = \begin{cases} \varepsilon_{abcd}, & for \quad A = [ab], \quad B = [cd] \\ 0 \quad, & otherwise \end{cases} \tag{2.8}$$

then we obtain the action of the General Relativity (GR).

We develop a gauge theory of the *DS* group in a 4-dimensional Minkowski space-time, endowed with spherical symmetry:

$$ds^2 = dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{2.9}$$

3. A model with particular gauge fields

With particular forms of spherically gauge fields of the *DS* group $e_\mu^a(x)$ and $\omega_\mu^{ab}(x)$ we obtain some nonsingular solutions. We use these expressions to compute the components of the tensors, $F_{\mu\nu}^a$ and $F_{\mu\nu}^{ab}$. With the components of the strength tensor we calculate the quantities $F_\mu^a = F_{\mu\nu}^{ab}\bar{e}_b^\nu$, and the scalar $F = F_{\mu\nu}^{ab}\bar{e}_a^\mu\bar{e}_b^\nu$. Here, $\bar{e}_a^\rho$ denotes the inverse of e_μ^a with the properties: $e_\mu^a\bar{e}_b^\mu = \delta_b^a$, $\quad e_\mu^a\bar{e}_a^\nu = \delta_\mu^\nu$.

We consider a particular form of spherically gauge fields of the de-Sitter group DS, $e_\mu^a(x)$ and $\omega_\mu^{ab}(x)$ given by the following ansatz:

$$e_\mu^0 = 2a\left(1, 0, 0, \sqrt{2}A^2(r,\theta)\right), \qquad e_\mu^1 = 2a(0,1,0,0), \tag{3.1}$$

$$e_\mu^2 = 2a(0,0,r,0), \qquad e_\mu^3 = 2a\left(0,0,0,A\sqrt{1+A^2(r,\theta)}\right),$$

respectively

$$\omega_\mu^{01} = \left(0,\ 0,\ 0,\ B(r,\theta)\right), \qquad \omega_\mu^{02} = \left(0,\ 0,\ 0,\ C(r,\theta)\right),$$

$$\omega_\mu^{03} = \left(0,\ P(r,\theta),\ E(r,\theta),\ 0\right), \qquad \omega_\mu^{12} = \left(0,0,W(r,\theta),0\right), \tag{3.2}$$

$$\omega_\mu^{13} = \left(Q(r,\theta),\ 0,\ 0,\ G(r,\theta)\right), \qquad \omega_\mu^{23} = \left(H(r,\theta),\ 0,\ 0,\ J(r,\theta)\right),$$

where *A, B, C, P, E, W, Q, G, H, J* are functions of three-dimensional radius r and θ. With this choice we obtain Gödel type solutions, i.e. nonsingular solutions. The non-null components of $F_{\mu\nu}^a$ are:

$$F_{01}^3 = 2a(Q+P), \qquad F_{02}^3 = 2a(Hr+E),$$

$$F_{03}^1 = -2a(QA\sqrt{1+A^2}+B), \qquad F_{03}^2 = -2a(HA\sqrt{1+A^2}+C),$$

$$F_{12}^2 = 2a(1-W), \qquad F_{13}^0 = 2a\left(2\sqrt{2}A\frac{\partial A}{\partial r} + B - PA\sqrt{1+A^2}\right),$$

$$F_{13}^2 = \frac{2a}{\sqrt{1+A^2}}\left(\frac{\partial A}{\partial r} + 2A^2\frac{\partial A}{\partial r} - G\sqrt{1+A^2} - P\sqrt{2}A^2\sqrt{1+A^2}\right),$$

$$F_{23}^0 = 2a\left(2\sqrt{2}A\frac{\partial A}{\partial \theta} + Cr - EA\sqrt{1+A^2}\right), \tag{3.3}$$

$$F_{23}^2 = \frac{2a}{\sqrt{1+A^2}}\left(\frac{\partial A}{\partial \theta} + 2A^2\frac{\partial A}{\partial \theta} - Jr\sqrt{1+A^2} - E\sqrt{2}A^2\sqrt{1+A^2}\right),$$

In order to write the Einstein equations, we calculate the curvature scalar $F = F_{\mu\nu}^{ab}\bar{e}_a^\mu\bar{e}_b^\nu$:

$$\begin{aligned}F = {} & \frac{1}{4a^2rA^2\sqrt{1+A^2}}(-2rA^2\sqrt{1+A^2}PQ - 256\lambda^2a^2rA^2\sqrt{1+A^2} - \\ & 2A^2\sqrt{1+A^2}EH + 2rABQ + 2rACH + 2AEC + 2AGW + 2rA\frac{\partial G}{\partial r} + \\ & 2A^2\sqrt{1+A^2}\frac{\partial W}{\partial r} - \sqrt{2}\sqrt{1+A^2}\frac{\partial C}{\partial \theta} - \sqrt{2}r\sqrt{1+A^2}\frac{\partial B}{\partial r} - \sqrt{2}r\sqrt{1+A^2}PG - \\ & \sqrt{2}r\sqrt{1+A^2}EJ - \sqrt{2}r\sqrt{1+A^2}BW + 2A\frac{\partial J}{\partial \theta} + 2rAPB).\end{aligned} \tag{3.4}$$

The calculations are performed using analytical programs conceived by us and running for the particular gauge fields presented in this paper. The basic program for the developed model in the gauge theory of gravitation is:

Program ”godel-in-gtg.mws”

```
restart: grtw():
grload(minknou, ‘c:/grtii(6)/metrics/minknou.mpl‘);
grdef(‘ev{^a miu}‘); grcalc(ev(up,dn));
grdef(‘omega{[^a ^b] miu}‘);          grcalc(omega(up,up,dn));
grdef(‘eta1{(a b)}‘); grcalc(eta1(dn,dn));
grdef(‘Famn{^a miu niu} := ev{^a niu,miu} - ev{^a miu,niu}
                  + omega{^a^b miu}*ev{^c niu}*eta1{b c}
                  - omega{^a^b niu}*ev{^c miu}*eta1{b c}‘);
grcalc(Famn(up,dn,dn)); grdisplay(_);
grdef(‘Fabmn{^a^b miu niu} := omega{^a^b niu, miu} -omega{^a^b miu, niu}
        + (omega{^a^c miu}*omega{^d^b niu}
        - omega{^a^c niu}*omega{^d^b miu})*eta1{c d}
        - 4*lambda^2*(ev{^a miu}*ev{^b niu}-ev{^b miu}*ev{^a niu})‘);
grcalc(Fabmn(up,up,dn,dn)); grdisplay(_);
grdef(‘evi{^miua}‘);grcalc(evi(up,dn));
grdef(‘F:=Fabmn{^a ^b miu niu}*einv{^miu a}*einv{^niu b}‘);
grcalc(F); grdisplay(_);
grdef(‘Fam{^a miu}:=Fabmn{^a^b miu niu}*einv{^niu b}‘);
grcalc(Fam(up,dn)); grdisplay(_);
grdef(‘Faminv{a^miu}:=Fam{^b niu}*eta1{a b}
              *einv{^miu c}*einv{^niu d}*eta1{^c^d}‘);
grcalc(Faminv(dn,up)); grdisplay(_);
grdef(`u {^miu}`);  grcalc(u(up));
grdef(`ui {miu}:=ev{^c miu}*ev{^d niu}*eta1{c d}*u{^niu}`);  grcalc(ui(dn));
grdef(`Ec{^a miu}=Fam{^a miu}-(1/2)*F*ev{^a miu}
           -(1/(4*a^4))*ui{miu}*u{^niu}*ev{^a niu}`);
grcalc(Ec(up,dn));  grdisplay(_);
```

4. The case of null torsion

If we assume that all the components $F^a_{\mu\nu}$ of the strength tensor vanish and if we remember the Riemann-Cartan theory of gravitation, then the torsion tensor $T^\rho_{\mu\nu} = \bar{e}^\rho_a F^a_{\mu\nu}$ vanish, in accord with *GR* theory. Here $\bar{e}^\rho_a$ denotes, also, the inverse of e^a_μ. From this condition we obtain the some constraints for gauge fields and its component functions. Using the obtained constraints, the analytical program compute the resulting components of $F^{ab}_{\mu\nu}$, F^a_μ, the scalar *F*.

If we consider the case of the model with null torsion, then, from (3.3) the constraints are:

$$B(r,\theta) = -\sqrt{2}A\frac{\partial A}{\partial r},\quad C(r,\theta) = -\sqrt{2}\frac{A}{r}\frac{\partial A}{\partial \theta},\quad P(r,\theta) = \frac{\sqrt{2}}{\sqrt{1+A^2}}\frac{\partial A}{\partial r},$$

$$E(r,\theta) = \frac{\sqrt{2}}{\sqrt{1+A^2}}\frac{\partial A}{\partial \theta},\quad W(r,\theta) = 1,\quad Q(r,\theta) = -\frac{\sqrt{2}}{\sqrt{1+A^2}}\frac{\partial A}{\partial r}, \tag{4.1}$$

$$G(r,\theta) = \frac{1}{\sqrt{1+A^2}}\frac{\partial A}{\partial r},\quad H(r,\theta) = -\frac{\sqrt{2}}{r\sqrt{1+A^2}}\frac{\partial A}{\partial \theta},\quad J(r,\theta) = \frac{1}{r\sqrt{1+A^2}}\frac{\partial A}{\partial \theta}.$$

We observe that the spin connection components ω^{ab}_μ are determined by tetrads e^a_μ (are not independently):

$$\omega^{01}_\mu = \left(0,\ 0,\ 0,\ -\sqrt{2}A\frac{\partial A}{\partial r}\right),\qquad \omega^{02}_\mu = \left(0,\ 0,\ 0, -\sqrt{2}\frac{A}{r}\frac{\partial A}{\partial \theta}\right),$$

$$\omega^{03}_\mu = \frac{\sqrt{2}}{\sqrt{1+A^2}}\left(0,\ \frac{\partial A}{\partial r},\ \frac{\partial A}{\partial \theta},\ 0\right),\qquad \omega^{12}_\mu = (0,0,1,0), \tag{4.2}$$

$$\omega^{13}_\mu = \frac{1}{\sqrt{1+A^2}}\left(-\sqrt{2}\frac{\partial A}{\partial r},\ 0,\ 0,\ \frac{\partial A}{\partial r}\right),\qquad \omega^{23}_\mu = \frac{1}{r\sqrt{1+A^2}}\left(-\sqrt{2}\frac{\partial A}{\partial \theta},\ 0,\ 0,\ \frac{\partial A}{\partial \theta}\right),$$

The equations of Einstein for a pressure-free perfect fluid with no vanishing density can be written in the form:

$$F^a_\mu - \frac{1}{2}Fe^a_\mu - \frac{1}{4a^2}u_\mu u^\nu e^a_\nu = 0 \tag{4.3}$$

where $u^\mu = (1,0,0,0)$, and $u_\mu = e^a_\mu e^b_\nu \eta_{ab} u^\nu$.

With the resulting constraints (4.1) we obtain the Einstein equations with the solution $A = \sinh(r\sin\theta)$ for $-1+24\lambda^2 a^2=0$, which means that $\frac{1}{2a^2} = 12\lambda^2$.

In the case of this solution we obtain the scalar $F = F^{ab}_{\mu\nu}\bar{e}^{\mu}_{a}\bar{e}^{\nu}_{b}$ in the form:

$$F = \frac{1}{a^2} - 48\lambda^2 \tag{4.4}$$

The solution corresponds to a metric $g_{\mu\nu} = e^{a}_{\mu}e^{b}_{\nu}\eta_{ab}$ which, in cylindrical coordinates, has the form:

$$ds^2 = a^2[dt^2 - dr^2 + \sinh^2 r(\sinh^2 r - 1)d\phi^2 + 2\sqrt{2}\sinh^2 r d\phi dt] - dz^2. \tag{4.5}$$

where $\omega^2 = \frac{1}{2a^2}$ is a positive constant which represent in fact the magnitude of the vorticity of the perfect fluid (the matter for the Gödel universe).

5. Concluding remarks

The gauge theory of gravitation allows a complementary description of the gravitational effects in which the mathematical structure of the underlying space-time is not affected by physical events. Only the gauge potentials $e^{a}_{\mu}(x)$ and $\omega^{ab}_{\mu}(x)$ of the gravitational field change as functions of coordinates. This is important when we consider a quantum gauge theory of gravitation.

In this paper the cosmological model represent a solution of Einstein equations with cosmological constant and energy momentum tensor for a perfect fluid.

For the Gödel type model in the gauge theory of gravitation of the *DS* group in a 4-dimensional spherical symmetric Minkowski space-time, $\Lambda = -12\lambda^2$ is interpreted as cosmological constant. This model is a solution for the equivalent of Einstein equations in the gauge theory of gravitation if $\frac{1}{2a^2} = 12\lambda^2$ which correspond to $\Lambda = -\frac{1}{2a^2}$ [2]. The Gödel type model has no singularities and it is valid if the cosmological constant is negative.

References

[1] S.W. MacDowell, F. Mansouri, Unified Geometric Theory of Gravity and Supergravity, Phys. Rev. Lett. 38, No.14, 739 (1977);

[2] K. Gödel, An Example of a New Type of Cosmological Solutions of Eistein's Field Equations of Gravitation, Rev.Mod.Phys, 21, p.447, 1949;

[3] G. Zet, V.Manta, S.Babeti, DeSitter gauge theory of gravitation, Int.J.Mod.Phys. C, Vol.14, No.1, pp. 41-48, 2003;

[4] G. Zet, C.D. Oprisan, S. Babeti, Solutions without sing. in gauge theory of gravitation, Int.J.Mod. Phys. C, Vol. 15, No. 7, pp.1031 - 1038, 2004, arXiv: gr-qc/0403104;
[5] S.Babeti, V. Chiritoiu, G.Zet, Methods and models with nonsingular solutions in a gauge theory of gravitation, Int. Conf. of Fundamental and Appl. Research inPhysics - FARPhys 2005, Einstein 100 - Theory of Relativity Symposium, Iasi, 27-29 October 2005.