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# INTERACTIONS BETWEEN ONE TENSOR FIELD (3,1) AND A SINGLE VECTOR FIELD*

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**Abstract**
The main BRST cohomological properties of a free theory, describing an abelian vector field and a massless tensor field that transforms in an irreducible representation of $GL(D,\mathbf{R})$ according to a two-column Young diagram of the type (3,1), are systematically used in order to determine the consistent interactions that can be added to this free theory. Our approach is mainly based on the deformation of the solution to the master equation from the antifield-BRST formalism by means of the local cohomology of the BRST differential. We constantly work under the hypotheses that the emerging interactions are local, smooth, Lorentz-covariant, Poincaré-invariant, and of maximum derivative order equal to two. The main result obtained by us is that the deformed theory modifies (at order one in the coupling constant) only the gauge transformations of the vector field with a term involving some of the gauge parameters from the (3,1) sector and consequently its field strength with a term linear in the first-order derivatives of the (3,1) tensor field. Accordingly, the deformed Lagrangian contains pieces of order one and two in the coupling constant. The gauge algebra and the reducibility ingredients present within the free theory are not affected by the deformation procedure.
***Keywords***: BRST symmetry, BRST quantization, consistent interactions.

## 1. Introduction

Tensor fields in ``exotic" representations of the Lorentz group, characterized by a mixed Young symmetry type held the attention lately on some important issues, like the dual formulation of field theories of spin two or higher [1]-[6], the impossibility of consistent cross-interactions in the dual formulation of linearized gravity [7] or a Lagrangian first-order approach [8,9] to some classes of massless or partially massive mixed symmetry-type tensor gauge fields, suggestively resembling to the tetrad formalism of General Relativity. An important matter related to mixed symmetry-type tensor fields is the study of their consistent interactions, among themselves, as well as with higher-spin gauge theories [10]--[13]. The best approach to this problem is the cohomological one, based on the deformation of the

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solution to the master equation [14]. The purpose of this paper is to investigate the consistent interactions between a single free massless tensor gauge field $t_{\lambda\mu\nu x}$ with the mixed symmetry of a two-column Young diagram of the type (3,1) and one vector field $A_{\mu}$. Our analysis relies on the deformation of the solution to the master equation by means of cohomological techniques with the help of the local BRST cohomology, whose component in the (3,1) sector has been reported in detail in [15]. Under the hypotheses of smoothness, locality, Lorentz covariance, and Poincaré invariance of the deformations, combined with the preservation of the number of derivatives on each field, we prove that the deformation of the solution to the master equation is non-trivial only in five spacetime dimensions, in which case it stops at order two in the coupling constant. From the fully deformed solution to the master equation we determine the Lagrangian formulation of the interacting model. The interacting Lagrangian action contains only mixing-component terms of order one and two in the coupling constant that can be written in a form similar to the free action of the vector field, but in terms of a deformed field strength to which the tensor field (3,1) brings a non-trivial contribution only at order one in the deformation parameter. At the level of the gauge transformations, only those of the vector fields are modified at order one in the coupling constant with a term linear in the antisymmetrized first-order derivatives of some gauge parameters from the (3,1) sector. Consequently, the gauge algebra and the reducibility structure of the coupled model are not modified during the deformation procedure, being the same like in the case of the starting free action. It is interesting to note that if we impose that the deformed theory is PT-invariant, then we obtain no interactions at all.

***Free model***

We begin with the Lagrangian action

$$\begin{aligned}S_0[t_{\lambda\upsilon\nu|\alpha},A_{\mu}] = \int d^D x\{\tfrac{1}{2}[(\partial^{\rho}t^{\lambda\mu\nu|a})(\partial_{\rho}t_{\lambda\mu\nu|a}) - (\partial^{\beta}t^{\lambda\mu\nu|a})(\partial_{\beta}t_{\lambda\mu\nu|a})] \\ -\tfrac{3}{2}[(\partial_{\lambda}t^{\lambda\mu\nu|\alpha})(\partial^{\rho}t_{\rho\mu\nu|a}) + (\partial^{\rho}t^{\lambda\mu})(\partial_{\rho}t_{\lambda\mu})] + 3(\partial_{\alpha}t^{\lambda\mu\nu|\nu})(\partial_{\lambda}t_{\mu\nu}) \\ +3(\partial_{\rho}t^{\rho\mu})(\partial^{\lambda}t_{\lambda\mu}) - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\} = S_0[t_{\lambda\mu\nu|\alpha}] + S_0[A_{\mu}],\end{aligned} \quad (1)$$

in $D \geq 5$ space-time dimensions. The massless tensor field $t_{\mu\nu!x}$ has the mixed symmetry (3,1), and hence transforms according to an irreducible representation of $GL(D,\mathbf{R})$ corresponding to a Young diagram with two columns and three rows. It is thus completely antisymmetric in its first three indices, $t_{\lambda\mu\nu|\alpha} = -t_{\mu\lambda\nu|\alpha} = -t_{\lambda\nu\mu|\alpha} = -t_{\nu\mu\lambda|\alpha}$, and satisfies the identity $t_{[\lambda\mu\nu|\alpha]}=0$. The tensor field $A_\mu$ is an abelian vector field with the field strength defined in the standard manner by

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\nu=\partial_{[\mu A\nu]}. \tag{2}$$

Everywhere in this paper we understand that the notation $[\lambda...\alpha]$ signifies complete antisymmetry with respect to the (Lorentz) indices between brackets, with the conventions that the minimum number of terms is always used and the result is never divided by the number of terms. The trace of $t_{\lambda\mu\nu|\alpha}$ is defined by $t_{\lambda\mu}=\sigma^{\nu\alpha}t_{\lambda\mu\nu|\alpha}$ and it is obviously an antisymmetric tensor. Everywhere in this paper we employ the flat Minkowski metric of `mostly plus' signature $\sigma^{\mu\nu}=\sigma_{\mu\nu}=(-,+,+,+...)$.

A generating set of gauge transformations for the action (1) can be taken of the form

$$\delta_{\varepsilon,\chi}t_{\lambda\mu\nu|a}=-3\partial_{[\lambda\mathrm{E}\upsilon\nu\alpha]}+4\partial_{[\lambda\mathrm{E}\upsilon\nu]\alpha}+\partial_{[\lambda\chi\mu\nu]!\alpha}. \tag{3}$$

$$\partial_\varepsilon A_\mu=\partial_\mu\varepsilon \tag{4}$$

where the gauge parameters $\varepsilon_{\lambda\mu\nu}$ determine a completely antisymmetric tensor, the other set of gauge parameters displays the mixed symmetry $(2,1)$, such that they are antisymmetric in the first two indices, $\chi_{\upsilon\nu|\alpha}=-\chi_{\nu\mu|\alpha}$, and satisfy the identity $\chi_{\mu\nu|\alpha}=0$, while the gauge parameter $\varepsilon$ is a scalar. The generating set of gauge transformations (3)-(4) is off-shell, second-stage reducible, the accompanying gauge algebra being obviously abelian.

***Construction of consistent interactions***

***Setting the problem***

We begin with a ``free" gauge theory, described by a Lagrangian action $S_0[\Phi^{\alpha 0}]$, which is assumed to be invariant under some gauge transformations

$$\delta_\varepsilon\Phi^{\alpha 0}=Z^{\alpha 0}{}_{\alpha 1}(\Phi)\varepsilon^{\alpha 1},\frac{\delta S_0}{\delta\Phi^{\alpha 0}}Z^{\alpha 0}{}_{\alpha 1}(\Phi)=0 \tag{5}$$

such that the gauge algebra reads as

$$\begin{aligned}&Z^{\beta 0}{}_{\alpha 1}(\Phi)\frac{\delta Z^{\alpha 0}{}_{\beta 1}(\Phi)}{\delta\Phi^{\beta 0}}-Z^{\beta 0}{}_{\beta 1}(\Phi)\frac{\delta Z^{\alpha 0}{}_{\alpha 1}(\Phi)}{\delta\Phi^{\beta 0}}=\\&C^{\lambda 1}{}_{\alpha 1\beta 1}(\Phi)Z^{\alpha 0}{}_{\lambda 1}(\Phi)+M^{\alpha 0\beta 0}{}_{\alpha 1\beta 1}(\Phi)\frac{\delta S_0}{\delta\Phi^{\beta 0}}.\end{aligned} \tag{6}$$

In (5-6) $\delta S_0/\delta\Phi^{\alpha 0}$ denotes the Euler-Lagrange derivatives of $S_0[\Phi^{\alpha 0}]$. If some of the functions $M^{\alpha 0\beta 0}{}_{\alpha 1\beta 1}$ are not vanishing, we say that the gauge algebra (6) is open or, in other words, that the gauge algebra only closes on-shell. We consider the problem of constructing consistent interactions among the fields $\Phi^{\alpha 0}$ such that the couplings preserve the field spectrum and the original number of gauge symmetries. In view of this, we deform the original action $S_0$

$$S_0 \to \bar{S}_0 = S_0 + g\overset{(1)}{S}_0 + g^2\overset{(2)}{S}_0 + .... \tag{7}$$

and the original gauge symmetries,

$$Z^{\alpha 0}{}_{\alpha 1} \to \bar{Z}^{\alpha 0}{}_{\alpha 1} = Z^{\alpha 0}{}_{\alpha 1} + g\overset{(1)}{Z}{}^{\alpha 0}{}_{\alpha 1} + g^2\overset{(2)}{Z}{}^{\alpha 0}{}_{\alpha 1} + ... \tag{8}$$

in such a way that the new gauge transformations $\bar{\delta}_\varepsilon\Phi^{\alpha 0} = \bar{Z}^{\alpha 0}{}_{\alpha 1}\varepsilon^{\alpha 1}$ are indeed gauge symmetries of the full action (7)

$$\frac{\delta(S_0 + g\overset{(1)}{S}_0 + g^2\overset{(2)}{S}_0 + ....)}{\delta\Phi^{\alpha 0}}\left(Z^{\alpha 0}{}_{\alpha 1} + g\overset{(1)}{Z}{}^{\alpha 0}{}_{\alpha 1} + g^2\overset{(2)}{Z}{}^{\alpha 0}{}_{\alpha 1} + ...\right) = 0. \tag{9}$$

In the above $g$ stands for the deformation parameter, also known as the coupling constant. In the case where the original gauge transformations are reducible, one should also demand that (8) remain reducible. By projecting the equation (9) on the various powers in the coupling constant we obtain an equivalent tower of equations. The equation corresponding to the order $g^0$ is satisfied by assumption, being nothing but the second relation in (5). The higher-order equations are rather intricate because they are non-linear and involve simultaneously not only the deformed action, but also all the deformed gauge generators.

As it will be seen below, a more convenient way to construct the consistent interactions relies on the cohomological approach, based on the BRST symmetry. The cohomological approach systematizes the recursive construction to co-cycles of the BRST differential. Finally, by reformulating the problem of consistent interactions at a cohomological level, one can bring in the powerful tools of homological algebra.

***Cohomological reformulation***

At the level of the BRST formalism, the entire gauge structure of a theory is completely captured by the BRST differential, $s$. The main feature of $s$ is its nilpotency, $s^2 = 0$. Denoting by (,) the antibracket, and by $S$ the canonical generator of the Lagrangian BRST symmetry

$$sF = (F, S) \tag{10}$$

the nilpotency of $s$ is equivalent to the classical master equation

$$(S, S) = 0. \tag{11}$$

In agreement with the structure (5-6) of the gauge algebra, the solution to the master equation (11) starts like

$$S = S_0 + \Phi^*{}_{\alpha 0} Z^{\alpha 0}{}_{\alpha 1} \eta^{\alpha 1} + \frac{1}{2}\left( \eta^*{}_{\lambda 1} C^{\lambda 1}{}_{\alpha 1 \beta 1} - \frac{1}{2} \Phi^*{}_{\alpha 0} \Phi^*{}_{\beta 0} M^{\alpha 0 \beta 0}{}_{\alpha 1 \beta 1} \right) \eta^{\alpha 1} \eta^{\beta 1} + ..., \tag{12}$$

where $\Phi^*{}_{\alpha 0}$ represent the antifields associated with the original fields, $\eta^{\alpha 1}$ are the ghosts corresponding to the gauge parameters $\varepsilon^{\alpha 1}$, and $\eta^*{}_{\lambda 1}$ denote the antifields of the ghosts.

Due to the fact that the solution to the master equation contains all the information on the gauge structure of a given theory, we can reformulate the problem of introducing consistent interactions as a deformation problem of the solution to the master equation corresponding to the ``free'' theory. If an interacting gauge theory can be consistently constructed, then the solution $S$ to the master equation associated with the ``free'' theory can be deformed into a solution $\bar{S}$

$$\begin{aligned} S \rightarrow \bar{S} + gS_1 + g^2 S_2 + ... \\ = S + g\int d^D xa + g^2 \int d^D xb + ... \end{aligned} \tag{13}$$

of the master equation for the deformed theory

$$(\bar{S}, \bar{S}) = 0, \tag{14}$$

such that both the ghost and antifield spectra of the initial theory are preserved. The equation (14) splits, according to the various orders in $g$, into

$$(S, S) = 0, \tag{15}$$

$$2(S_1, S) = 0, \tag{16}$$

$$2(S_2, S) + (S_1, S_1) = 0, \tag{17}$$

$$(S_3, S) + (S_1, S_2) = 0, \tag{18}$$

…..

The equation (15) is fulfilled by hypothesis. The next one requires that the first-order deformation of the solution to the master equation, $S_1$, is a co-cycle of the ``free'' BRST differential. However, only cohomologically non-trivial solutions to (16) should be taken into account, as the BRST-exact ones (BRST co-boundaries) correspond to trivial interactions. This means that $S_1$ pertains to the ghost number zero cohomological space of $s$, $H^0(s)$, which is generically nonempty due to its isomorphism to the space of physical observables of the ``free'' theory. It has been shown (on behalf of the triviality of the antibracket map in the cohomology of the BRST differential) that there are no obstructions in finding solutions to the remaining equations ((17-18), etc.). However, the resulting interactions may be nonlocal, and there might even appear obstructions if one insists on their locality. The analysis of these obstructions can be done with the help of cohomological techniques.

## 2. Results

There are three main types of consistent interactions that can be added to a given gauge theory: (i) the first type deforms only the Lagrangian action, but not its gauge transformations; (ii) the second kind modifies both the action and its transformations, but not the gauge algebra; (iii) the third, and certainly most interesting category, changes everything, namely, the action, its gauge symmetries and the accompanying algebra. From the full deformed solution to the master equation we can identify the interacting theory, as well as its gauge structure. In the sequel we list the main characteristics of the Lagrangian formulation of the coupled theory resulting from the corresponding deformed solution to the master equation, the detailed analyzed being reported in [16].

Our main result is given by the following theorem.

**Theorem**

*Under the hypotheses of smoothness, locality, Lorentz covariance, and Poincaré invariance of the deformations, combined with the preservation of the number of derivatives on each field, we obtain that the only non-trivial interacting theory whose free limit is (1) and (3)-(4) possesses the following features:*

*(a) its Lagrangian action is given by*

$$\bar{S}_0[t_{\lambda\mu\nu|\alpha}.A_\mu] = S_0[t_{\lambda\mu\nu|\alpha}] + S_0[A_\mu] + \int d^D x\{-\frac{2g}{3}\varepsilon^{\mu\nu\alpha\beta\gamma}F_{\mu\nu}\partial_{[\rho t\alpha\beta\gamma]|}{}^{\rho} + \frac{16g^2}{3}(\partial_{[\rho^t{}_{\alpha\beta\gamma}]|}{}^{\rho})(\partial^{[\rho^t{}_{\alpha\beta\gamma}]|}{}_{\lambda})\}; \tag{19}$$

*(b) the deformed gauge symmetries read as*

$$\bar{\delta}_{\varepsilon,\chi^t\lambda\mu\nu|\alpha} = -3\partial_{[\lambda^\varepsilon\mu\nu]\alpha} + \partial_{[\lambda\chi\mu\nu]|\alpha}, \tag{20}$$

$$\bar{\delta}_{\underset{e}{(1)}A_\mu = \partial_\mu \overset{(1)}{\varepsilon} + 4g\varepsilon_{\mu\alpha\beta\gamma\delta}\partial^\alpha\varepsilon^{\beta\gamma\delta};} \tag{21}$$

*(c) the corresponding gauge algebra is not modified if compared with the original one;*
*(d) the reducibility relations are not changed with respect to the free theory.*

The above theorem emphasizes that the coupled model describes in our case precisely some type (ii) interactions.

We observe that the action (deflag) contains only mixing-component terms and can be equivalently written as

$$\bar{S}_0[t_{\lambda\mu\nu|\alpha}, A_\mu] = S_0[t_{\lambda\mu\nu|\alpha}] - \frac{1}{4}\int d^D x \bar{F}_{\mu\nu}\bar{F}^{\mu\nu}, \qquad (22)$$

in terms of the deformed field strength

$$\bar{F}^{\mu\nu} = F^{\mu\nu} + \frac{4g}{3}\varepsilon^{\mu\nu\alpha\beta\gamma}\partial_{[\rho}t_{\alpha\beta\gamma]|}{}^{\rho}. \qquad (23)$$

It is easy to see from (19) and (20)-(21) that if we impose the PT-invariance at the level of the coupled model, then we obtain no interactions (we must set $g = 0$ in these formulas).

## 3. Conclusion

In this paper we have discussed a cohomological approach to the problem of constructing consistent interactions between a single massless tensor field $t_{\mu\nu|x}$ with the mixed symmetry $(3,1)$ and one vector field. Under the general assumptions of smoothness of the deformations in the coupling constant, locality, (background) Lorentz invariance, Poincaré invariance, PT invariance and preservation of the number of derivatives on each field, we have exhausted all the consistent, non-trivial couplings.

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