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DIRAC QUANTUM FIELD EQUATION AND COMPUTER ALGEBRA*

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Abstract
The Dirac field on curved space-times is an interesting subject that is developed in this paper. The article presents algorithm computer algebra procedures and routines without using GrTensorII package. The main parts are exposed. Some results are provided for the usual metrics.
Keywords: Dirac equation, curved space-time.

1. Introduction

The Dirac fields in flat space-time, was built in order to write a linear equation from the Klein – Gordon relativistic field equation. In order to introduce Dirac equation on curved space-time we could use an anholonomic orthonormal frame because at any point of space-time we need an orthonormal reference frame in order to describe the spinor field as it is already pointed before [1, 2, 3 and 4]. The Dirac equation in a general reference of frame, defined by an anholonomic tetrad field is [2]:

$$i\eta\gamma^{\mu} D_{\mu}\Psi = mc\Psi \tag{1}$$

where the covariant Dirac derivative D_{μ} is

$$\mathrm{D}_{\mu} = \partial_{\mu} + \Gamma_{\mu} = \frac{\partial}{\partial x^{\mu}} + \Gamma_{\mu} \tag{2}$$

In the last few years, the most used package in developing this kind of algorithms is GrTensorII [1, 2, 4, 5]. The flexibility and the facilities offered by this set of procedures that was the reasons to study and to work with it.

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In this paper is presented a different way in building the fermions' evolution equation, without the GrTensorII using. For curved manifold, the geometrical structure could be described considering the line element given as

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu}$$

To set the stage for building the Dirac equation in a curved space-time, should be introduced a local tetrad field

$$g^{\mu\nu} = e^{\mu}{}_{\alpha} e^{\nu}{}_{\beta} \eta^{\alpha\beta}$$

where the metric tensor $\eta_{\alpha\beta}$ is considered to be of minkowskian kind,

$$\eta_{\alpha\beta} = diag\begin{bmatrix}1 & 1 & 1 & -1\end{bmatrix}$$

The fermions' covariant evolution equation has be written as

$$\left[-i\gamma^{\mu}(x)\frac{\partial}{\partial x^{\mu}} - i\gamma^{\mu}(x)\Gamma_{\mu}(x) \right]\psi(x) = m\psi(x) \tag{3}$$

The $\left\{\gamma^{\mu}(x)\right\}_{\mu=\overline{1,4}}$ matrices are curvature dependent and could be presented in terms of tetrad fields as $\gamma^{\mu} = e^{\mu}{}_{\alpha}\gamma^{\alpha}$ and the spin connection $\Gamma_{\mu}(x)$ has to satisfy the equation

$$\left[\Gamma_{\mu}(x), \gamma^{\nu}(x)\right] = \frac{\partial \gamma^{\nu}(x)}{\partial x^{\mu}} + \Gamma^{\nu}{}_{\mu\rho}\gamma^{\rho}(x) \tag{4}$$

where $\Gamma^{\nu}{}_{\mu\rho}$ are Christoffel symbols computed as

$$\Gamma^{\nu}{}_{\mu\rho} = \frac{1}{2} g^{\nu\tau}\left(\frac{\partial g_{\tau\rho}}{\partial x^{\mu}} + \frac{\partial g_{\tau\mu}}{\partial x^{\rho}} - \frac{\partial g_{\mu\rho}}{\partial x^{\tau}} \right) \tag{5}$$

Building the $\Gamma_{\mu}(x)$ coefficients' equation, can be derived the fermions' evolution Dirac relation on curved space-time.

2. Overview on program structure

This section is devoted to a simple description of the used algorithm. First of all, it has to call the packages. The program includes only the usual groups of functions. So, it could be included in every other future program. After the inclusion of the used functions, it should be declared the involved co-ordinates and, of course the metric tensor functions. This first step ends with a set of instructions in order to compute the necessary Christoffel symbols coefficients and the inverse metric tensor $g^{\mu\nu}$:

```
> restart; with(tensor):with(PDEtools);
> coord:=[t,r,theta,phi];
> coord1:=[w1,w2,w3,w4];
> #The g11 element of the metric tensor
> F1(coord[1],coord[2],coord[3],coord[4]):=F(t)^2;
> ...... ;
> Mg:= array (1..4, 1..4, symmetric, [(1,1) =F1 (coord[1], coord[2],
coord[3], coord[4]) ,(1,2) =F2 (coord[1], coord[2], coord[3],
coord[4]) , (1,3) =……]);
> g:=create([-1,-1],op(Mg));
> ...... ;
> Cf1:=Christoffel1(D1g);
> ...... ;
> g_inv:=invert(g,'det_g');
> ...... ;
> Cf2:=Christoffel2(g_inv,Cf1);
```

The next level includes some essential program elements. In this point is computed the linear transformations to and from the tetradic frame. These tensors are to be used in transforming the Dirac equation between the co-ordinates and rigid frames:

```
> ...... ;
> frame (g,h1_inv,const_g,coord);
> eval(const_g);
> ...... ;
```

Next, it has to introduce the Dirac $\{\gamma^{\mu}\}_{\mu=0,3}$ matrices, which allow building the Dirac equation. It could be used the Kramers representation. Further, can construct the entire set of Dirac representation matrices, and could transform them in the rigid frame.

An important advantage of building these matrices in a tensorial form is given by the simplest expression of transforming the vectorial operators between the two considered frames. In the last instructions part is shown some checking procedures in order to ensure about the validity of obtained results.

```
> Gst:=prod(H_transf,Gs,[2,1]);
> M_Gst:=Gst[compts];
> ...... ;
```

With these transformed Dirac matrices, should be built the entire 16 Dirac's representation matrices set in order to obtain a complete picture of the state vector space structure. Next section is devoted to build the spin connection coefficients equation. In order to write the obtained coefficients, in a symbolic manner, it has to use a supplementary procedure to build the spin connection expressions. In this point of view, it could be built the spin coefficients matrix.

```
> ...... ;
> TBV1:=create([+1], op(BV1)); AMV1:=TV1[compts];
> TBV2:=prod(TBV1,TBV1);
> ATV2:=TBV2[compts];
> MATV2:=Matrix(1..4,1..4);
> for I from 1 to 4 do for j from 1 to 4 do MATV2[I,j]:=ATV2[I,j]
end do end do;
```

This algorithm structure offers a real possibility in computing the spin connection coefficients, for a large set of curved space-times. It was checked on a series of cases [1, 2, 6, 7, 8 and 9] and in comparison with other software programs' results. The compatibility with the consecrated works allows considering it as a useful tool.

3. Results and Discussions

Let's consider a Robertson-Walker space –time metric of the form

$$ds^2 = R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + \sin^2\theta d\varphi^2\right] - dt^2 \tag{6}$$

where $R(t)$ is an unknown function of time and k is a constant, which can be chosen, by a suitable units for r to have the value –1, 0 or +1.

The spin connection coefficients are [3]

$$\Gamma_1 = \frac{\dot{R}(t)}{2\sqrt{1-kr^2}}\gamma_0\gamma_1$$

$$\Gamma_2 = \frac{1}{2}\dot{R}(t) r\gamma_0\gamma_2 + \frac{1}{2}\sqrt{1-kr^2}\gamma_2\gamma_1 \tag{7}$$

$$\Gamma_3 = \frac{1}{2}\dot{R}(t) r\sin\theta\gamma_0\gamma_3 + \frac{1}{2}\sqrt{1-kr^2}\sin\theta\gamma_3\gamma_1 + \frac{1}{2}\cos\theta\gamma_3\gamma_2 \qquad \Gamma_4 = 0$$

and, thus, can be written the Dirac equations on curved manifold

$$\begin{aligned}&\left[iR\gamma_0\left(\frac{\partial}{\partial t} + \frac{3\dot{R}}{2R}\right) - i\sqrt{1-kr^2}\gamma_1\left(\frac{\partial}{\partial r} + \frac{1}{r}\right) + \frac{1}{r}\gamma_2\left(\frac{\partial}{\partial\theta} + \frac{\cos\theta}{2\sin\theta}\right) + \right.\\ &\left. + \frac{1}{r\sin\theta}\gamma_2\frac{\partial}{\partial\varphi} - mR\right]\psi(r,\theta,\varphi,t) = 0\end{aligned} \tag{8}$$

For a modified space –time metric tensor as

$$ds^2 = F(t)^2 dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + \sin^2\theta d\varphi^2\right] \tag{9}$$

the spin connection coefficients are

$$\Gamma_0 = 0 \qquad \Gamma_1 = \frac{\dot{R}(t)}{2F(t)\sqrt{1-kr^2}}\gamma_0\gamma_1$$

$$\Gamma_2 = \frac{1}{2}\frac{\dot{R}(t)}{F(t)} r\gamma_0\gamma_2 + \frac{1}{2}\sqrt{1-kr^2}\gamma_2\gamma_1 \tag{10}$$

$$\Gamma_3 = \frac{1}{2}\frac{\dot{R}(t)}{F(t)} r\sin\theta\gamma_0\gamma_3 + \frac{1}{2}\sqrt{1-kr^2}\sin\theta\gamma_3\gamma_1 + \frac{1}{2}\cos\theta\gamma_3\gamma_2$$

Let us consider a spherically symmetric configuration describe by a static conformal metric tensor type, expressed in Schwarzchild coordinates as

$$ds^2 = e^{2(\Xi t+\Sigma)}\left[a(r)^2 dr^2 + \left(r^2 d\theta^2 + r^2\sin(\theta)d\varphi^2\right) - b(r)^2 dt^2\right] \tag{11}$$

The spin connection coefficients are [2, 14]

$$\Gamma_1 = \frac{1}{2a(r)}\frac{\partial b(r)}{\partial r}\gamma_0\gamma_1 \qquad \Gamma_2 = \frac{1}{2}\Xi\frac{a(r)}{b(r)}\gamma_0\gamma_2$$

$$\Gamma_3 = \frac{1}{2}\Xi\frac{r}{b(r)}\gamma_0\gamma_2 + \frac{1}{2}\frac{1}{a(r)}\gamma_2\gamma_1 \tag{12}$$

$$\Gamma_4 = 0$$

and, thus, can be written the Dirac equations on curved manifold

$$\begin{aligned}&\left[iR\gamma_0\left(\frac{\partial}{\partial t} + \frac{3\dot{R}}{2R}\right) - i\sqrt{1-kr^2}\gamma_1\left(\frac{\partial}{\partial r} + \frac{1}{r}\right) + \frac{1}{r}\gamma_2\left(\frac{\partial}{\partial\theta} + \frac{\cos\theta}{2\sin\theta}\right) + \right.\\ &\left. + \frac{1}{r\sin\theta}\gamma_2\frac{\partial}{\partial\varphi} - mR\right]\psi(r,\theta,\varphi,t) = 0\end{aligned} \tag{13}$$

For the metric [10] with the line element of the form

$$ds^2 = dx^1dx^1 + dx^2dx^2 + \frac{\left(dx^3\right)^2}{1-2\lambda x^3} - \left(1-2\lambda x^3\right)\left(dt^2\right) \tag{14}$$

where

$$x^3 \in \left\{-\infty, \frac{1}{2\lambda}\right\}$$

could be derived that the single non-vanishing component of the 1 – form spin connection is

$$\Gamma_3^* = \frac{\lambda}{1-\lambda z}$$

and the Dirac equation becomes

$$\gamma^1\Psi_{,1} + \gamma^2\Psi_{,2} + \gamma^3\Psi_{,3} + \gamma^4\Psi_{,4} + m_0\Psi = \frac{\lambda/2}{1-\lambda z}\gamma^3\Psi \qquad (15)$$

The obtained result is same as in [10].

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